

Bilal Article

Chapter 7.  
**SCALAR AND VECTORS**

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# Contents

Exercise NO 7.1.....	2
Exercise NO 7.2.....	7
Exercise NO 7.3.....	10
Exercise NO 7.4.....	16
Exercise NO 7.5.....	21

**Scalar:**

A quantity which has magnitude only is called scalar. e. g time, volume, speed and work e.tc the scalars are denoted by letters.

**Vectors:**

A quantity which has both magnitude and direction is called vector. e. g Velocity, displacement, force and torque e. t. c. A vector say  $V$  is denoted by  $\vec{V}$  or  $V$  e.t.c. or by bold face letter  $v$ .

**Geometrical interpretation of Vector:**

Geometrically, A vector is represented by a line segment  $AB$  with A its initial point and B its terminal point. It is often found convenient to denoted a vector by an arrow and written either as  $\vec{AB}$  or as a bold face symbol like  $v$  or in underlined form  $\underline{v}$ .

**Magnitude of a vector.**

Let  $v$  be a vector, then its magnitude is denoted by  $|\vec{v}|$  or simply  $v$ . it is also called norm or length of vector if the line  $\vec{AB}$  represent a vector, then distance from Point A to point B will be magnitude of  $\vec{AB}$  and is Denoted by  $|\vec{AB}|$

**Unit vector:**

A vector whose magnitude is one (unity) is called unit vector. Unit vector of  $\underline{v}$  is written as  $\hat{v}$  (read as v hat) and is defined as  $\hat{v} = \frac{v}{|v|}$

**Null vector:**

A vector whose magnitude is zero but no specific direction is called null or zero vector. It is denoted by  $\vec{o}$

**Negative Vectors:**

Two vectors are said to be negative of each other if they have the same magnitude but opposite direction. If  $\vec{AB} = \underline{v}$  then  $\vec{BA} = -\vec{AB} = -\underline{v}$  and  $|\vec{BA}| = |-\vec{AB}|$  ( $\because$  the magnitude of a vector is a non negative member)

**Multiplication of vector by a scalar:** let  $k$  be a scalar

Number ( $k \in R$  and  $\underline{v}$  be a vector. then  $k\underline{v}$  is a vector Which is k times to vector  $\underline{v}$  .

- (i)  $\underline{v}$  and  $k\underline{v}$  are in same direction if  $k > 0$ .
- (ii)  $\underline{v}$  and  $k\underline{v}$  are in opposite direction if  $k < 0$

**Equal Vectors:**

Two vectors  $\vec{AB}$  and  $\vec{CD}$  are said to be equal, if they have same magnitude and same direction i. e  $|\vec{AB}| = |\vec{CD}|$

**Parallel vectors:**

Two vectors  $\vec{u}$ , and  $\vec{v}$  are said to be parallel if  $\vec{u} \times k \vec{v}$  Or  $\vec{v} \times k \vec{u}$  if  $\vec{u} \times k \vec{v}$  then  $\vec{u}$  and  $\vec{v}$  are same direction if  $k > 0$  also  $\vec{u}$  and  $\vec{v}$  are in opposite direction if

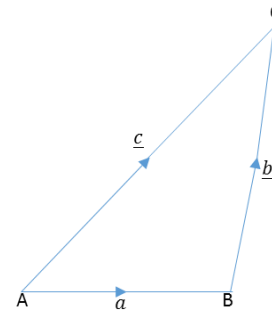
$k < 0$

**Addition of two vectors:**

Addition of two vectors is explained by following two laws.

**i. Triangle law of addition:**

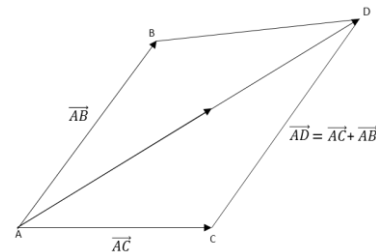
If two vectors  $\underline{a}$  and  $\underline{b}$  are represented by two Sides  $AB$  and  $BC$  of  $\Delta ABC$  s. that the terminal point  $\underline{a}$  coincide with the initial point of  $\underline{b}$ . Then the third side  $AC$  of the triangle gives Vector sum  $AC$  of the triangle gives vector sum  $\underline{a} + \underline{b}$  i. e  $\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \underline{a} + \underline{b} = \vec{AC}$



**ii. Parallelogram law of addition:**

If two vectors  $\underline{u}$  and  $\underline{v}$  are represented by two Sides  $AB$  and  $AC$  of ||gram as shown in figure. then Diagonal  $AD$  give the sum of  $\vec{AB}$  and  $\vec{AC}$  i. e

$$\vec{AD} = \vec{AB} + \vec{AC} = \underline{u} + \underline{v}$$



**Subtraction of two vectors:**

let ABCD be a ||gram.

$$\vec{AB} = \vec{DC} = \underline{a}$$

$$\vec{BC} = \vec{AD} = \underline{b}$$

In  $\Delta ABC$

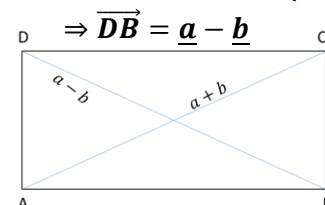
$$\vec{AC} = \vec{AB} - \vec{BC}$$

$$\vec{AC} = \underline{a} - \underline{b}$$

In  $\Delta ABD$ ,  $\vec{AB} + \vec{BD} = \vec{AD}$

$$\Rightarrow \vec{BD} = \vec{AD} - \vec{AB}$$

$$\Rightarrow \vec{BD} = \underline{b} - \underline{a} \Rightarrow \vec{DB} = -(\underline{b} - \underline{a})$$



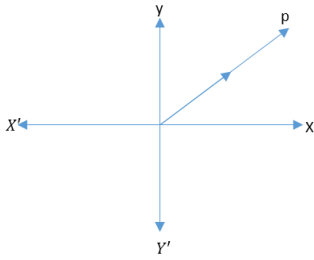
**Position vector:**

The vector whose initial point is at called position vector of point p.

Position vector of p =  $\vec{OP}$

$$\vec{AB} = \vec{DC} = \vec{a}$$

$$\vec{AB} = \vec{DC} = \vec{a}$$

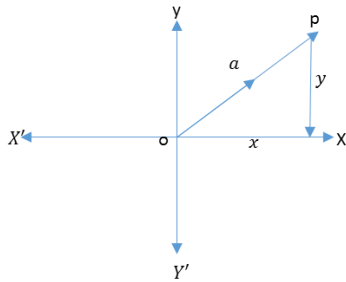


**Vectors in plane:**

let  $\vec{OP} = \vec{a}$  be a vector in xy = plane. we resolve x

and y be the horizontal and vertical components of a respectively. Then  $\vec{a}$

= [x, y] is a vector in xy – plane



**Addition:**

let  $\vec{a} = [x_1, y_1]$ ,  $\vec{b} = [x_2, y_2]$  be two vectors in plane then  $\vec{a} + \vec{b} = [x_1, y_1] + [x_2, y_2], [x_1 + x_2, y_1 + y_2]$

**Subtraction:**

let  $\vec{a} = [x_1, y_1]$ ,  $\vec{b} = [x_2, y_2]$  be two vectors in plane then  $\vec{a} - \vec{b} = [x_1, y_1] - [x_2, y_2], [x_1 - x_2, y_1 - y_2]$

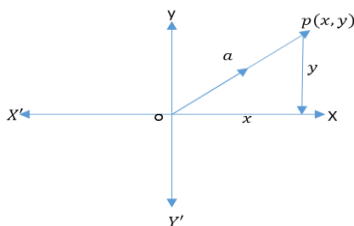
**Magnitude of a vector in plane:**

Let  $\vec{a} = [x, y]$  be vector in plane as shown in fig. Then by pat agora’s theorem

$$|\vec{OP}|^2 = |\vec{OA}|^2 + |\vec{AP}|^2$$

$$|\vec{a}|^2 = x^2 + y^2$$

$\Rightarrow a = \sqrt{x^2 + y^2}$  which is magnitude of  $\vec{a}$ .



**Another notation for represent vectors in plane:**

let  $\vec{OA}$  and  $\vec{OB}$  be two unit vectors along x – axis and y – axis respectively. then  $|\vec{OA}| = 1$   $|\vec{OB}| = 1$

as  $\vec{OA} = [1,0]$  ,  $\vec{OB} = [0,1]$

Then  $\hat{i} = [1,0]$ ,  $\hat{j} = [0,1]$

**The ratio formula**

**Statement:**

let A and B be two points whose position vectors  $\vec{a}$  and  $\vec{b}$  respectively. if a point p divides AB in the ratio p : q then the position vector of p is given

By

$$\vec{r} = \frac{qa + pb}{p + q}$$

**Proof:**

In fig.

$$\vec{OA} = \vec{a}, \quad \vec{OB} = \vec{b}$$

$$\vec{OP} = \vec{r}$$

Given that

$$\vec{AP} : \vec{PB} = p : q \Rightarrow \frac{\vec{AP}}{\vec{PB}} = \frac{p}{q} \rightarrow (i)$$

$$\Delta OAP, \quad \vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{AP} = \vec{OA} - \vec{OP} = \vec{r} - \vec{a}$$

In  $\Delta OAB$ ,  $\vec{OB} = \vec{OP} + \vec{PB}$

$$\vec{PB} = \vec{OB} - \vec{OP} = \vec{b} - \vec{r}$$

By (i)

$$\Rightarrow \frac{\vec{r} - \vec{a}}{\vec{b} - \vec{r}} = \frac{p}{q} \Rightarrow q(\vec{r} - \vec{a}) = p(\vec{b} - \vec{r})$$

$$\Rightarrow q\vec{r} - q\vec{a} = p\vec{b} - p\vec{r} \Rightarrow q\vec{r} + p\vec{r} = p\vec{b} + q\vec{a}$$

$$\Rightarrow (q + p)\vec{r} = p\vec{b} + q\vec{a}$$

$$\Rightarrow \vec{r} = \frac{p\vec{b} + q\vec{a}}{(q + p)}$$

$$\text{Hence } \Rightarrow \vec{r} = \frac{q\vec{b} + p\vec{a}}{(q + p)}$$

**Corollary:**

if p is the mid point of AB then p : q = 1 : 1

$$\text{So, } \vec{r} = \frac{\vec{a} + \vec{b}}{2}$$

**Exercise NO 7.1**

**Question No.1**

Write the vector  $\vec{PQ}$  in the form  $x\hat{i} + y\hat{j}$ .

(i)  $P(2, 3)$ ,  $Q(6, -2)$

$$\vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$\vec{PQ} = (6-2)\hat{i} + (-2-3)\hat{j}$$

$$\vec{PQ} = 4\hat{i} - 5\hat{j}$$

(ii)  $P(0, 5)$ ,  $Q(-1, -6)$

$$\vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$\vec{PQ} = (-1-0)\hat{i} + (-6-5)\hat{j}$$

$$\vec{PQ} = -1\hat{i} - 11\hat{j}$$

**Q.2: Find the magnitude of the vector u:**

(i)  $\underline{u} = 2\underline{i} - 7\underline{j}$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(2)^2 + (-7)^2}$$

$$|\underline{u}| = \sqrt{4 + 49}$$

$$|\underline{u}| = \sqrt{53}$$

(ii)  $\underline{u} = \underline{i} + \underline{j}$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(1)^2 + (1)^2}$$

$$|\underline{u}| = \sqrt{1 + 1}$$

$$|\underline{u}| = \sqrt{2}$$

(iii)  $\underline{u} = [3, -4]$

$\underline{u} = 3\underline{i} - 4\underline{j}$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(3)^2 + (-4)^2}$$

$$|\underline{u}| = \sqrt{9 + 16}$$

$$|\underline{u}| = \sqrt{25}$$

$$|\underline{u}| = 5$$

**Q.3: if  $\underline{u} = 2\underline{i} - 7\underline{j}$ ,  $\underline{v} = \underline{i} - 6\underline{j}$ ,  $\underline{w} = -\underline{i} + \underline{j}$ . Find the following vectors.**

(i)  $\underline{u} + \underline{v} - \underline{w}$

$$= (2\underline{i} - 7\underline{j}) + (\underline{i} - 6\underline{j}) - (-\underline{i} + \underline{j})$$

$$= 2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} + \underline{i} - \underline{j}$$

$$= 4\underline{i} - 14\underline{j}$$

(ii)  $2\underline{u} - 3\underline{v} + 4\underline{w}$

$$= 2(2\underline{i} - 7\underline{j}) - 3(\underline{i} - 6\underline{j}) + 4(-\underline{i} + \underline{j})$$

$$= 4\underline{i} - 14\underline{j} - 3\underline{i} + 18\underline{j} - 4\underline{i} + 4\underline{j}$$

$$= -3\underline{i} + 8\underline{j}$$

(iii)  $\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} + \frac{1}{2}\underline{w}$

$$= \frac{1}{2}(\underline{u} + \underline{v} + \underline{w}) = \frac{1}{2}(2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} - \underline{i} + \underline{j})$$

$$= \frac{1}{2}(2\underline{i} - 12\underline{j}) = \underline{i} - 6\underline{j}$$

**Q.4: Find the sum of the vectors AB and CD, given the four points A(1, -1), B(2,0), C(-1,3), D(-2,2).**

**SOLUTION:** Given: A(1, -1), B(2,0), C(-1,3), D(-2,2)

$$\overline{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overline{AB} = (2-1)\underline{i} + (0+1)\underline{j} = \underline{i} + \underline{j}$$

$$\overline{CD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overline{CD} = (-2+1)\underline{i} + (2-3)\underline{j} = -\underline{i} - \underline{j}$$

NOW  $\overline{AB} + \overline{CD} = \underline{i} + \underline{j} - \underline{i} - \underline{j} = 0\underline{i} + 0\underline{j} = \underline{0}$  (Null Vector)

**Q.5: find the vector from the point A to the origin where  $\underline{AB} = 4\underline{i} - 2\underline{j}$  and B is the point (-2,5).**

**SOLUTION:**  $\overline{AO} = ?$  here o = (0,0)

Given that:  $\overline{AB} = 4\underline{i} - 2\underline{j}$ , B(-2,5)  $\Rightarrow \overline{OB} = -2\underline{i} + 5\underline{j}$

AS we know that  $\overline{AB} = \overline{OB} - \overline{OA}$

$$\Rightarrow \overline{OA} = \overline{OB} - \overline{AB}$$

$$\Rightarrow \overline{OA} = (-2\underline{i} + 5\underline{j}) - (4\underline{i} - 2\underline{j})$$

$$= -2\underline{i} + 5\underline{j} - 4\underline{i} + 2\underline{j}$$

$$= -6\underline{i} + 7\underline{j}$$

$$\Rightarrow -\overline{AO} = -6\underline{i} + 7\underline{j} \Rightarrow \overline{AO} = 6\underline{i} - 7\underline{j} \quad (\because$$

$$\overline{OA} = -\overline{AO})$$

**Q.9: if "O" is the origin and  $\overline{OP} = \overline{AB}$ , find the point when A and B are (-3,7) and (1,0) respectively. (2019 I S.Q)**

**SOLUTION:** NOW we find point P(x, Y). Given that (-3,7) and (1,0) and here O(0,0).

As  $\overline{OP} = \overline{AB}$

$$\Rightarrow (x - 0)\underline{i} + (y - 0)\underline{j} = (1 + 3)\underline{i} + (0 - 7)\underline{j}$$

$$x\underline{i} + y\underline{j} = 4\underline{i} - 7\underline{j}$$

Comparing both sides, we have

$$x = 4 \text{ and } y = -7$$

$$\text{Thus } P(x, y) = P(4, -7)$$

**Q.6: Find a unit vector in the direction of the vector given below.**

(i)  $\underline{v} = 2\underline{i} - \underline{j}$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

NOW  $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$

$$= \frac{2\underline{i} - \underline{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$$

$$= \frac{1}{\sqrt{5}}(2\underline{i} - \underline{j})$$

(ii)  $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

$$|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

NOW  $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$

$$= \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

(iii)  $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

$$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

NOW  $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$

$$= \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$$

**Q.7: If A, B, C are respectively the points (2,-4), (4,0), (1,6). Use vector method to find the coordinates of the point D**

(i) ABCD is a parallelogram

**SOLUTION:** we find D(x, y)

Given: A (2, -4), B (4,0), C (1,6)

Since ABCD is a parallelogram, so

$$\overline{AB} = \overline{DC} \text{ and } \overline{AB} \parallel \overline{DC}$$

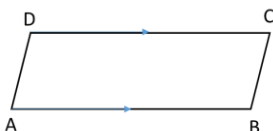
$$\overline{AB} = [4,0] - [2,-4] = [4 - 2, 0 + 4] = [2,4]$$

$$\overline{DC} = [1,6] - [x,y] = [1 - x, 6 - y]$$

$$\because \overline{AB} = \overline{DC} \Rightarrow [2,4] = [1 - x, 6 - y]$$

$$\Rightarrow 2 = 1 - x, \quad 4 = 6 - y \Rightarrow x = 1 - 2, y = 6 - 4$$

$$\Rightarrow x = -1, y = 2 \text{ so } D(x, y) = (-1, 2)$$



(ii)  $\therefore A(2, -4), B(4, 0)$  and  $C(1, 6)$

Let  $D(x, y)$  be the req. pt

as  $ADBC$  is ||gram.

So from fig  $\therefore \overrightarrow{AD} = \overrightarrow{CB}$  and  $\overrightarrow{AD} \parallel \overrightarrow{CB}$

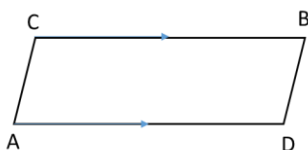
$$\overrightarrow{AD} = [x, y] - [2, -4] = [x - 2, y + 4]$$

$$\overrightarrow{CB} = [4, 0] - [1, 6] = [4, -1, 0, -6] = [3, -6]$$

$$\therefore \overrightarrow{AD} = \overrightarrow{CB} \Rightarrow [x - 2, y + 4] = [3, -6]$$

$$\Rightarrow x - 2 = 3, y + 4 = -6 \Rightarrow x = 5, y = -10$$

$$\Rightarrow \text{so } D(x, y) = (5, -10)$$



**Q.8:** If  $B, C, D$  are respectively the points  $(4, 1), (-2, 3), (-8, 0)$ . Use vector method to find the coordinates of the point:

(i) **A**, if  $ABCD$  is a parallelogram

**SOLUTION:** we find  $A(x, y)$

Given:  $B(4, 1), C(-2, 3), D(-8, 0)$

Since  $ABCD$  is a parallelogram, so

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$(4 - x)\underline{i} + (1 - y)\underline{j} = (-2 + 8)\underline{i} + (3 - 0)\underline{j}$$

$$(4 - x)\underline{i} + (1 - y)\underline{j} = 6\underline{i} + 3\underline{j}$$

Comparing both sides

$$4 - x = 6 \quad \& \quad 1 - y = 3$$

$$\Rightarrow x = -2 \quad \& \quad y = -2$$

Thus

$$A(-2, -2)$$

(ii) **E**,  $AEBC$  is a parallelogram

**SOLUTION:** we find  $E(x, y)$

Given:  $A(-2, -2), B(4, 1), D(-8, 0)$

Since  $AEBC$  is a parallelogram, so

$$\overrightarrow{AE} = \overrightarrow{BD}$$

$$(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$(x + 2)\underline{i} + (y + 2)\underline{j} = (4 + 8)\underline{i} + (1 - 0)\underline{j}$$

$$(x + 2)\underline{i} + (y + 2)\underline{j} = 12\underline{i} + \underline{j}$$

Comparing both sides

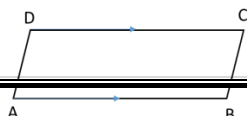
$$x + 2 = 12 \quad \& \quad y + 2 = 1$$

$$\Rightarrow x = 10 \quad \& \quad y = -1 \quad \text{Thus } E(10, -1)$$

**Q.10:** Use vectors, to show that  $ABCD$  is a parallelogram, when the points  $A, B, C$  and  $D$  are respectively  $(0, 0), (a, 0), (b, c), (b - a, c)$ .

**SOLUTION:** Given that:  $(0, 0), (a, 0), (b, c), (b - a, c)$

$$\overrightarrow{AB} = (a - 0)\underline{i} + (0 - 0)\underline{j} = a\underline{i} + 0\underline{j}$$



$$\overrightarrow{DC} = (b - b + a)\underline{i} + (c - c)\underline{j} = a\underline{i} + 0\underline{j}$$

$$\overrightarrow{AD} = (b - a - 0)\underline{i} + (c - 0)\underline{j} = (b - a)\underline{i} + c\underline{j}$$

$$\overrightarrow{BC} = (b - a)\underline{i} + (c - 0)\underline{j} = (b - a)\underline{i} + c\underline{j}$$

$$\text{As } \overrightarrow{AB} = \overrightarrow{DC} \quad \& \quad \overrightarrow{AD} = \overrightarrow{BC}$$

So  $ABCD$  is a parallelogram.

**Q.11:** if  $\overrightarrow{AB} = \overrightarrow{CD}$ . Find the coordinates of point  $A$ , when points  $B, C, D$  are  $(1, 2), (-2, 5), (4, 11)$  respectively.

**SOLUTION:** NOW we find point  $A(x, Y)$ . Given that  $B(1, 2), C(-2, 5), D(4, 11)$

As  $\overrightarrow{AB} = \overrightarrow{CD}$

$$\Rightarrow (1 - x)\underline{i} + (2 - y)\underline{j} = (4 + 2)\underline{i} + (11 - 5)\underline{j}$$

$$(1 - x)\underline{i} + (2 - y)\underline{j} = 6\underline{i} + 6\underline{j}$$

Comparing both sides, we have

$$1 - x = 6 \quad \text{and} \quad 2 - y = 6$$

$$x = 5 \quad \text{and} \quad y = -4$$

$$\text{Thus } A(x, y) = A(5, -4)$$

**Q.12:** Find the position vectors of the point of division of the line segments joining the following pair of points, in the given ratio:

(i) point " $C$ " with position vector  $2\underline{i} - 3\underline{j}$  and point  $D$  with position vector  $3\underline{i} + 2\underline{j}$  in the ratio  $4 : 3$

**SOLUTION:** Given that  $\overrightarrow{OC} = a = 2\underline{i} - 3\underline{j}, \overrightarrow{OD} = b = 3\underline{i} + 2\underline{j}, p : q = 4 : 3$

Let  $\overrightarrow{OP} = \underline{r} = ?$ , Let  $P$  be the point which divides  $CD$  in ratio  $4 : 3$

Using ratio formula, we have

$$\underline{r} = \frac{qa + pb}{p + q} = \frac{3(2\underline{i} - 3\underline{j}) + 4(3\underline{i} + 2\underline{j})}{3 + 4} = \frac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7}$$

$$= \frac{18\underline{i} - \underline{j}}{7} = \frac{18}{7}\underline{i} - \frac{1}{7}\underline{j}$$

(ii) point " $E$ " with position vector  $5\underline{i}$  and point  $F$  with position vector  $4\underline{i} + \underline{j}$  in the ratio  $2 : 5$ .

**SOLUTION:** Given that  $\overrightarrow{OE} = a = 5\underline{j}, \overrightarrow{OF} = b = 4\underline{i} + \underline{j}, p : q = 2 : 5$

Let  $\overrightarrow{OP} = \underline{r} = ?$

Let  $P$  be the point which divides  $CD$  in ratio  $2 : 5$

Using ratio formula, we have

$$\underline{r} = \frac{qa + pb}{p + q} = \frac{5(5\underline{j}) + 2(4\underline{i} + \underline{j})}{2 + 5}$$

$$= \frac{25\underline{j} + 8\underline{i} + 2\underline{j}}{7} = \frac{8\underline{i} + 27\underline{j}}{7}$$

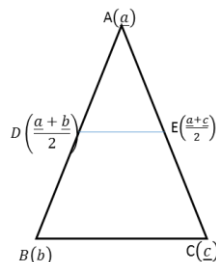
$$= \frac{8}{7}\underline{i} + \frac{27}{7}\underline{j}$$

**Q.14:** Prove that line segment joining the mid points of two sides of a triangle is parallel to third side & half as long.

**SOLUTION:** consider a triangle ABC, such that  $\vec{OA} = \underline{a}$ ,  
 $\vec{OB} = \underline{b}$ ,  $\vec{OC} = \underline{c}$

Let D & E be the mid points of sides  $\vec{AB}$  &  
 $\vec{AC}$  respectively.

$$\vec{OD} = \frac{\underline{a} + \underline{b}}{2} ; \vec{OE} = \frac{\underline{a} + \underline{c}}{2}$$



NOW  $\vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b} \rightarrow$  (i)

AND  $\vec{DE} = \vec{OE} - \vec{OD} = \frac{\underline{a} + \underline{b}}{2} - \frac{\underline{a} + \underline{c}}{2} = \frac{(\underline{a} + \underline{b}) - (\underline{a} + \underline{c})}{2} =$   
 $\frac{\underline{a} + \underline{b} - \underline{a} - \underline{c}}{2} = \frac{\underline{b} - \underline{c}}{2} = \frac{1}{2}(\underline{c} - \underline{b})$  (ii)

From (i) & (ii) it is clear that BC & DE are parallel and  $\vec{DE}$  is half to  $\vec{BC}$ .

**Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.**

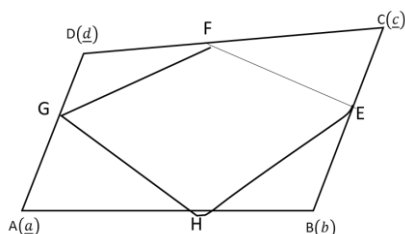
**Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.**

**SOLUTION:** consider the quadrilateral ABCD, such that  $\vec{OA} = \underline{a}$ ,  $\vec{OB} = \underline{b}$ ,  $\vec{OC} = \underline{c}$ ,  $\vec{OD} = \underline{d}$

Let E, F, G, H be the mid points of sides  $\vec{AB}$ ,  $\vec{BC}$ ,  $\vec{CD}$ ,  $\vec{AD}$  respectively. D(d)

$$\vec{OE} = \frac{\underline{a} + \underline{b}}{2}, \vec{OF} = \frac{\underline{b} + \underline{c}}{2}, \vec{OG} = \frac{\underline{c} + \underline{d}}{2}, \vec{OH} = \frac{\underline{a} + \underline{d}}{2}$$

NOW  $\vec{EF} = \vec{OF} - \vec{OE} = \frac{\underline{b} + \underline{c}}{2} - \frac{\underline{a} + \underline{b}}{2}$



$$= \frac{(\underline{b} + \underline{c}) - (\underline{a} + \underline{b})}{2} = \frac{\underline{b} + \underline{c} - \underline{a} - \underline{b}}{2} = \frac{\underline{c} - \underline{a}}{2}$$

$$\vec{HG} = \vec{OG} - \vec{OH} = \frac{\underline{c} + \underline{d}}{2} - \frac{\underline{a} + \underline{d}}{2} = \frac{(\underline{c} + \underline{d}) - (\underline{a} + \underline{d})}{2} = \frac{\underline{c} + \underline{d} - \underline{a} - \underline{d}}{2} = \frac{\underline{c} - \underline{a}}{2}$$

$$\vec{EH} = \vec{OH} - \vec{OE} = \frac{\underline{a} + \underline{d}}{2} - \frac{\underline{a} + \underline{b}}{2} = \frac{(\underline{a} + \underline{d}) - (\underline{a} + \underline{b})}{2} = \frac{\underline{a} + \underline{d} - \underline{a} - \underline{b}}{2} = \frac{\underline{d} - \underline{b}}{2}$$

$$\vec{FG} = \vec{OG} - \vec{OF} = \frac{\underline{c} + \underline{d}}{2} - \frac{\underline{b} + \underline{c}}{2} = \frac{(\underline{c} + \underline{d}) - (\underline{b} + \underline{c})}{2} = \frac{\underline{c} + \underline{d} - \underline{b} - \underline{c}}{2} = \frac{\underline{d} - \underline{b}}{2}$$

As  $\vec{EF} = \vec{HG}$  and  $\vec{EH} = \vec{FG}$   
 $\Rightarrow \vec{EF} \parallel \vec{HG}$  and  $\vec{EH} \parallel \vec{FG}$

Thus EFGH is a parallelogram.

**Concept of vector in space**

If three dimensional space  $XOX'$  and  $ZOZ'$  are called Coordinate axis.

The planes made by

$XOY$ ,  $YOZ$  and  $ZOX$  are called  $XY$

Plane.  $YZ$  – plane and  $ZX$  plane. respectively

These planes are mutually orthogonal to each other.

**Note:**

If P is point in space then it will have three coordinates along  $x$  – axis,  $y$  – axis and  $z$  – axis respectively.

if  $a, b, c$  are distance along  $x$  – axis,  $y$  – axis and  $z$  – axis respectively

Then coordinates of point p are  $p(a, b, c)$  as shown in fig. let  $\vec{u}$  be vector in space whose position vector is  $\vec{OP}$  then  $\vec{u} = [x, y, z]$

**Now (i) Addition:**

$$\text{let } \vec{u} = [x_1, y_1, z_1], \vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1, y_1, z_1] + [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

**(ii) Difference:**

$$\text{let } \vec{u} = [x_1, y_1, z_1], \vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} - \vec{v} = [x_1, y_1, z_1] - [x_2, y_2, z_2]$$

$$\vec{u} - \vec{v} = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$$

**(iii) Scalar multiplication:**

if  $k$  be any scalar and  $\vec{u} = [x, y, z]$  then  $k\vec{u}$

$$= k[x, y, z]$$

$$= [kx, ky, kz]$$

**Magnitude of vector (in space)**

let  $\vec{u} = [x, y, z]$  be a vector in space then its length or norm or magnitude denoted by  $|\vec{u}|$  and defines as

$$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$$

**Properties of a vector:**

**(i) Commutative property:**

for any two vectors  $\vec{u}$  and  $\vec{v}$

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

**Proof:**

$$\text{let } \vec{u} = [x_1, y_1, z_1], \vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1, y_1, z_1] + [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$= [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$= [x_2 + x_1, y_2 + y_1, z_2 + z_1]$$

$$= [x_2, y_2, z_2] + [x_1, y_1, z_1]$$

$$\equiv \vec{v} + \vec{u}$$

(in real numbers commutatives law hold)

Thus  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

**(ii) Associative Property:**

for any three vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

**Proof:**

Let



$$\begin{aligned}
 (\vec{u} + \vec{v}) + \vec{w} &= (x_1, y_1, z_1 + x_2, y_2, z_2) + (x_3, y_3, z_3) \\
 &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3) \\
 &= (x_1, y_1, z_1) + (x_2 + x_3, y_2 + y_3, z_2 + z_3) \\
 &= (x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3)) \\
 &= \vec{u} + (\vec{v} + \vec{w})
 \end{aligned}$$

Hence  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

**Inverse property:**

For a vector  $\vec{u}$ ,  $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = 0$

*Proof:*

let  $\vec{u} = [x, y, z]$  then

$$-\vec{u} = [-x, -y, -z]$$

then  $\vec{u} + (-\vec{u}) = [x, y, z] + [-x, -y, -z]$

$$= [x - x, y - y, z - z]$$

$$(0, 0, 0) = 0$$

Similarly:

$$(-\vec{u}) + \vec{u} = [-x, -y, -z] + [x, y, z]$$

$$= [-x + x, -y + y, -z + z] = [0, 0, 0] = 0$$

Thus for  $\vec{u}$  its additive inverse is  $-\vec{u}$

**(iv) Distributive Property:**

if  $k$  be any scalar and  $\vec{u}, \vec{v}$  be two vectors then

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

*Proof:*

let  $\vec{u} = [x_1, y_1, z_1]$ ,  $\vec{v} = [x_2, y_2, z_2]$

$$k(\vec{u} + \vec{v}) = k[(x_1, y_1, z_1) + (x_2, y_2, z_2)]$$

$$= k[x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$= [kx_1 + kx_2, ky_1 + ky_2, kz_1 + kz_2]$$

$$= k(x_1, y_1, z_1) + k(x_2, y_2, z_2)$$

$$= k\vec{u} + k\vec{v}$$

**(v) Scalar Multiplication property:**

for  $a, b$  being scalars and  $\vec{u}$  be a vector. then

$$a(b\vec{u}) = (ab)\vec{u}$$

*Proof:*

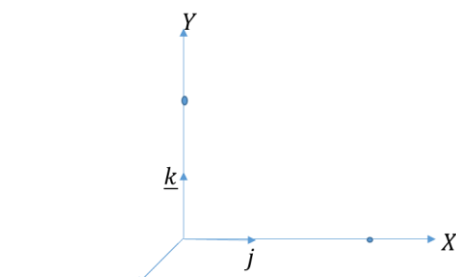
Let  $\vec{u} = [x, y, z]$

$$\Rightarrow a(b\vec{u}) = a(b[x, y, z]) = a[bx, by, bz]$$

$$\Rightarrow [abx, aby, abz] = (ab)\vec{u}$$

**Another Notation for represented vectors in space:**

Let  $\vec{OA}$ ,  $\vec{OB}$  and  $\vec{OC}$  be vectors along  $x$ -axis,  $y$ -axis and  $z$ -axis respectively.



As

$$\vec{OA} = [1, 0, 0], \vec{OB} = [0, 1, 0], \vec{OC} = [0, 0, 1]$$

$$\text{then } \underline{i} = [1, 0, 0], \underline{j} = [0, 1, 0], \underline{k} = [0, 0, 1]$$

where  $\underline{i}, \underline{j}$ , and  $\underline{k}$  are called unit vectors along

$x$ -axis,  $y$ -axis and  $z$ -axis.

Now if  $\vec{u} = [x, y, z]$  then  $\vec{u}$  can be written as

$$\vec{u} = [x, 0, 0] + [0, y, 0] + [0, 0, z]$$

$$= x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1]$$

$$\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$$

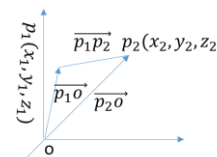
**Distance between two points in space:**

if  $\vec{OP}_1$  and  $\vec{OP}_2$  are the position vectors of the points  $p_1(x_1, y_1, z_1)$  and  $p_2(x_2, y_2, z_2)$

The vector  $\vec{p_1p_2}$  is given by

$$\begin{aligned}
 \vec{p_1p_2} &= \vec{OP}_2 - \vec{OP}_1 \\
 &= [x_2, y_2, z_2] - [x_1, y_1, z_1] \\
 &= [x_2 - x_1, y_2 - y_1, z_2 - z_1]
 \end{aligned}$$

By



$$\text{Distance b/w } p_1 \text{ and } p_2 = |\vec{p_1p_2}|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is called distance formula b/w two points

$p_1$  and  $p_2$  in  $R$

**Distance angles and direction cosine of a vector:**

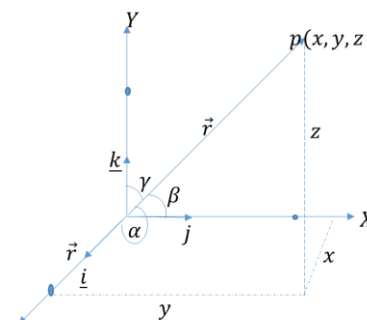
Let  $C = \vec{OP} = [x, y, z] = x\hat{i} + y\hat{j} + z\hat{k}$  be a vector

such that it makes angles  $\alpha, \beta$  and  $\gamma$  along

coordinates axes the  $\alpha, \beta$  and  $\gamma$  are called direction angles

of vectors  $\vec{r}$  while  $\cos\alpha, \cos\beta$  and  $\cos\gamma$  are called direction

cosines of  $\vec{r}$



**Important result**

Prove that  $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

*Proof:*

$$\text{let } \vec{r} = [x_1, y_1, z_1] = x\underline{i} + y\underline{j} + z\underline{k}$$

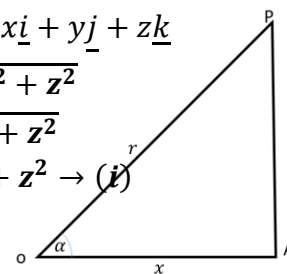
$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2 \rightarrow (i)$$

in right  $\Delta OAP$

$$\cos\alpha = \frac{OA}{OP} = \frac{z}{r}$$



$$\text{similarly } \cos\beta = \frac{y}{r} = \cos\gamma = \frac{x}{r}$$

$$\text{So } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \text{ hence proved.}$$



## Exercise NO 7.2

**Q.1:** Let  $A = (2, 5)$ ,  $B = (-1, 1)$ ,  $C = (2, -6)$ . Find

(i)  $\overrightarrow{AB} = ?$

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overrightarrow{AB} = (-1-2)\underline{i} + (1-5)\underline{j}$$

$$\overrightarrow{AB} = -3\underline{i} - 4\underline{j}$$

(ii)  $2\overrightarrow{AB} - \overrightarrow{CB} = ?$

$$2\overrightarrow{AB} - \overrightarrow{CB} = 2[(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}] - [(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}]$$

$$2\overrightarrow{AB} - \overrightarrow{CB} = 2[(-1-2)\underline{i} + (1-5)\underline{j}] - [(-1-2)\underline{i} + (1+6)\underline{j}]$$

$$2\overrightarrow{AB} - \overrightarrow{CB} = 2[-3\underline{i} - 4\underline{j}] - [-3\underline{i} + 7\underline{j}] = -6\underline{i} - 8\underline{j} + 3\underline{i} - 7\underline{j} = -3\underline{i} - 15\underline{j}$$

(iii)  $2\overrightarrow{CB} - 2\overrightarrow{CA} = ?$

$$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = 2\{[(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}] - [(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}]\}$$

$$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = 2\{(-1-2)\underline{i} + (1+6)\underline{j}\} - \{(-2-2)\underline{i} + (5+6)\underline{j}\} = 2\{(-3\underline{i} + 7\underline{j})\} - \{0\underline{i} + 11\underline{j}\}$$

$$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = -6\underline{i} + 14\underline{j} - 0\underline{i} - 11\underline{j} = -6\underline{i} - 3\underline{j}$$

**Q.2:** Let  $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$ ,  $\underline{v} = 3\underline{i} - 2\underline{j} + 2\underline{k}$ ,  $\underline{w} =$

$5\underline{i} - \underline{j} + 3\underline{k}$ . Find the indicated vectors or numbers:

(i)  $\underline{u} + 2\underline{v} + \underline{w} = ?$

$$\underline{u} + 2\underline{v} + \underline{w} = (\underline{i} + 2\underline{j} - \underline{k}) + 2(3\underline{i} - 2\underline{j} + 2\underline{k}) + (5\underline{i} - \underline{j} + 3\underline{k}) = \underline{i} + 2\underline{j} - \underline{k} + 6\underline{i} - 4\underline{j} + 4\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k} = 12\underline{i} - 3\underline{j} + 6\underline{k}$$

(ii)  $\underline{v} - 3\underline{w} = ?$

$$\underline{v} - 3\underline{w} = (3\underline{i} - 2\underline{j} + 2\underline{k}) - 3(5\underline{i} - \underline{j} + 3\underline{k}) = 3\underline{i} - 2\underline{j} + 2\underline{k} - 15\underline{i} + 3\underline{j} - 9\underline{k} = -12\underline{i} + \underline{j} - 7\underline{k}$$

(iii)  $|3\underline{v} + \underline{w}| = ?$

$$3\underline{v} + \underline{w} = 3(3\underline{i} - 2\underline{j} + 2\underline{k}) + 5\underline{i} - \underline{j} + 3\underline{k} = 9\underline{i} - 6\underline{j} + 6\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k} = 14\underline{i} - 7\underline{j} + 9\underline{k}$$

$$|3\underline{v} + \underline{w}| = \sqrt{(14)^2 + (-7)^2 + (9)^2} = \sqrt{196 + 49 + 81} = \sqrt{326}$$

**Q.3:** Find the magnitude of the vector  $\underline{v}$  and write the direction cosine of  $\underline{v}$ :

(i)  $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

**MAGNITUDE**

$$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\underline{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2}$$

$$|\underline{v}| = \sqrt{4 + 9 + 16} = \sqrt{29}$$

**DIRECTION COSINE**

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 2, 3, 4$$

$$\text{So, Direction cosine} = \frac{2, 3, 4}{\sqrt{29}}$$

$$\text{Direction cosine} = \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

(ii)  $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

**MAGNITUDE**

$$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\underline{v}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2}$$

$$|\underline{v}| = \sqrt{1 + 1 + 1} = \sqrt{3}$$

**DIRECTION COSINE**

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 1, -1, -1$$

$$\text{So, Direction cosine} = \frac{1, -1, -1}{\sqrt{3}}$$

$$\text{Direction cosine} = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

(iii)  $\underline{v} = 4\underline{i} - 5\underline{j}$

**MAGNITUDE**

$$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\underline{v}| = \sqrt{(4)^2 + (-5)^2 + (0)^2}$$

$$|\underline{v}| = \sqrt{16 + 25 + 0} = \sqrt{41}$$

**DIRECTION COSINE**

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 4, -5, 0$$

$$\text{So, Direction cosine} = \frac{4, -5, 0}{\sqrt{41}}$$

$$\text{Direction cosine} = \frac{4}{\sqrt{41}}, \frac{-5}{\sqrt{41}}, \frac{0}{\sqrt{41}}$$

**Q.4:** Find  $\alpha$ , so that

$$|\alpha\underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$$

**SOLUTION:**

$$|\alpha\underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$$

$$\sqrt{(\alpha)^2 + (\alpha + 1)^2 + (2)^2} = 3$$

$$\sqrt{\alpha^2 + \alpha^2 + 1 + 2\alpha + 4} = 3$$

Squaring on both side

$$\alpha^2 + \alpha^2 + 1 + 2\alpha + 4 = 9$$

$$2\alpha^2 + 2\alpha + 5 = 9$$

$$2\alpha^2 + 2\alpha + 5 - 9 = 0$$

$$2\alpha^2 + 2\alpha - 4 = 0$$

Dividing by 2

$$\alpha^2 + \alpha - 2 = 0$$

$$\alpha^2 + 2\alpha - \alpha - 2 = 0$$

$$\alpha(\alpha + 2) - 1(\alpha + 2) = 0$$

$$(\alpha + 2)(\alpha - 1) = 0$$

Either  $\alpha + 2 = 0$  or  $\alpha - 1 = 0$

**Q.5:** Find a unit vector in the direction of  $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

**SOLUTION:**

Let  $\hat{V}$  be unit vector in the direction of  $v$

$$\text{Then, } \underline{v} = \underline{i} + 2\underline{j} - \underline{k}$$

$$|\underline{v}| = \sqrt{(1)^2 + (2)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Required unit vector is:

$$\hat{V} = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} + 2\underline{j} - \underline{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\underline{i} + \frac{2}{\sqrt{6}}\underline{j} - \frac{1}{\sqrt{6}}\underline{k}$$

**Q.6:** if  $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$ ,  $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ ,  $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$ . find a unit vector parallel to  $3\underline{a} - 2\underline{b} + 4\underline{c}$ .

**SOLUTION:**

$$\text{Let } \underline{v} = 3\underline{a} - 2\underline{b} + 4\underline{c} = 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$$

$$= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k} = 17\underline{i} + 13\underline{j} - 10\underline{k}$$

$$|\underline{v}| = \sqrt{(17)^2 + (13)^2 + (-10)^2} = \sqrt{289 + 169 + 100} = \sqrt{558}$$

$$\text{Let } \underline{u} \text{ be a unit vector parallel to } \underline{v}, \text{ then}$$

$$\underline{u} = \frac{\underline{v}}{|\underline{v}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}} = \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

**Q.7:** Find a vector whose:

i) Magnitude is 4 and is parallel to  $2\underline{i} - 3\underline{j} + 6\underline{k}$  (2019 I

S.Q)

**SOLUTION:**

$$U = ? \text{ Given: } |u| = 4, \underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \frac{\underline{v}}{|\underline{v}|}$$

$$U = |u| \frac{\underline{v}}{|\underline{v}|} = (4) \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{4[2\underline{i} - 3\underline{j} + 6\underline{k}]}{\sqrt{49}}$$

$$= \frac{8\underline{i} - 12\underline{j} + 24\underline{k}}{7}$$

$$U = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

ii) magnitude is 2 and is parallel to  $-\underline{i} + \underline{j} + \underline{k}$

**SOLUTION:**

$$U = ? \text{ Given: } |u| = 2, \underline{v} = -\underline{i} + \underline{j} + \underline{k}$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \frac{\underline{v}}{|\underline{v}|}$$

$$U = |u| \frac{\underline{v}}{|\underline{v}|} = (2) \frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2[-\underline{i} + \underline{j} + \underline{k}]}{\sqrt{49}}$$

$$= \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{7}$$

$$U = -\frac{2}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{2}{7}\underline{k}$$

**Q.8:** if  $\underline{u} = 2\underline{i} + 3\underline{j} + 4\underline{k}$ ,  $\underline{v} = -\underline{i} + 3\underline{j} - \underline{k}$ ,

$\underline{w} = \underline{i} + 6\underline{j} + z\underline{k}$  represents the sides of a triangle. Find the value of  $Z$ .

**SOLUTION:**

As  $u, v, w$  represents the sides of triangle

$$\text{So } \underline{u} + \underline{v} = \underline{w}$$

$$2\underline{i} + 3\underline{j} + 4\underline{k} - \underline{i} + 3\underline{j} - \underline{k} = \underline{i} + 6\underline{j} + z\underline{k}$$

$$\underline{i} + 6\underline{j} + 8\underline{k} = \underline{i} + 6\underline{j} + z\underline{k}$$

By comparing

$$\Rightarrow z = 8$$

**Q.9:** The position vectors of the points  $A, B, C$  and  $D$  are  $2\underline{i} - \underline{j} + \underline{k}$ ,  $3\underline{i} + \underline{j}$ ,  $2\underline{i} + 4\underline{j} - 2\underline{k}$  and  $-\underline{i} - 2\underline{j} + \underline{k}$  respectively. Show that  $AB$  is parallel to  $CD$ .

**SOLUTION:**

$$\text{Given that: } OA = 2\underline{i} - \underline{j} + \underline{k}, OB = 3\underline{i} + \underline{j}, OC = 2\underline{i} + 4\underline{j} - 2\underline{k}, OD = -\underline{i} - 2\underline{j} + \underline{k}$$

$$AB = OB - OA = (3\underline{i} + \underline{j}) - (2\underline{i} - \underline{j} + \underline{k}) = 3\underline{i} + \underline{j} - 2\underline{i} + \underline{j} - \underline{k} = \underline{i} + 2\underline{j} - \underline{k}$$

$$CD = OD - OC = (-\underline{i} - 2\underline{j} + \underline{k}) - (2\underline{i} + 4\underline{j} - 2\underline{k}) = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k} = -3\underline{i} - 6\underline{j} + 3\underline{k} = -3AB$$

$$\Rightarrow AB \parallel CD \text{ Hence } AB \text{ is parallel to } CD.$$

**Q.10:** We say that vectors  $v$  and  $w$  are parallel if there is a scalar  $c$  such that  $v = cw$ . The vector point in the same direction if  $c > 0$  and the vector point in the opposite direction if  $c < 0$ .

a) Find two vectors of length 2 parallel to vector  $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

**SOLUTION:**

$$\text{Given: } |\underline{u}| = 2, \underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k} \underline{u} = ?$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \pm \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \pm \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm |\underline{u}| \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm (2) \frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{\sqrt{2^2 + (-4)^2 + 4^2}}$$

$$\underline{u} = \pm \frac{2[2\underline{i} - 4\underline{j} + 4\underline{k}]}{6}$$

$$\underline{u} = \pm \frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{3}$$

$$\underline{u} = \pm \left( \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k} \right)$$

b) Find the constant  $a$  so that the vectors  $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$  &  $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$  are parallel.

**SOLUTION:**

$$\text{As } \underline{v} \parallel \underline{w}$$

$$\therefore \frac{a}{1} = \frac{9}{-3} \text{ (Direction ratio are same)}$$

$$\Rightarrow a = -3$$

c) Find a vectors of length 5 in the direction opposite that of  $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$  (2019 S.Q)

**SOLUTION:**  $\underline{u} = ?$

$$\text{Given: } |\underline{u}| = 5, \underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\begin{aligned} \text{So } \underline{u} &= -\frac{v}{|v|} \\ \Rightarrow \frac{\underline{u}}{|\underline{u}|} &= -\frac{v}{|v|} \\ \underline{u} &= -|\underline{u}| \frac{v}{|v|} \\ \underline{u} &= -(5) \frac{i-2j+3k}{\sqrt{1^2+(-2)^2+3^2}} \\ \underline{u} &= -\frac{5[i-2j+3k]}{\sqrt{14}} \\ \underline{u} &= \frac{-5i+10j-15k}{\sqrt{14}} \end{aligned}$$

d) Find a & b so that the vectors

$$3\underline{i} - \underline{j} + 4\underline{k}, a\underline{i} + b\underline{j} - 2\underline{k} \text{ are parallel.}$$

**SOLUTION:** As  $\underline{v} \parallel \underline{w}$

$$\therefore \frac{a}{3} = \frac{b}{-1} = \frac{-2}{4} \Rightarrow \frac{a}{3} = \frac{b}{-1} = \frac{-1}{2}$$

$$\Rightarrow \frac{a}{3} = \frac{-1}{2} \text{ and } \frac{b}{-1} = \frac{-1}{2}$$

$$\Rightarrow a = -\frac{3}{2} \text{ and } b = \frac{1}{2}$$

Q.11: Find the direction cosines for the given vector.

i)  $3\underline{i} - \underline{j} + 2\underline{k}$

**SOLUTION:**

$$\text{Let } \underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$$

$$|\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$$

$$|\underline{v}| = \sqrt{9 + 1 + 4} = \sqrt{14}$$

Direction ratio of v are 3, -1, 2

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$$

ii)  $6\underline{i} - 2\underline{j} + \underline{k}$  (2017S.Q)

**SOLUTION:**

$$\text{Let } \underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{36 + 4 + 1} = \sqrt{41}$$

Direction ratio of v are 6, -2, 1

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}$$

iii)  $\overrightarrow{PQ}$ , where  $\mathbf{p} = (2,1,5)$  &  $\mathbf{Q} = (1,3,1)$

$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PQ} = (1-2)\underline{i} + (3-1)\underline{j} + (1-5)\underline{k}$$

$$\overrightarrow{PQ} = -\underline{i} + 2\underline{j} - 4\underline{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (2)^2 + (-4)^2}$$

$$|\overrightarrow{PQ}| = \sqrt{1 + 4 + 16} = \sqrt{21}$$

Direction ratio of v are -1, 2, -4

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}$$

Q.12: Which of the following triples can be the direction angles of a single vector:

i)  $45^\circ, 45^\circ, 60^\circ$  (2018S.Q)

**SOLUTION:**

$$\text{Let } \alpha = 45^\circ, \beta = 45^\circ, \gamma = 60^\circ$$

If  $\alpha, \beta, \gamma$  are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{2+2+1}{4} = \frac{5}{4} \neq \text{R.H.S}$$

Thus,  $\beta, \gamma$  are not the direction angles of a single vector.

ii)  $30^\circ, 45^\circ, 60^\circ$

**SOLUTION:**

$$\text{Let } \alpha = 30^\circ, \beta = 45^\circ, \gamma = 60^\circ$$

If  $\alpha, \beta, \gamma$  are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \left(\frac{3}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{9}{16} + \frac{1}{2} + \frac{1}{4} = \frac{9+8+4}{16} = \frac{21}{16} \neq \text{R.H.S}$$

Thus,  $\beta, \gamma$  are not the direction angles of a single vector.

iii)  $45^\circ, 60^\circ, 60^\circ$

**SOLUTION:**

$$\text{Let } \alpha = 45^\circ, \beta = 60^\circ, \gamma = 60^\circ$$

If  $\alpha, \beta, \gamma$  are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ$$

$$= \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{2+1+1}{4} = \frac{4}{4} = 1 \text{ R.H.S}$$

Thus,  $\beta, \gamma$  are the direction angles of a single vector.

**The Scalar product of two vectors (Dot product)**

**Definition:**

The scalar product of two non-zero vectors  $\vec{u}$  and  $\vec{v}$  is denoted by  $\vec{u} \cdot \vec{v}$  (read as  $\vec{u}$  dot  $\vec{v}$ ) and defined as ;

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta$$

Where  $\theta$  is angle from  $\vec{u}$  to  $\vec{v}$  and  $0 \leq \theta \leq \pi$

**Note: the dot product is also referred to the scalar product or the inner product.**

**Deduction of the important results:**

For unit vector  $\underline{i}, \underline{j}, \underline{k}$  we have.

$$(a) \underline{i} \cdot \underline{i} = |\underline{i}| |\underline{i}| \cos 0^\circ = 1$$

$$\underline{j} \cdot \underline{j} = |\underline{j}| |\underline{j}| \cos 0^\circ = 1$$

$$\underline{k} \cdot \underline{k} = |\underline{k}| |\underline{k}| \cos 0^\circ = 1$$

$$(b) \underline{i} \cdot \underline{j} = |\underline{i}| |\underline{j}| \cos 90^\circ = 0$$

$$\underline{j} \cdot \underline{k} = |\underline{j}| |\underline{k}| \cos 90^\circ = 0$$

$$\underline{k} \cdot \underline{i} = |\underline{k}| |\underline{i}| \cos 90^\circ = 0$$

$$(c) \text{ prove that } \underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos\theta$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos(-\theta)$$

$$\underline{u} \cdot \underline{v} = |\underline{v}| |\underline{u}| \cos\theta$$

$$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$$

Hence dot product of two vectors is commutative.

**Perpendicular (orthogonal) vectors**

Two non-zero vectors  $\underline{u}$  and  $\underline{v}$  are perpendicular if

$$\text{and only if } \underline{u} \cdot \underline{v} = 0$$

(∵ angle b/w  $\vec{u}$  and  $\vec{v}$  is  $\frac{\pi}{2}$  so  $\cos \frac{\pi}{2} = 0$ )

$$\text{thus } \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta \cos \frac{\pi}{2} \Rightarrow \vec{u} \cdot \vec{v} = 0$$

**Properties of Dot Product:**

Let  $\underline{u}, \underline{v}$  and  $\underline{w}$  be vectors and let  $c \in R$

then (i)  $\underline{u} \cdot \underline{v} = 0 \Rightarrow \underline{u} = 0$  or  $\underline{v} = 0$

(ii)  $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$  (Commutative Property)

(iii)  $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$  (dis. property)

(iv)  $(c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v})$  ( $c$  is scalar)

(v)  $\underline{u} \cdot \underline{u} = |\underline{u}|^2$

**Analytical Expression of dot product  $\underline{u} \cdot \underline{v}$ :**

(Dot product of vectors in their components from)

let  $\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$  and

$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$

Be two vectors

Now

$$\begin{aligned} \underline{u} \cdot \underline{v} &= (a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}) \cdot (a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}) \\ &= (a_1 a_2)(\underline{i} \cdot \underline{i}) + a_1 b_2 (\underline{i} \cdot \underline{j}) + a_1 c_2 (\underline{i} \cdot \underline{k}) \\ &\quad + b_1 a_2 (\underline{j} \cdot \underline{i}) \end{aligned}$$

$$\begin{aligned} &+ b_1 b_2 (\underline{j} \cdot \underline{j}) + b_1 c_2 (\underline{j} \cdot \underline{k}) + c_1 a_2 (\underline{k} \cdot \underline{i}) + c_1 b_2 (\underline{k} \cdot \underline{j}) \\ &\quad + c_1 c_2 (\underline{k} \cdot \underline{k}) \end{aligned}$$

$$\Rightarrow \underline{u} \cdot \underline{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

**Angle b/w two vectors:**

For two vectors  $\underline{u}$  and  $\underline{v}$

$$(a) \quad \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta \quad \because \quad 0 \leq \theta \leq \pi$$

$$\therefore \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\underline{u}| |\underline{v}|}$$

(b) if  $\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$  and

$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$

$$\Rightarrow \underline{u} \cdot \underline{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

$$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2} \quad , \quad |\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

Since

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\underline{u}| |\underline{v}|}$$

$$\Rightarrow \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

**Corollaries:**

if  $\theta$

$= 0^\circ$  or  $\pi$  the vector  $\underline{u}$  and  $\underline{v}$  are collinear.

(ii) if  $\theta = \frac{\pi}{2}$ ,  $\cos\theta = 0 \Rightarrow \underline{u} \cdot \underline{v} = 0$

Then vector  $\underline{u}$  and  $\underline{v}$  are  $\perp$  or othogonal.

**Projection of a vector upon another vector:**

Let  $\vec{OA} = \underline{u}$  and  $\vec{OB} = \underline{v}$

let  $\theta$  be the angle between them such that

$$0 \leq \theta \leq \pi \text{ Draw } \overline{BM}$$

$\perp \overline{OA}$  then  $\overline{OM}$  is called the

projection of  $\underline{v}$  along  $\underline{u}$  Now in  $\Delta OMB$ .

$$\cos\theta = \frac{|\overline{OM}|}{|\overline{OB}|} \Rightarrow |\overline{OM}| = |\overline{OB}| \cos\theta$$

$$|\overline{OM}| = |\underline{v}| \cos\theta \quad \therefore \cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\therefore \text{projction of } \underline{v} \text{ along } \underline{u} = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}|}$$

Similarly,

$$\therefore \text{projction of } \underline{u} \text{ along } \underline{v} = \frac{\underline{u} \cdot \underline{v}}{|\underline{v}|}$$

**Exercise NO 7.3**

**Question No1**

Find the cosine of angle  $\theta$  between  $\underline{u}$  and  $\underline{v}$

i)  $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$ ,  $\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$  (2016 S.Q)

**SOLUTION:**

$$\underline{u} = 3\underline{i} + \underline{j} - \underline{k}, \underline{v} = 2\underline{i} - \underline{j} + \underline{k}$$

If  $\theta$  is the angle between  $\underline{u}$  and  $\underline{v}$ , then

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos\theta = \frac{(3\underline{i} + \underline{j} - \underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k})}{\sqrt{3^2 + 1^2 + (-1)^2} \cdot \sqrt{2^2 + (-1)^2 + 1^2}}$$

$$\cos\theta = \frac{6 - 1 - 1}{\sqrt{9 + 1 + 1} \cdot \sqrt{4 + 1 + 1}}$$

$$\cos\theta = \frac{4}{\sqrt{11} \cdot \sqrt{6}} = \frac{4}{\sqrt{66}}$$

ii)  $\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}$ ,  $\underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$

**SOLUTION:**

$$\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}, \underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$$

If  $\theta$  is the angle between  $\underline{u}$  and  $\underline{v}$ , then

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos\theta = \frac{(\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (4\underline{i} - \underline{j} + 3\underline{k})}{\sqrt{1^2 + (-3)^2 + 4^2} \cdot \sqrt{(4)^2 + (-1)^2 + 3^2}}$$

$$\cos\theta = \frac{4 + 3 + 12}{\sqrt{1 + 9 + 16} \cdot \sqrt{16 + 1 + 9}}$$

$$\cos\theta = \frac{19}{\sqrt{26} \cdot \sqrt{26}} = \frac{19}{26}$$

iii)  $\underline{u} = [-3, 5]$ ,  $\underline{v} = [6, -2]$

**SOLUTION:**

$$\underline{u} = -3\underline{i} + 5\underline{j}, \underline{v} = 6\underline{i} - 2\underline{j}$$

If  $\theta$  is the angle between  $\underline{u}$  and  $\underline{v}$ , then

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos\theta = \frac{(-3\underline{i} + 5\underline{j}) \cdot (6\underline{i} - 2\underline{j})}{\sqrt{(-3)^2 + (5)^2} \cdot \sqrt{(6)^2 + (-2)^2}}$$

$$\cos\theta = \frac{-18 - 10}{\sqrt{9 + 25} \cdot \sqrt{36 + 4}}$$

$$\cos \theta = \frac{-28}{\sqrt{34} \cdot \sqrt{40}} = \frac{-28}{\sqrt{1360}}$$

$$\cos \theta = \frac{-28}{\sqrt{16} \cdot \sqrt{85}} = \frac{-28}{4\sqrt{85}} = \frac{-7}{\sqrt{85}}$$

iv)  $\underline{u} = [2, -3, 1], \underline{v} = [2, 4, 1]$

**SOLUTION:**

$\underline{u} = 2\underline{i} - 3\underline{j} + \underline{k}, \underline{v} = 2\underline{i} + 4\underline{j} + \underline{k}$

If  $\theta$  is the angle between  $u$  and  $v$ , then

$$\cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos \theta = \frac{(2\underline{i} - 3\underline{j} + \underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})}{\sqrt{2^2 + (-3)^2 + 1^2} \cdot \sqrt{(2)^2 + (4)^2 + 1^2}}$$

$$\cos \theta = \frac{4 - 12 + 1}{\sqrt{4 + 9 + 1} \cdot \sqrt{4 + 16 + 1}}$$

$$\cos \theta = \frac{-7}{\sqrt{14} \cdot \sqrt{21}} = \frac{-7}{\sqrt{294}}$$

$$\cos \theta = \frac{-7}{\sqrt{49 \times 6}} = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

**Q.2: Calculate the projection of  $a$  along  $b$  and  $b$  along  $a$  when:**

i)  $\underline{a} = \underline{i} - \underline{k}, \underline{b} = \underline{j} + \underline{k}$

**SOLUTION:**

Projection of  $\underline{a}$  along  $\underline{b} = \underline{a} \cdot \hat{b} = \underline{a} \cdot \frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$

$$\frac{(\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k})}{\sqrt{1^2 + 1^2}} = \frac{0 + 0 + 1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Projection of  $\underline{b}$  along  $\underline{a} = \underline{b} \cdot \hat{a} = \underline{b} \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|}$

$$\frac{(\underline{j} + \underline{k}) \cdot (\underline{i} - \underline{k})}{\sqrt{1^2 + (-1)^2}} = \frac{0 + 0 - 1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

ii)  $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}, \underline{b} = -2\underline{i} - \underline{j} + \underline{k}$  (2018 S.Q)

**SOLUTION:**

Projection of  $\underline{a}$  along  $\underline{b} = \underline{a} \cdot \hat{b} = \underline{a} \cdot \frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$

$$\frac{(3\underline{i} + \underline{j} - \underline{k}) \cdot (-2\underline{i} - \underline{j} + \underline{k})}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{-6 - 1 - 1}{\sqrt{4 + 1 + 1}} = -\frac{8}{\sqrt{6}}$$

Projection of  $\underline{b}$  along  $\underline{a} = \underline{b} \cdot \hat{a} = \underline{b} \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|}$

$$\frac{(-2\underline{i} - \underline{j} + \underline{k}) \cdot (3\underline{i} + \underline{j} - \underline{k})}{\sqrt{3^2 + (1)^2 + (-1)^2}} = \frac{-6 - 1 - 1}{\sqrt{9 + 1 + 1}} = -\frac{8}{\sqrt{11}}$$

**Q.3(7.3): Find a real number ' $\alpha$ ' so that the vectors  $u$  and  $v$  are perpendicular.**

i)  $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}, \underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$

**SOLUTION:**

Since  $\underline{u}$  and  $\underline{v}$  are perpendicular, so

$$\underline{u} \cdot \underline{v} = 0$$

$$(2\alpha\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$$

$$2\alpha + \alpha - 4 = 0$$

$$3\alpha = 4 \Rightarrow \alpha = \frac{4}{3}$$

ii)  $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}, \underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

**SOLUTION:**

Since  $\underline{u}$  and  $\underline{v}$  are perpendicular, so

$$\underline{u} \cdot \underline{v} = 0$$

$$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0$$

$$2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$$

$$\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$

$$(2\alpha + 3)(\alpha - 1) = 0$$

$$2\alpha + 3 = 0 \quad \text{or} \quad \alpha - 1 = 0$$

$$\alpha = -\frac{3}{2} \quad \text{or} \quad \alpha = 1$$

**Q.4: Find the number 'Z' so that the triangle with vertices A(1,-1,0), B(-2,2,1), C(0,2,Z) is a right triangle with right angle at C.**

**SOLUTION:**

**A(1,-1,0), B(-2,2,1), C(0,2,Z)**

$$\overrightarrow{CA} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{CA} = (1-0)\underline{i} + (-1-2)\underline{j} + (0-Z)\underline{k}$$

$$\overrightarrow{CA} = \underline{i} - 3\underline{j} - Z\underline{k}$$

$$\overrightarrow{CB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{CB} = (-2-0)\underline{i} + (2-2)\underline{j} + (1-Z)\underline{k}$$

$$\overrightarrow{CB} = -2\underline{i} + 0\underline{j} + (1-Z)\underline{k}$$

As  $m\angle ACB = 90^\circ$ , so  $\overrightarrow{CB} \perp \overrightarrow{CA}$

$$\Rightarrow \overrightarrow{CA} \cdot \overrightarrow{CB} = 0$$

$$(\underline{i} - 3\underline{j} - Z\underline{k}) \cdot (-2\underline{i} + 0\underline{j} + (1-Z)\underline{k}) = 0$$

$$-2 - 0 - Z(1-Z) = 0$$

$$-2 - Z + Z^2 = 0$$

$$Z^2 - Z - 2 = 0$$

$$Z^2 - 2Z + Z - 2 = 0$$

$$Z(Z-2) + 1(Z-2) = 0$$

$$(Z-2)(Z+1) = 0$$

$$Z-2 = 0 \quad \text{or} \quad Z+1 \Rightarrow Z = -1$$

**Q.5: if  $\underline{v}$  is a vector for which  $\underline{v} \cdot \underline{i} = 0, \underline{v} \cdot \underline{j} = 0, \underline{v} \cdot \underline{k} =$**

**0. Find  $\underline{v}$ . (2018 S.Q)**

**SOLUTION:**

Let the required vector  $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$  (i)

$$\text{As } \underline{v} \cdot \underline{i} = 0 \Rightarrow (x\underline{i} + y\underline{j} + z\underline{k}) \cdot \underline{i} = 0 \Rightarrow x = 0$$

$$\text{As } \underline{v} \cdot \underline{j} = 0 \Rightarrow (x\underline{i} + y\underline{j} + z\underline{k}) \cdot \underline{j} = 0 \Rightarrow y = 0$$

$$\text{As } \underline{v} \cdot \underline{k} = 0 \Rightarrow (x\underline{i} + y\underline{j} + z\underline{k}) \cdot \underline{k} = 0 \Rightarrow z = 0$$

Putting the values in (i)

$$\underline{v} = 0\underline{i} + 0\underline{j} + 0\underline{k} = 0 \text{ (Null vector)}$$

**Q.6: Show that the vectors  $3\underline{i} - 2\underline{j} + \underline{k}, \underline{i} - 3\underline{j} + 5\underline{k}$  and  $2\underline{i} + \underline{j} - 4\underline{k}$  form a right triangle.**

**SOLUTION:**

(i) Let  $u = 3\underline{i} - 2\underline{j} + \underline{k}, v = \underline{i} - 3\underline{j} + 5\underline{k}, w = 2\underline{i} + \underline{j} - 4\underline{k}$

$$\text{Since } v + w = \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} - 4\underline{k} = 3\underline{i} - 2\underline{j} + \underline{k} = u$$

Therefore  $u, v$  and  $w$  are the sides of a triangle.

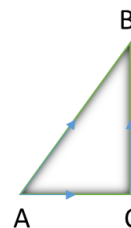
$$\text{Since } u \cdot w = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (2\underline{i} + \underline{j} - 4\underline{k}) = 6 - 2 - 4 = 0$$

$$\Rightarrow u \perp w = 0$$

$\therefore u, v$  and  $w$  are the sides of a right triangle.

(ii) Show that the set of points  $P = (1,3,2), Q = (4,1,4), R = (6,5,5)$  form a right triangle.

**SOLUTION:**



Let  $p = (1,3,2)$  ,  $Q = (4,1,4)$  ,  $R = (6,5,5)$

$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PQ} = (4-1)\underline{i} + (1-3)\underline{j} + (4-2)\underline{k}$$

$$\overrightarrow{PQ} = 3\underline{i} - 2\underline{j} + 2\underline{k}$$

$$\overrightarrow{QR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{QR} = (6-4)\underline{i} + (5-1)\underline{j} + (5-4)\underline{k}$$

$$\overrightarrow{QR} = 2\underline{i} + 4\underline{j} + \underline{k}$$

$$\overrightarrow{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PR} = (6-1)\underline{i} + (5-3)\underline{j} + (5-2)\underline{k}$$

$$\overrightarrow{PR} = 5\underline{i} + 2\underline{j} + 3\underline{k}$$

$$\therefore \overrightarrow{PQ} + \overrightarrow{QR} = 3\underline{i} - 2\underline{j} + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k} = 5\underline{i} + 2\underline{j} + 3\underline{k} = \overrightarrow{PR}$$

$\therefore$  P, Q, R are the vertices of a triangle.

$$\therefore \overrightarrow{PQ} \cdot \overrightarrow{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k}) = 6 - 8 + 2 = 0$$

$\therefore$  P, Q, R are the vertices of a right triangle.

**Q.7: Show that mid point of hypotenuse of a right triangle is equidistance from its vertices.**

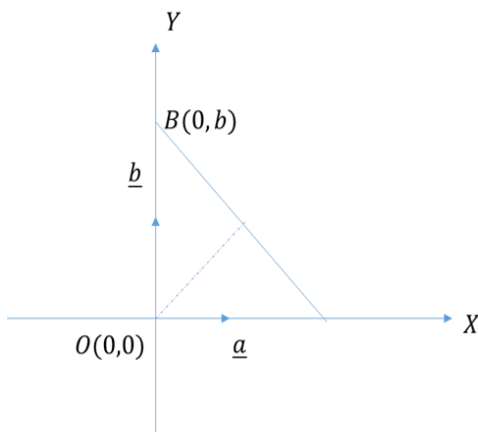
**SOLUTION:**

Consider a right triangle AOB such that  $O(0,0)$  ,  $A(a,0)$  ,  $B(0,b)$ .

Let M be the mid point of hypotenuse  $\overline{AB}$  such that

$$M\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = M\left(\frac{a}{2}, \frac{b}{2}\right)$$

We want to prove:  $|\overrightarrow{AM}| = |\overrightarrow{BM}| = |\overrightarrow{OM}|$



$$\overrightarrow{AM} = \left(\frac{a}{2} - a\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$$

$$\overrightarrow{AM} = \left(-\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$$

$$|\overrightarrow{AM}| = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$|\overrightarrow{AM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$|\overrightarrow{AM}| = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$\overrightarrow{BM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - b\right)\underline{j}$$

$$\overrightarrow{BM} = \left(\frac{a}{2}\right)\underline{i} - \left(\frac{b}{2}\right)\underline{j}$$

$$|\overrightarrow{BM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$

$$|\overrightarrow{BM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$|\overrightarrow{BM}| = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$\overrightarrow{OM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$$

$$\overrightarrow{OM} = \left(\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$$

$$|\overrightarrow{OM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$|\overrightarrow{OM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$|\overrightarrow{OM}| = \frac{1}{2}\sqrt{a^2 + b^2}$$

Clearly  $|\overrightarrow{AM}| = |\overrightarrow{BM}| = |\overrightarrow{OM}|$  , which is the required result.

**Q.8: Prove that the perpendicular bisectors of the sides of a triangle are concurrent.**

**SOLUTION:**

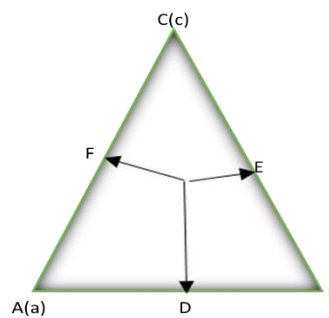
Consider the triangle ABC such that D , E , F are the mid points of sides  $\overline{AB}$  ,  $\overline{BC}$  ,  $\overline{AC}$  respectively. **C**

**(c)**

Then  $\overrightarrow{OA} = \underline{a}$  ,  $\overrightarrow{OB} = \underline{b}$  ,  $\overrightarrow{OC} = \underline{c}$  ,  $\overrightarrow{OD} = \frac{a+b}{2}$  ,  $\overrightarrow{OE} = \frac{b+c}{2}$  ,  $\overrightarrow{OF} = \frac{a+c}{2}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a} \quad , \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - \underline{b} \quad , \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$$

Let  $\overrightarrow{OD} \perp \overline{AB}$  ,  $\overrightarrow{OE} \perp \overline{BC}$  be the right bisectors which meet at 'O'



As  $\overrightarrow{OD} \perp \overline{AB}$  , so

$$\overrightarrow{OD} \cdot \overrightarrow{AB} = 0$$

$$\left(\frac{a+b}{2}\right) \cdot (\underline{b} - \underline{a}) = 0$$

$$(\underline{b} + \underline{a}) \cdot (\underline{b} - \underline{a}) = 0$$

$$b^2 - a^2 = 0 \rightarrow (i)$$

As  $\overrightarrow{OE} \perp \overline{BC}$  , so

$$\overrightarrow{OE} \cdot \overrightarrow{BC} = 0$$

$$\left(\frac{b+c}{2}\right) \cdot (\underline{c} - \underline{b}) = 0$$

$$(c + b) \cdot (c - b) = 0$$

$$c^2 - b^2 = 0 \rightarrow (ii)$$



Adding (i) and (ii)

$$b^2 - a^2 + c^2 - b^2 = 0$$

$$\Rightarrow c^2 - a^2 = 0$$

$$\Rightarrow (c+a) \cdot (c-a) = 0$$

Divide both side by 2

$$\Rightarrow \left(\frac{c+a}{2}\right) \cdot (c-a) = \frac{0}{2}$$

$$\Rightarrow \vec{OF} \cdot \vec{AC} = 0$$

$$\Rightarrow \vec{OF} \perp \vec{AC}$$

So,  $\vec{OF}$  is also a right bisector of  $\vec{AC}$ .

Thus all the right bisector are concurrent at 'O'

**Q.9: Prove that the altitudes of a triangle are concurrent.**

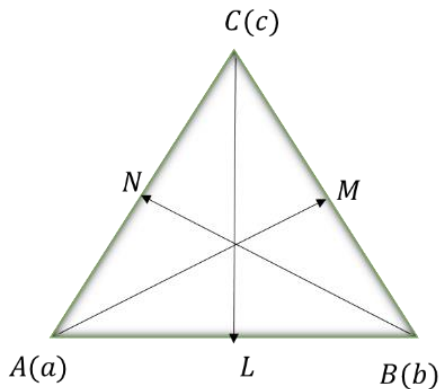
**SOLUTION:**

Consider the triangle ABC such that

$$\text{Then } \vec{OA} = \underline{a}, \vec{OB} = \underline{b}, \vec{OC} = \underline{c}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \underline{b} - \underline{a}, \quad \vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \underline{c} - \underline{a}$$

Let  $\vec{AM} \perp \vec{BC}$ ,  $\vec{CL} \perp \vec{AB}$  be the altitudes of a triangle which meet at 'O'



$$\text{As } \vec{AM} \perp \vec{BC}$$

$$\text{so } \vec{OA} \perp \vec{BC}$$

$$\Rightarrow \vec{OA} \cdot \vec{BC} = 0$$

$$\underline{a} \cdot (\underline{c} - \underline{b}) = 0$$

$$\underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} = 0 \rightarrow (i)$$

$$\text{As } \vec{CL} \perp \vec{AB}$$

$$\text{so } \vec{OC} \perp \vec{AB}$$

$$\Rightarrow \vec{OC} \cdot \vec{AB} = 0$$

$$\underline{c} \cdot (\underline{b} - \underline{a}) = 0$$

$$\underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} = 0 \rightarrow (ii)$$

Adding (i) and (ii)

$$\underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} + \underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} = 0$$

$$\Rightarrow \underline{c} \cdot \underline{b} - \underline{a} \cdot \underline{b} = 0$$

$$\Rightarrow (\underline{c} - \underline{a}) \cdot \underline{b} = 0$$

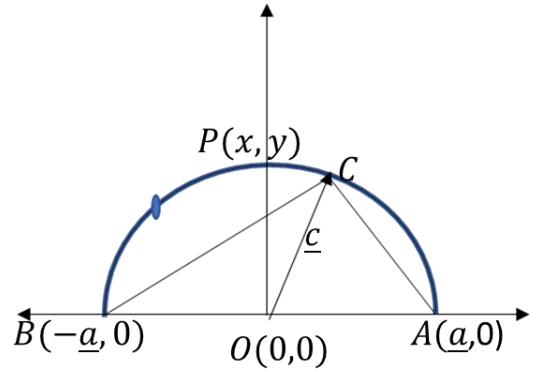
$$\Rightarrow \vec{AC} \cdot \vec{OB} = 0$$

$$\Rightarrow \vec{OB} \perp \vec{AC}$$

$$\Rightarrow \vec{BN} \perp \vec{AC} \text{ \& passing through; O'}$$

So altitude of the triangle are concurrent at 'o'

**Q.10: Prove that the angle in a semi circle is a right angle. (2017 L.Q)**



Consider a semicircle APB with center at origin O(0,0) and A, B are ends of diameter having position vectors  $\underline{a}$ ,  $-\underline{a}$  respectively.

Let C be any point on semicircle having position vector  $\underline{c}$ .

$$\text{Clearly } |\underline{a}| = |-\underline{a}| = |\underline{c}| \quad (\text{radii of semicircle}) \quad (i)$$

$$\text{Now } \vec{AC} = \vec{OC} - \vec{OA} = \underline{c} - \underline{a}$$

$$\text{And } \vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - (-\underline{a}) = \underline{c} + \underline{a}$$

$$\text{Consider } \vec{AC} \cdot \vec{BC} = (\underline{c} - \underline{a}) \cdot (\underline{c} + \underline{a})$$

$$\vec{AC} \cdot \vec{BC} = \underline{c} \cdot (\underline{c} + \underline{a}) - \underline{a} \cdot (\underline{c} + \underline{a})$$

$$\vec{AC} \cdot \vec{BC} = \underline{c} \cdot \underline{c} + \underline{c} \cdot \underline{a} - \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{a}$$

$$\vec{AC} \cdot \vec{BC} = c^2 + \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{c} - a^2 \quad \because \underline{a} \cdot \underline{c} =$$

$$\underline{c} \cdot \underline{a}$$

$$\vec{AC} \cdot \vec{BC} = c^2 - a^2$$

$$\vec{AC} \cdot \vec{BC} = c^2 - c^2 \quad \because |\underline{a}| =$$

$$|\underline{c}|$$

$$\vec{AC} \cdot \vec{BC} = 0$$

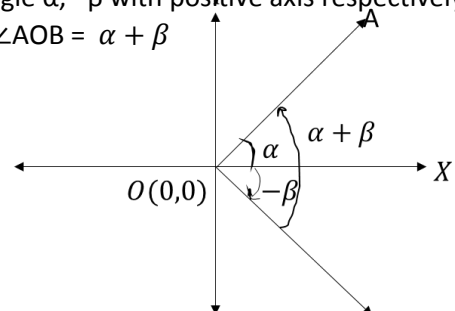
$$\Rightarrow \vec{AC} \perp \vec{BC}$$

Thus the angle in a semi circle is a right angle

**Q.11: Prove that  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$**

**SOLUTION:**

Consider two unit vectors  $\vec{OA}$  and  $\vec{OB}$  in XY-plane making angle  $\alpha$ ,  $-\beta$  with positive axis respectively So that  $m\angle AOB = \alpha + \beta$





$$\vec{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\vec{OB} = \cos(-\beta) \underline{i} + \sin(-\beta) \underline{j} = \cos \beta \underline{i} - \sin \beta \underline{j}$$

Taking dot product of  $\vec{OA}$  and  $\vec{OB}$

$$\vec{OA} \cdot \vec{OB} = (\cos \alpha \underline{i} + \sin \alpha \underline{j}) \cdot (\cos \beta \underline{i} - \sin \beta \underline{j})$$

$$|\vec{OA}| |\vec{OB}| \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$1 \cdot 1 \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \because |\vec{OA}| = |\vec{OB}| = 1$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

**Q.12: Prove that in any triangle ABC**

**i)  $b = c \cos A + a \cos C$**

**SOLUTION:**

Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{b} = -\underline{a} - \underline{c}$$

Taking dot product with  $\underline{b}$

$$\underline{b} \cdot \underline{b} = (-\underline{a} - \underline{c}) \cdot \underline{b}$$

$$b^2 = -\underline{a} \cdot \underline{b} - \underline{c} \cdot \underline{b}$$

$$b^2 = -|\underline{a}| |\underline{b}| \cos(\pi - C) - |\underline{c}| |\underline{b}| \cos(\pi - A)$$

$$\because \cos(\pi - C) = (-\cos C)$$

$$\because \cos(\pi - A) = (-\cos A)$$

$$b^2 = -ab(-\cos C) - cb(-\cos A)$$

$$b^2 = ab \cos C + cb \cos A$$

Taking b common

$$b^2 = b[a \cos C + c \cos A]$$

Cancel b on both sides

$$b = a \cos C + c \cos A$$

**ii)  $c = a \cos B + b \cos A$**

**SOLUTION:**

Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{c} = -\underline{a} - \underline{b}$$

Taking dot product with  $\underline{c}$

$$\underline{c} \cdot \underline{c} = (-\underline{a} - \underline{b}) \cdot \underline{c}$$

$$c^2 = -\underline{a} \cdot \underline{c} - \underline{b} \cdot \underline{c}$$

$$\because \cos(\pi - B) = (-\cos B)$$

$$\because \cos(\pi - A) = (-\cos A)$$

$$c^2 = -|\underline{a}| |\underline{c}| \cos(\pi - B) - |\underline{b}| |\underline{c}| \cos(\pi - A)$$

$$c^2 = -ac(-\cos B) - bc(-\cos A)$$

$$c^2 = ac \cos B + bc \cos A$$

Taking c common

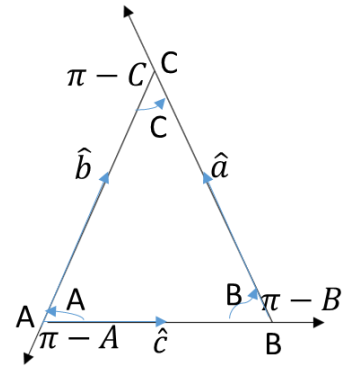
$$c^2 = c[a \cos C + c \cos A]$$

Cancel c on both sides

$$c = a \cos B + c \cos A$$

**iii)  $b^2 = c^2 + a^2 - 2ca \cos B$**

**SOLUTION:**



Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{b} = -\underline{a} - \underline{c} = -(\underline{a} + \underline{c})$$

Squaring on both sides

$$(\underline{b})^2 = [-(\underline{a} + \underline{c})]^2$$

$$(\underline{b})^2 = (\underline{a} + \underline{c})^2$$

$$\underline{b} \cdot \underline{b} = (\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c})$$

$$b^2 = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{c}$$

$$b^2 = a^2 + c^2 + 2\underline{a} \cdot \underline{c}$$

$$b^2 = a^2 + c^2 + 2|\underline{a}| |\underline{c}| \cos(\pi - B)$$

$$\because \cos(\pi - B) = (-\cos B)$$

$$b^2 = a^2 + c^2 + 2ca(-\cos B)$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

**iii)  $c^2 = a^2 + b^2 - 2ab \cos C$**

**SOLUTION:**

Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{c} = -\underline{a} - \underline{b} = -(\underline{a} + \underline{b})$$

squaring on both sides

$$(\underline{c})^2 = [-(\underline{a} + \underline{b})]^2$$

$$(\underline{c})^2 = (\underline{a} + \underline{b})^2$$

$$\underline{c} \cdot \underline{c} = (\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b})$$

$$c^2 = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$$

$$c^2 = a^2 + b^2 + 2\underline{a} \cdot \underline{b}$$

$$c^2 = a^2 + b^2 + 2|\underline{a}| |\underline{b}| \cos(\pi - C)$$

$$\because \cos(\pi - C) = (-\cos C)$$

$$c^2 = a^2 + b^2 + 2ab(-\cos C)$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

**The cross product (vector product)**

let  $\underline{u}$  and  $\underline{v}$  be two non

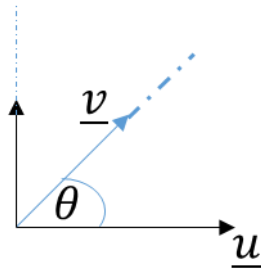
zero vectors. The cross or

vector product of  $\underline{u}$  and  $\underline{v}$  written as  $\underline{u}$

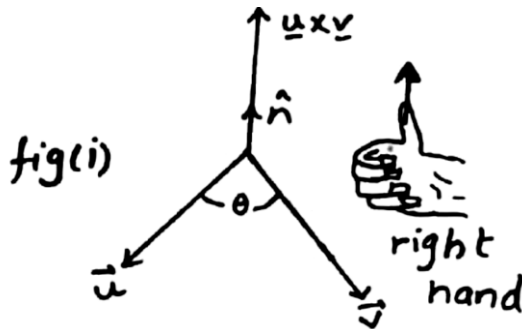
$\times \underline{v}$  is define

$$\underline{u} \times \underline{v} = (|\underline{u}| |\underline{v}| \sin \theta) \hat{n}$$

Where  $\theta$  is the angle b/w the vectors  $s$ . that  $0 \leq \theta \leq \pi$  and  $\hat{n}$  is a unit vector  $\perp$  ar to the plane of  $\underline{u}$  and  $\underline{v}$  written direction is given by the right hand rule.



Right hand rule:

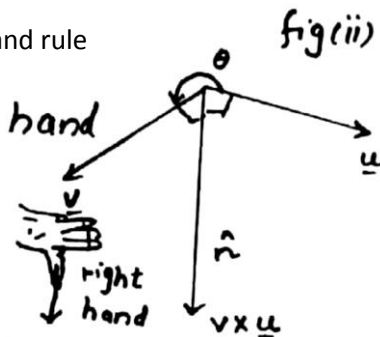


if the figure right hand point along the vectors  $\underline{u}$  and the curl towards the vector  $\underline{v}$ , then the thumbs will give

The direction of  $\hat{n}$  which is  $\underline{u} \times \underline{v}$

In fig.(ii) the right hand rule Shows the direction

Of  $\underline{v} \times \underline{u}$



Derivative of uses full results of cross products:

For unit vector  $\underline{i}, \underline{j}, \underline{k}$  we have.

$$(a) \underline{i} \times \underline{i} = |\underline{i}| |\underline{i}| \sin 0^\circ \hat{n} = 0$$

$$\underline{j} \times \underline{j} = |\underline{j}| |\underline{j}| \sin 0^\circ \hat{n} = 0$$

$$\underline{k} \times \underline{k} = |\underline{k}| |\underline{k}| \sin 0^\circ \hat{n} = 0$$

$$(b) \underline{i} \times \underline{j} = |\underline{i}| |\underline{j}| \sin 90^\circ \hat{k} = \hat{k}$$

$$\underline{j} \times \underline{k} = |\underline{j}| |\underline{k}| \sin 90^\circ \hat{i} = \hat{i}$$

$$\underline{k} \times \underline{i} = |\underline{k}| |\underline{i}| \sin 90^\circ \hat{j} = \hat{j}$$

(c) Prove that  $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

$$\underline{u} \times \underline{v} = |\underline{u}| |\underline{v}| \sin \theta \hat{n}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \sin(-\theta) \hat{n}$$

$$\underline{u} \cdot \underline{v} = |\underline{v}| |\underline{u}| \sin \theta \hat{n}$$

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$\underline{u} \times \underline{u} = |\underline{u}| |\underline{u}| \sin 0^\circ \hat{n} = 0$$

Note:

The cross product of  $\underline{i}, \underline{j}$  and  $\underline{k}$

are written in cyclic pattern

It is helpful to remember.

Properties of cross product:

$$(i) \underline{u} \times \underline{v} = 0 \text{ if } \underline{u} = 0 \text{ or } \underline{v} = 0$$

$$(ii) \underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$(iii) \underline{u} \times (\underline{v} \times \underline{w})$$

$$= \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$$

(distributive property)

$$(iv) \underline{u} \times (k\underline{v}) = k(\underline{u} \times \underline{v}) = k(\underline{u} \times \underline{v})$$

$$(v) \underline{u} \times \underline{u} = 0$$

Analytical Expression of  $\underline{u} \times \underline{v}$

Let

$$\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k} \text{ and}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$$

$$\underline{u} \times \underline{v} = (a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}) \times (a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k})$$

$$= (a_1 a_2)(\underline{i} \times \underline{i}) + a_1 b_2 (\underline{i} \times \underline{j}) + a_1 c_2 (\underline{i} \times \underline{k})$$

$$+ b_1 a_2 (\underline{j} \times \underline{i})$$

$$+ b_1 b_2 (\underline{j} \times \underline{j}) + b_1 c_2 (\underline{j} \times \underline{k}) + c_1 a_2 (\underline{k} \times \underline{i})$$

$$+ c_1 b_2 (\underline{k} \times \underline{j}) + c_1 c_2 (\underline{k} \times \underline{k})$$

$$\Rightarrow a_1 b_2 \underline{k} + a_1 c_2 \underline{j} - b_1 a_2 \underline{k} + b_1 a_2 \underline{i} + c_1 a_2 \underline{j} - c_1 b_2 \underline{i}$$

Rearrange we have,

$$\Rightarrow \underline{u} \times \underline{v} = (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} + (a_1 b_2 - a_2 b_1) \underline{k}$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} + (a_1 b_2 - a_2 b_1) \underline{k}$$

$$\Rightarrow (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} + (a_1 b_2 - a_2 b_1) \underline{k}$$

Parallel vectors:

If  $\underline{u}$  and  $\underline{v}$  are || vectors  $\theta = 0, \sin 0^\circ = 0$  then

$$\underline{u} \times \underline{v} = |\underline{u}| |\underline{v}| \sin \theta \hat{n} = 0 \Rightarrow \underline{u} \times \underline{v} = 0$$

and if  $\underline{u} \times \underline{v} = 0$  then either  $\sin \theta = 0$

or  $\underline{u} = 0$  or  $\underline{v} = 0$

(i) if  $\sin \theta = 0 \Rightarrow \theta = 180^\circ$  or  $0^\circ$  which shows That vectors  $\underline{u}$  and  $\underline{v}$  are parallel.

if  $\underline{u}$

= 0 or  $\underline{v}$  then since zero vector has no specific direction, so zero vector is || to every vector.

Note:

Zero vector is both || to  $\perp$  ar to every vector.

Area of parallelogram:

## Exercise NO 7.4

**Q.1:** Compute the cross product  $\underline{a} \times \underline{b}$  and  $\underline{b} \times \underline{a}$ .  
Check your answer by showing that each  $\underline{a}$  and  $\underline{b}$  is perpendicular to  $\underline{a} \times \underline{b}$  and  $\underline{b} \times \underline{a}$ .

i)  $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}, \underline{b} = \underline{i} - \underline{j} + \underline{k}$

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$= 0\underline{i} - 3\underline{j} - 3\underline{k}$$

Now  $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$$= \underline{i}(-1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$$

$$= 0\underline{i} + 3\underline{j} + 3\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So  $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$$

$$= 0 + 3 - 3 = 0$$

So  $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So  $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So  $\underline{b} \perp \underline{b} \times \underline{a}$

ii)  $\underline{a} = \underline{i} + \underline{j}, \underline{b} = \underline{i} - \underline{j}$

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= 0\underline{i} - 0\underline{j} - 2\underline{k}$$

Now  $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(1+1)$$

$$= 0\underline{i} - 0\underline{j} + 2\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (\underline{i} + \underline{j}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= 0 - 0 - 0 = 0$$

So  $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (\underline{i} + \underline{j}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= 0 + 0 + 0 = 0$$

So  $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (\underline{i} - \underline{j}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= 0 + 0 - 0 = 0$$

So  $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (\underline{i} - \underline{j}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= 0 + 0 - 0 = 0$$

So  $\underline{b} \perp \underline{b} \times \underline{a}$

iii)  $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}, \underline{b} = \underline{i} + \underline{j}$

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2)$$

$$= -1\underline{i} + \underline{j} + 5\underline{k}$$

Now  $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$

$$= \underline{i}(1+0) - \underline{j}(1-0) + \underline{k}(-2-3)$$

$$= 1\underline{i} - \underline{j} - 5\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (-1\underline{i} + \underline{j} + 5\underline{k})$$

$$= -3 - 2 + 5 = 0$$

So  $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (1\underline{i} - \underline{j} - 5\underline{k})$$

$$= 3 + 2 - 5 = 0$$

So  $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (\underline{i} + \underline{j}) \cdot (-1\underline{i} + \underline{j} + 5\underline{k})$$

$$= -1 + 1 + 0 = 0$$

So  $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (\underline{i} + \underline{j}) \cdot (1\underline{i} - \underline{j} - 5\underline{k})$$

$$= 1 - 1 + 0 = 0$$

So  $\underline{b} \perp \underline{b} \times \underline{a}$

**Q.2:** Find a unit vector perpendicular to the plane containing  $\underline{a}$  and  $\underline{b}$ . Also find sine of the angle between them:

i)  $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}, \underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$  (2019 S.Q)

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$$= \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24)$$

$$= 15\underline{i} - 10\underline{j} + 30\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{15^2 + (-10)^2 + 30^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{225 + 100 + 900} = \sqrt{1225} = 35$$

Required unit vector =  $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35} = \frac{15}{35}\underline{i} + \frac{10}{35}\underline{j} + \frac{30}{35}\underline{k} = \frac{3}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}$$

If  $\theta$  is the angle between  $\underline{a}$  and  $\underline{b}$ , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{35}{\sqrt{2^2 + (-6)^2 + (-3)^2} \cdot \sqrt{4^2 + 3^2 + (-1)^2}}$$

$$\sin \theta = \frac{35}{\sqrt{4+36+9} \cdot \sqrt{16+9+1}} = \frac{35}{\sqrt{49} \cdot \sqrt{26}} = \frac{35}{7 \cdot \sqrt{26}} = \frac{5}{\sqrt{26}}$$

ii)  $\underline{a} = -\underline{i} - \underline{j} - \underline{k}, \underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$$

$$= -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + 2^2 + 5^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{49 + 4 + 25} = \sqrt{78}$$

Required unit vector =  $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}} = \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$$

If  $\theta$  is the angle between  $\underline{a}$  and  $\underline{b}$ , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{78}}{\sqrt{(-1)^2 + (-1)^2 + (-1)^2} \cdot \sqrt{2^2 + (-3)^2 + 4^2}}$$

$$\sin \theta = \frac{\sqrt{78}}{\sqrt{1+1+1} \cdot \sqrt{4+9+16}} = \frac{\sqrt{78}}{\sqrt{3} \cdot \sqrt{29}} = \sqrt{\frac{78}{3 \cdot 29}} = \sqrt{26} \cdot \frac{1}{\sqrt{29}} =$$

$$\sqrt{\frac{26}{29}}$$

iii)  $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}, \underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$  (2016 S.Q)

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0} \quad (\text{null vector})$$

$$|\underline{a} \times \underline{b}| = \sqrt{0^2 + 0 + 0^2}$$

$$|\underline{a} \times \underline{b}| = 0$$

As  $\underline{a} \times \underline{b} = \underline{0}$

$\Rightarrow \underline{a}$  and  $\underline{b}$  are parallel.

Also  $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} = -2(-\underline{i} + \underline{j} - 2\underline{k}) = -2\underline{b}$

$\Rightarrow \underline{a}$  and  $\underline{b}$  are parallel but opposite in direction.

Thus the angle between them is  $180^\circ$

$$\therefore \theta = 180^\circ$$

$$\Rightarrow \sin \theta = 0$$

iv)  $\underline{a} = \underline{i} + \underline{j}, \underline{b} = \underline{i} - \underline{j}$

**SOLUTION:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0^2 + (-0)^2 + (-2)^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0 + 0 + 4} = \sqrt{4} = 2$$

Required unit vector =  $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{0\underline{i} - 0\underline{j} - 2\underline{k}}{2} = \frac{0}{2}\underline{i} + \frac{0}{2}\underline{j} + \frac{-2}{2}\underline{k} = 0\underline{i} - 0\underline{j} - \underline{k} = -\underline{k}$$

If  $\theta$  is the angle between  $\underline{a}$  and  $\underline{b}$ , then

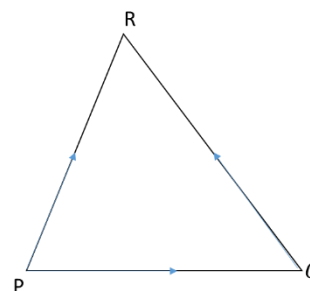
$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{2}{\sqrt{1^2 + 1^2 + 0} \cdot \sqrt{1^2 + (-1)^2 + 0^2}}$$

$$\sin \theta = \frac{2}{\sqrt{1+1+0} \cdot \sqrt{1+1+0}} = \frac{2}{\sqrt{2} \cdot \sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1$$

**Q.3: Find the area of the triangle, determined by the points P, Q, R.**

(i)  $P(0,0,0) ; Q(2,3,2) ; R(-1,1,4)$

**SOLUTION:**



$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PQ} = (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k}$$

$$\overrightarrow{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\overrightarrow{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PR} = (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k}$$

$$\overrightarrow{PR} = -\underline{i} + \underline{j} + 4\underline{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$$

$$= 10\underline{i} - 10\underline{j} + 5\underline{k}$$

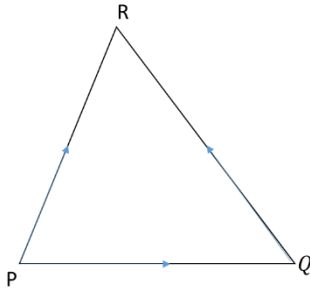
$$|\vec{PQ} \times \vec{PR}| = \sqrt{10^2 + (-10)^2 + 5^2}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{100 + 100 + 25} = \sqrt{225} = 15$$

$$\text{Area of } \Delta PQR = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \cdot 15 = \frac{15}{2} \text{ square unit}$$

(ii) P(1,-1,-1) ; Q(2,0,-1) ; R(0,2,1)

SOLUTION:



$$\vec{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{PQ} = (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k}$$

$$\vec{PQ} = \underline{i} + \underline{j} + 0\underline{k}$$

$$\vec{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{PR} = (0-1)\underline{i} + (2+1)\underline{j} + (1+1)\underline{k}$$

$$\vec{PR} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= \underline{i}(2-0) - \underline{j}(2+0) + \underline{k}(3+1)$$

$$= 2\underline{i} - 2\underline{j} + 4\underline{k}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{2^2 + (-2)^2 + 4^2}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{4+4+16} = \sqrt{24} = \sqrt{2 \times 6} = 2\sqrt{6}$$

$$\text{Area of } \Delta PQR = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \cdot 2\sqrt{6} = \sqrt{6} \text{ square unit}$$

Q.4: Find the area of the parallelogram whose vertices are:

i) A(0,0,0) ; B(1,2,3) ; C(2, -1,1) ; D(3,1,4)

SOLUTION:

$$\vec{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AB} = (1-0)\underline{i} + (2-0)\underline{j} + (3-0)\underline{k}$$

$$\vec{AB} = \underline{i} + 2\underline{j} + 3\underline{k}$$

$$\vec{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AD} = (3-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k}$$

$$\vec{AD} = 3\underline{i} + \underline{j} + 4\underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= \underline{i}(8-3) - \underline{j}(4-9) + \underline{k}(1-6)$$

$$= 5\underline{i} + 5\underline{j} - 5\underline{k}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{5^2 + 5^2 + (-5)^2}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{25 + 25 + 25} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area of } \Delta PQR = |\vec{AB} \times \vec{AD}| = 5\sqrt{3} \text{ sq. unit}$$

ii) A(1,2, -1) ; B(4,2, -3) ; C(6, -5,2) ; D(9, -5,0)

SOLUTION:

$$\vec{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AB} = (4-1)\underline{i} + (2-2)\underline{j} + (-3+1)\underline{k}$$

$$\vec{AB} = 3\underline{i} + 0\underline{j} - 2\underline{k}$$

$$\vec{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AD} = (9-1)\underline{i} + (-5-2)\underline{j} + (0+1)\underline{k}$$

$$\vec{AD} = 8\underline{i} - 7\underline{j} + \underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$$= \underline{i}(0-14) - \underline{j}(3+16) + \underline{k}(-21-0)$$

$$= -14\underline{i} - 19\underline{j} - 21\underline{k}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{196 + 361 + 441} = \sqrt{998}$$

$$\text{Area of } \Delta PQR = |\vec{AB} \times \vec{AD}| = \sqrt{998} \text{ sq. unit}$$

iii) A(-1,1,1) ; B(-1,2,2) ; C(-3, 4, -5) ; D(-3,5, -4)

SOLUTION:

$$\vec{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AB} = (-1+1)\underline{i} + (2-1)\underline{j} + (2-1)\underline{k}$$

$$\vec{AB} = 0\underline{i} + \underline{j} + \underline{k}$$

$$\vec{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\vec{AD} = (-3+1)\underline{i} + (5-1)\underline{j} + (-4-1)\underline{k}$$

$$\vec{AD} = -2\underline{i} + 4\underline{j} - 5\underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$$= \underline{i}(-5-4) - \underline{j}(0+2) + \underline{k}(0+2)$$

$$= -9\hat{i} - 2\hat{j} + 2\hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-9)^2 + (-2)^2 + 2^2}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{81 + 4 + 4} = \sqrt{89}$$

$$\text{Area of } \Delta PQR = |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{89} \text{ sq. unit}$$

**Q.5: Which vectors if any, are perpendicular or parallel:**

i)  $\underline{u} = 5\hat{i} - \hat{j} + \hat{k}$ ;  $\underline{v} = \hat{j} - 5\hat{k}$ ;  $\underline{w} = -15\hat{i} + 3\hat{j} - 3\hat{k}$

**SOLUTION:**

$$\text{As } \underline{u} \cdot \underline{v} = (5\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{j} - 5\hat{k}) = 0 - 1 - 5 = -6 \neq 0$$

So  $\underline{u}$  and  $\underline{v}$  are not perpendicular.

$$\text{As } \underline{u} \cdot \underline{w} = (5\hat{i} - \hat{j} + \hat{k}) \cdot (-15\hat{i} + 3\hat{j} - 3\hat{k}) = -75 - 3 - 3 = -81 \neq 0$$

So  $\underline{u}$  and  $\underline{w}$  are not perpendicular.

$$\text{As } \underline{v} \cdot \underline{w} = (\hat{j} - 5\hat{k}) \cdot (-15\hat{i} + 3\hat{j} - 3\hat{k}) = 0 + 3 + 15 = 18 \neq 0$$

So  $\underline{u}$  and  $\underline{w}$  are not perpendicular.

$$\text{Now } \underline{w} = -15\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\underline{w} = -3(5\hat{i} - \hat{j} + \hat{k})$$

$$\underline{w} = -3\underline{u}$$

$$\Rightarrow \underline{u} \parallel \underline{w}$$

ii)  $\underline{u} = \hat{i} + 2\hat{j} - \hat{k}$ ;  $\underline{v} = -\hat{i} + \hat{j} + \hat{k}$ ;  $\underline{w} = -\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k}$

**SOLUTION:**

$$\text{As } \underline{u} \cdot \underline{v} = (\hat{i} + 2\hat{j} - \hat{k}) \cdot (-\hat{i} + \hat{j} + \hat{k}) = -1 + 2 - 1 = 0$$

So  $\underline{u}$  and  $\underline{v}$  are perpendicular.

$$\text{As } \underline{u} \cdot \underline{w} = (\hat{i} + 2\hat{j} - \hat{k}) \cdot (-\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k}) = -\frac{\pi}{2} - 2\pi - \frac{\pi}{2} = -3\pi \neq 0$$

So  $\underline{u}$  and  $\underline{w}$  are not perpendicular.

$$\text{As } \underline{v} \cdot \underline{w} = (-\hat{i} + \hat{j} + \hat{k}) \cdot (-\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k}) = \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = 0$$

So  $\underline{u}$  and  $\underline{w}$  are perpendicular.

$$\text{Now } \underline{w} = -\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k} = \frac{-\pi\hat{i} - 2\pi\hat{j} + \pi\hat{k}}{2}$$

$$\underline{w} = -\frac{\pi}{2}(\hat{i} + 2\hat{j} - \hat{k})$$

$$\underline{w} = -\frac{\pi}{2}\underline{u}$$

$$\Rightarrow \underline{u} \parallel \underline{w} \text{ but opposite in direction.}$$

**Q.6: Prove that  $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$**

**SOLUTION: (2015 S.Q)**

$$\text{L.H.S} = \underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$$

$$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) + (\underline{b} \times \underline{a}) + (\underline{c} \times \underline{a}) + (\underline{c} \times \underline{b})$$

$$\because \underline{b} \times \underline{a} = -(\underline{a} \times \underline{b}) \text{ \& } \underline{c} \times \underline{a} = -(\underline{a} \times \underline{c}) \text{ \& } \underline{c} \times \underline{b} = -(\underline{b} \times \underline{c})$$

$$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) - (\underline{a} \times \underline{b}) - (\underline{a} \times \underline{c}) - (\underline{b} \times \underline{c})$$

$$= \underline{0}$$

**SOLUTION:** As  $\underline{a} + \underline{b} + \underline{c} = \underline{0}$

$$\underline{0} = \text{R.H.S}$$

Taking cross product with  $\underline{a}$

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{a}$$

$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$$

$$\because \underline{a} \times \underline{a} = \underline{0}$$

$$\underline{0} + \underline{a} \times \underline{b} - (\underline{c} \times \underline{a}) = \underline{0}$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (1)$$

Taking cross product with  $\underline{b}$

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{b}$$

$$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0}$$

$$\because \underline{b} \times \underline{b} = \underline{0}$$

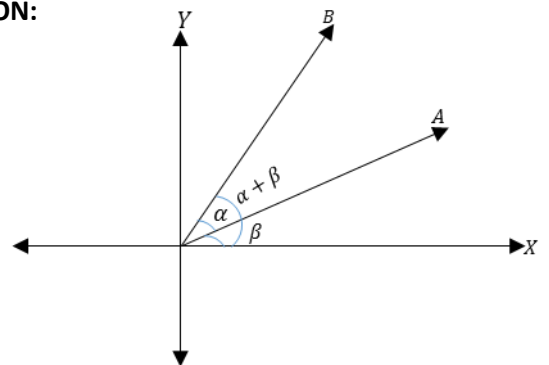
$$\underline{b} \times \underline{a} + \underline{0} - (\underline{c} \times \underline{b}) = \underline{0}$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (2)$$

From (1) & (2), we conclude that  $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

**Q.8: Prove that  $\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$  (2015 L.Q)**

**SOLUTION:**



Consider two unit vectors  $\overrightarrow{OA}$  and  $\overrightarrow{OB}$  in  $xy$ -plane making angle  $\alpha, \beta$  with positive  $x$ -axis respectively.

Such that  $m\angle AOB = \alpha - \beta$

$$\overrightarrow{OA} = \cos \alpha \hat{i} + \sin \alpha \hat{j}$$

$$\overrightarrow{OB} = \cos \beta \hat{i} + \sin \beta \hat{j}$$

Taking cross product of  $\overrightarrow{OB}$  and  $\overrightarrow{OA}$

$$\vec{OB} \times \vec{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix}$$

$$|\vec{OB}| |\vec{OA}| \sin(\alpha - \beta) \underline{k} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\sin \alpha \cos \beta - \cos \alpha \sin \beta)$$

$$1.1 \sin(\alpha - \beta) \underline{k} = \underline{k}(\sin \alpha \cos \beta - \cos \alpha \sin \beta)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

**Q.9: if  $\underline{a} \times \underline{b} = \underline{0}$  and  $\underline{a} \cdot \underline{b} = 0$  what conclusion can be drawn about  $\underline{a}$  or  $\underline{b}$**

**SOLUTION:**

$$\because \underline{a} \times \underline{b} = \underline{0}$$

$$\Rightarrow |\underline{a}| |\underline{b}| \sin \theta = 0 \Rightarrow \sin \theta = 0$$

$$\Rightarrow \theta = \sin^{-1} 0 = 0$$

$$\Rightarrow \theta = 0, \pi \text{ so } \underline{a} \text{ and } \underline{b} \text{ are parallel.}$$

$$\text{And } \underline{a} \cdot \underline{b} = 0$$

$$\Rightarrow |\underline{a}| |\underline{b}| \cos \theta = 0 \Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = 90^\circ \Rightarrow \theta = 90^\circ$$

so  $\underline{a}$  and  $\underline{b}$  are perpendicular. at the same time || and linear not possible so one vector should be zero or null.

**Triple product of vectors:**

There are two types of triple product of vectors:

(a) Scalar triple product:

$$(\underline{u} \times \underline{v}) \cdot \underline{w} \text{ or } \underline{u} \cdot (\underline{v} \times \underline{w})$$

(b) Vector triple product:

$$\underline{u} \times (\underline{v} \times \underline{w})$$

in this section we shall study the scalar triple product only.

**Definition:**

$$\text{Let } \underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k} \text{ and}$$

$$\underline{w} = a_3 \underline{i} + b_3 \underline{j} + c_3 \underline{k} \text{ be three vectors, the scalar}$$

Triple product of vectors  $\underline{u}, \underline{v}$  and  $\underline{w}$  is defined by

$$(\underline{u} \times \underline{v}) \cdot \underline{w} \text{ or } \underline{u} \cdot (\underline{v} \times \underline{w}) \text{ or } \underline{w} \cdot (\underline{u} \times \underline{v}). \text{ the scalar}$$

product  $\underline{u} \cdot (\underline{v} \times \underline{w})$  is written as;

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = [\underline{u}, \underline{v}, \underline{w}]$$

**Analytic Expression of  $\underline{u} \cdot (\underline{v} \times \underline{w})$**

$$\text{let } \underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k} \text{ and}$$

$$\underline{w} = a_3 \underline{i} + b_3 \underline{j} + c_3 \underline{k}$$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \underline{i}(b_2 c_3 - c_2 b_3) - \underline{j}(a_2 c_3 - c_2 a_3) + \underline{k}(a_2 b_3 - b_2 a_3)$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = a_1(b_2 c_3 - c_2 b_3) - b_1(a_2 c_3 - c_2 a_3) + c_1(a_2 b_3 - b_2 a_3)$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which determine

formula for scalar triple product of

$\underline{u}, \underline{v}$  and  $\underline{w}$

**Prove that:**

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

*Proof: we know that*

$$\text{let } \underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k} \text{ and}$$

$$\underline{w} = a_3 \underline{i} + b_3 \underline{j} + c_3 \underline{k}$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$\because \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u})$$

Now

$$\underline{v} \cdot (\underline{w} \times \underline{u}) = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$\because \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$\text{Hence } \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

**Note: (i)** Dot and cross may be interchanged with out altering the value

$$\text{i.e. } (\underline{u} \times \underline{v}) \cdot \underline{w} = \underline{u} \cdot (\underline{v} \times \underline{w}) = [\underline{u}, \underline{v}, \underline{w}]$$

$$(\underline{v} \times \underline{w}) \cdot \underline{u} = \underline{v} \cdot (\underline{w} \times \underline{u}) = [\underline{v}, \underline{w}, \underline{u}]$$

$$(\underline{w} \times \underline{u}) \cdot \underline{v} = \underline{w} \cdot (\underline{u} \times \underline{v}) = [\underline{w}, \underline{u}, \underline{v}]$$

(ii) the value of the product changes if the order is non cycle.

(iii)  $\underline{u} \cdot \underline{v} \cdot \underline{w}$  and  $\underline{u} \cdot (\underline{v} \times \underline{w})$  are meaningless.

**the volume of parallelepiped**

consider  $\vec{OA} = \underline{u}, \vec{OB} = \underline{v}$  and  $\vec{OC} = \underline{w}$  be the



adjacent edges of parallel piped  $OAFCDGEB$

let  $\theta$  be the angle b/w  $\underline{w}$  and  $(\underline{u} \times \underline{v})$

As  $|\underline{u} \times \underline{v}| = \text{area of } \parallel\text{gram } OAGB$

area of base of  $\parallel\text{parallelepiped}$

Resolve  $\underline{w}$  in components in  $\Delta COM$ ,  $\cos\theta = \frac{|\overline{OM}|}{|\overline{OC}|}$

$$\Rightarrow |\overline{OM}| = |\overline{OC}| \cos\theta$$

$$\Rightarrow |\overline{OM}| = w \cos\theta$$

As  $|\overline{OM}| = \text{height of } \parallel\text{piped}$

As Volume  $\parallel\text{piped} = (\text{area of base})(\text{height})$

$$= |\underline{u} \times \underline{v}| \cdot |\underline{w}| \cos\theta$$

$$\Rightarrow \text{Volume of } \parallel\text{piped} = \underline{w} \cdot (\underline{u} \times \underline{v})$$

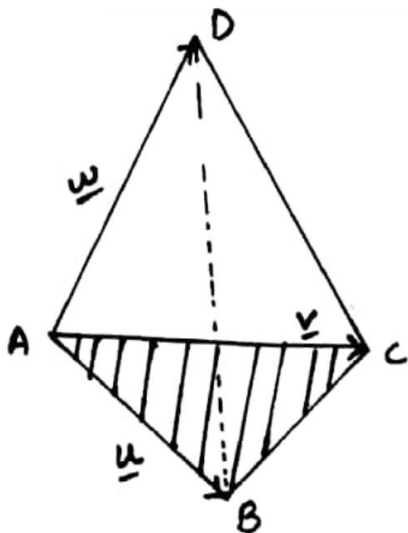
$$\therefore \underline{w} \cdot (\underline{u} \times \underline{v}) = \underline{u} \cdot (\underline{v} \times \underline{w}) \left( \begin{array}{l} \because \underline{w} \cdot (\underline{u} \times \underline{v}) \\ |\underline{u} \times \underline{v}| \cdot |\underline{w}| \cos\theta \end{array} \right)$$

so Volume of  $\parallel\text{piped} = \underline{u} \cdot (\underline{v} \times \underline{w})$

**The Volume of Tetrahedron:**

**Volume of tetrahedron ABCD**

$$= \frac{1}{3} (\Delta ABC) (\text{height of } D \text{ above the place } ABC)$$



Thus,  $\text{Volume} = \frac{1}{6} (\underline{u} \times \underline{v}) \cdot \underline{w} = \frac{1}{6} [\underline{u} \ \underline{v} \ \underline{w}]$

**Properties of scalar triple product:**

1. if  $\underline{u}$ ,

$\underline{v}$  and  $\underline{w}$  are coplaner, then the volume of

$\parallel\text{piped}$  so formed is zero. i.e the vector  $\underline{u}, \underline{v}, \underline{w}$

are coplaner  $\Leftrightarrow (\underline{u} \times \underline{v}) \cdot \underline{w} = 0$

if any two vectors of scalar triple product are

equal, then its value is zero. i.e

$$[\underline{u} \ \underline{v} \ \underline{w}] = [\underline{u} \ \underline{v} \ \underline{w}] = 0$$

**Application of vectors in physics and Engineering**

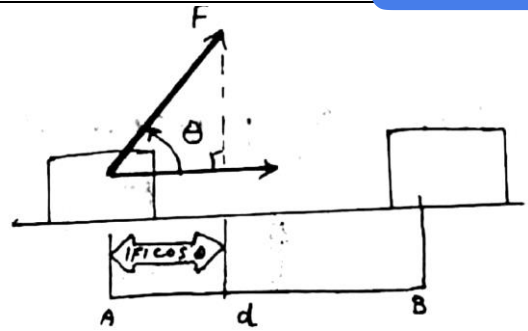
(a) Work done:

if  $d$  is displacement from A to B when force

$\vec{F}$  is applied on the particle then,

Workdone = (force)(Displacement)

$$\Rightarrow \underline{W} = \vec{F} \cdot \underline{AB} = \underline{F} \cdot \underline{d} \quad \underline{d} = \underline{AB}$$



**Exercise NO 7.5**

Q.1: find the volume of the parallelepiped for which the given vectors are three edges:

i)  $\underline{u} = 3\mathbf{i} + 2\mathbf{k}$

$$\underline{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\underline{w} = -\mathbf{j} + 4\mathbf{k}$$

Volume of a parallelepiped =  $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(8+1) - 0(4-0) + 2(-1-0)$$

$$= 27 - 0 - 2 = 25 \quad \text{cubic unit}$$

ii)  $\underline{u} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

$$\underline{v} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\underline{w} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

Volume of a parallelepiped =  $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -2 \\ -3 & 1 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= -7 + 20 + 1 = 14 \quad \text{cubic unit}$$

iii)  $\underline{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\underline{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\underline{w} = \mathbf{j} + \mathbf{k}$$

volume of a parallelepiped =  $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-1+1) + 2(2+0) + 3(2+0)$$

$$= 0+4 + 6 = 10 \quad \text{cubic unit}$$

**Q.2: verify that  $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$ , if**

$$\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}; \underline{b} = 4\underline{i} - 2\underline{j} + 4\underline{k}; \underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 3(3+10)+1(4+4)+5(20-6)$$

$$= 39+8+70 = 117$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix}$$

$$= 4(25+1)-3(10-3)-2(-2-15)$$

$$= 104-21+34 = 117$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -1 & 5 \\ 3 & -2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 2(2-15)-5(-6-20)+1(9+4)$$

$$= -26+130+13 = 117$$

**Q.3: Prove that the vectors  $\underline{i} - 2\underline{j} + 3\underline{k}$  ;  $-2\underline{i} + 3\underline{j} - 4\underline{k}$  and  $\underline{i} - 3\underline{j} + 5\underline{k}$  are coplanar.**

**SOLUTION:**

$$\text{Let } \underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}; \quad \underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}; \quad \underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$$

$$[\underline{u} \underline{v} \underline{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1 \begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix} - (-2) \begin{vmatrix} -2 & -4 \\ 1 & 5 \end{vmatrix} + 3 \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 1(15-12) + 2(-10+4) + 3(6-3) = 3-12+9 = 0$$

Thus the vectors are perpendicular.

**Q.4: Find the constant 'α' such that the vectors are coplanar.**

$$\text{i) } \underline{i} - \underline{j} + \underline{k} ; \underline{i} - 2\underline{j} - 3\underline{k} \text{ and } 3\underline{i} - \alpha\underline{j} + 5\underline{k}$$

**SOLUTION: (2017 S.Q)**

$$\text{Let } \underline{u} = \underline{i} - \underline{j} + \underline{k}; \quad \underline{v} = \underline{i} - 2\underline{j} - 3\underline{k}; \quad \underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$$

The vectors  $\underline{u}, \underline{v}, \underline{w}$  are coplanar if  $[\underline{u} \underline{v} \underline{w}] = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$$1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$$

$$-10-3\alpha+14-\alpha+6=0$$

$$-4\alpha+10=0$$

$$\Rightarrow \alpha = \frac{10}{4} \qquad \Rightarrow \alpha = \frac{5}{2}$$

$$\text{ii) } \underline{i} - 2\underline{a}\underline{j} - \underline{k} ; \underline{i} - \underline{j} + 2\underline{k} \text{ and } \alpha \underline{i} - 2\underline{j} + \underline{k}$$

**SOLUTION:**

$$\text{Let } \underline{u} = \underline{i} - 2\underline{a}\underline{j} - \underline{k}; \quad \underline{v} = \underline{i} - \underline{j} + 2\underline{k}; \quad \underline{w} = \alpha \underline{i} - 2\underline{j} + \underline{k}$$

The vectors  $\underline{u}, \underline{v}, \underline{w}$  are coplanar if  $[\underline{u} \underline{v} \underline{w}] = 0$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$$1(-1+2) - 1(2\alpha - 1) - \alpha(4\alpha - 1) = 0$$

$$\Rightarrow 1 + 2\alpha + 1 - 4\alpha^2 - \alpha = 0 \Rightarrow -4\alpha^2 + \alpha + 2 = 0$$

$$4\alpha^2 - \alpha - 2 = 0$$

Using quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \frac{1 \pm \sqrt{1+32}}{8}$$

$$\Rightarrow \alpha = \frac{1 \pm \sqrt{33}}{8}$$

**Q.5: (a) Find the value of i)  $2\underline{i} \times 2\underline{j} \cdot \underline{k}$  ii)  $3\underline{j} \cdot \underline{k} \times \underline{i}$  iii)  $[\underline{k} \underline{i} \underline{j}]$  (2015 S.Q) iv)  $[\underline{i} \underline{i} \underline{k}]$**

**SOLUTION:**

$$\text{i) } 2\underline{i} \times 2\underline{j} \cdot \underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(2-0) - 0(0-0) + 0(0-0) = 4 - 0 - 0 = 4$$

$$\text{ii) } 3\underline{j} \cdot \underline{k} \times \underline{i} = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0(0-0) - 3(0-1) + 0(0-1) = 0 + 3 + 0 = 3$$

$$\text{iii) } [\underline{k} \underline{i} \underline{j}] = \underline{k} \cdot \underline{j} \times \underline{i} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0(0-0) - 0(0-0) + 1(1-0) = 0 - 0 + 1 = 1$$

$$\text{iv) } [\underline{i} \underline{i} \underline{k}] = \underline{i} \cdot \underline{i} \times \underline{k} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(0-0) - 0(1-0) + 0(0-0) = 0 - 0 - 0 = 0$$

**(b) Prove that**

$$\underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v}) = 3\underline{u} \cdot (\underline{v} \times \underline{w})$$

**SOLUTION:**

L.H.S

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$\because \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w})$$

$$= 3\underline{u} \cdot (\underline{v} \times \underline{w})$$

= R.H.S

**Q.6:** Find the volume of the tetrahedron with the vertices.

i) (0,1,2) , (3,2,1) , (1,2,1) , and (5,5,6)

**SOLUTION:**

Let A(0,1,2) , B(3,2,1) , C(1,2,1) , D(5,5,6)

$$\vec{AB} = (3 - 0)\underline{i} + (2 - 1)\underline{j} + (1 - 2)\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$$

$$\vec{AC} = (1 - 0)\underline{i} + (2 - 1)\underline{j} + (1 - 2)\underline{k} = \underline{i} + \underline{j} - \underline{k}$$

$$\vec{AD} = (5 - 0)\underline{i} + (5 - 1)\underline{j} + (6 - 2)\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$$

$$\text{Volume of the tetrahedron ABCD} = \frac{1}{6} [\vec{AB} \ \vec{AC} \ \vec{AD}]$$

$$= \frac{1}{6} \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= \frac{1}{6} [3(4 + 4) - 1(4 + 5) + (-1)(4 - 5)]$$

$$= \frac{1}{6} [24 - 9 + 1] = \frac{1}{6} \cdot 16 = \frac{8}{3} \text{ Cubic unit}$$

ii) (2,1,8) , (3,2,9) , (2,1,4) , and (3,3,10) (2015 S.Q)(2018,19L.Q)

**SOLUTION:**

Let A(2,1,8) , B(3,2,9) , C(2,1,4) , D(3,3,10)

$$\vec{AB} = (3 - 2)\underline{i} + (2 - 1)\underline{j} + (9 - 8)\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

$$\vec{AC} = (2 - 2)\underline{i} + (1 - 1)\underline{j} + (4 - 8)\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$$

$$\vec{AD} = (3 - 2)\underline{i} + (3 - 1)\underline{j} + (10 - 8)\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$$

$$\text{Volume of the tetrahedron ABCD} = \frac{1}{6} [\vec{AB} \ \vec{AC} \ \vec{AD}]$$

$$= \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6} [1(0 + 8) - 1(0 + 4) + 1(0 - 0)]$$

$$= \frac{1}{6} [8 - 4 + 0] = \frac{1}{6} \cdot 4 = \frac{2}{3} \text{ Cubic unit}$$

**Q.7:** Find the work done, if the point at which the constant force  $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$  is applied to an object, moves from  $P_1(3, 1, -2)$  to  $P_2(2, 4, 6)$ .

**SOLUTION:** Given:  $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$  ,  $P_1(3,1, -2)$  ,  $P_2(2,4,6)$

**Displacement:**  $\underline{d} = \vec{P_1P_2} = (2 - 3)\underline{i} + (4 - 1)\underline{j} + (6 + 2)\underline{k} = -\underline{i} + 3\underline{j} + 8\underline{k}$

We know that Work done =  $\vec{F} \cdot \underline{d} = (4\underline{i} + 3\underline{j} + 5\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 8\underline{k}) = -4 + 9 + 40 = 45$  Units

**Q.8:** A particle, acted by constant forces  $4\underline{i} + \underline{j} - \underline{k}$  and  $3\underline{i} - \underline{j} - \underline{k}$  is displaced from A(1,2,3) to B(5,4,1). Find the work done.

**SOLUTION:** Given: let  $\vec{F}_1 = 4\underline{i} + \underline{j} - \underline{k}$  ;  $\vec{F}_2 = 3\underline{i} - \underline{j} - \underline{k}$  and A(1,2,3) , B(5,4,1)

**Total force:**  $\vec{F} = \vec{F}_1 + \vec{F}_2 = 4\underline{i} + \underline{j} - \underline{k} + 3\underline{i} - \underline{j} - \underline{k} = 7\underline{i} + 0\underline{j} - 4\underline{k}$

**Displacement:**  $\underline{d} = \vec{AB} = (5 - 1)\underline{i} + (4 - 2)\underline{j} + (1 - 3)\underline{k} = 4\underline{i} + 2\underline{j} - 2\underline{k}$

We know that Work done =  $\vec{F} \cdot \underline{d} = (3\underline{i} - \underline{j} - \underline{k}) \cdot (4\underline{i} + 2\underline{j} - 2\underline{k}) = 28 + 0 + 8 = 36$  units

**Q.9:** A particle is displaced from the point A(5,-5,-7) , to the point B(6,2,-2) under the action of constant forces defined by

$$10\underline{i} - 6\underline{j} + 11\underline{k}, 4\underline{i} + 5\underline{j} + 9\underline{k}, -2\underline{i} + \underline{j} - 9\underline{k}.$$

Show that the total work done by the forces is 67 units.

**SOLUTION:** Given: Let  $\vec{F}_1 = 10\underline{i} - 6\underline{j} + 11\underline{k}$  ,  $\vec{F}_2 = 4\underline{i} + 5\underline{j} + 9\underline{k}$   $\vec{F}_3 = -2\underline{i} + \underline{j} - 9\underline{k}$  and A(5,-5,-7) , B(6,2,-2)

**Total force :**  $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 10\underline{i} - 6\underline{j} + 11\underline{k} + 4\underline{i} + 5\underline{j} + 9\underline{k} + (-2\underline{i} + \underline{j} - 9\underline{k}) = 12\underline{i} + 0\underline{j} + 11\underline{k}$

**Displacement:**  $\underline{d} = \vec{AB} = (6 - 5)\underline{i} + (2 + 5)\underline{j} + (-2 + 7)\underline{k} = \underline{i} + 7\underline{j} + 5\underline{k}$

We know that Work done =  $\vec{F} \cdot \underline{d} = (12\underline{i} + 0\underline{j} + 11\underline{k}) \cdot (\underline{i} + 7\underline{j} + 5\underline{k}) = 12 + 0 + 55 = 67$  units

**Q.10:** A force of magnitude 6 units acting parallel to  $2\underline{i} - 2\underline{j} + \underline{k}$  displaces the point of application from (1,2,3) to (5,3,7). Find the work done.

**SOLUTION:** Let  $\vec{F} = ?$  ,  $|\vec{F}| = 6$  Given vector:  $\underline{v} = 2\underline{i} - 2\underline{j} + \underline{k}$  and let A(1,2,3) , B(5,3,7)

As  $\vec{F} \parallel \underline{v}$

So  $\hat{F} = \hat{v} \Rightarrow \frac{\vec{F}}{|\vec{F}|} = \frac{\underline{v}}{|\underline{v}|} \Rightarrow \vec{F} = |\vec{F}| \frac{\underline{v}}{|\underline{v}|} = 6 \cdot \frac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{4+4+1}} = 6 \cdot \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3} = 2(2\underline{i} - 2\underline{j} + \underline{k}) = 4\underline{i} - 4\underline{j} + 2\underline{k}$

**Displacement:**  $\underline{d} = \vec{AB} = (5 - 1)\underline{i} + (3 - 2)\underline{j} + (7 - 3)\underline{k} = 4\underline{i} + \underline{j} + 4\underline{k}$

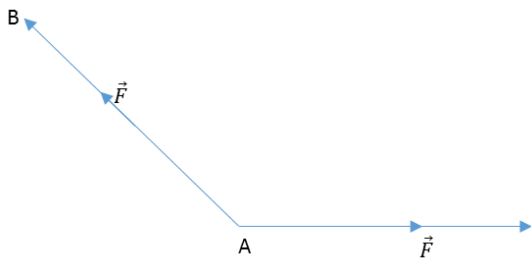
We know that

$$\text{Work done} = \vec{F} \cdot \underline{d} = (4\underline{i} - 4\underline{j} + 2\underline{k}) \cdot (4\underline{i} + \underline{j} + 4\underline{k})$$

$$= 16 - 4 + 8 = 20 \text{ Units}$$

**Q.11:** A force  $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$  is applied at the point (1, 1, 2). Find the moment of the force about the point (2, -1, 3).

**SOLUTION:** Given:  $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$  and Let A (1, 1, 2), B(2, -1, 3)



$$\underline{r} = \overline{AB} = (1 - 2)\hat{i} + (-1 + 1)\hat{j} + (2 - 3)\hat{k} = -\hat{i} - 0\hat{j} - \hat{k}$$

Moment of force about A =  $\underline{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = \hat{i}(0 + 2) - \hat{j}(4 + 3) + \hat{k}(-2 - 0) = 2\hat{i} - 7\hat{j} - 2\hat{k}$

**Q.12:** A force  $\vec{F} = 4\hat{i} - 3\hat{k}$  passes through the point A(2, -2, 5). Find the moment of  $\vec{F}$  about the point B(1, -3, 1).

**SOLUTION:** Given:  $\vec{F} = 4\hat{i} - 3\hat{k}$  and A (2, -2, 5), B(1, -3, 1)

$$\underline{r} = \overline{AB} = (2 - 1)\hat{i} + (-2 + 3)\hat{j} + (5 - 1)\hat{k} = \hat{i} + \hat{j} + 4\hat{k}$$

Moment of force about B =  $\underline{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} = \hat{i}(-3 - 0) - \hat{j}(-3 - 16) + \hat{k}(0 - 4) = -3\hat{i} + 19\hat{j} - 4\hat{k}$

**Q.13:** Given a force  $\vec{F} = 2\hat{i} + \hat{j} - 3\hat{k}$  acting at a point A(1, -2, 1). Find the moment of  $\vec{F}$  about the point B(2, 0, -2). (19 S.Q)

**SOLUTION:** Given:  $\vec{F} = 2\hat{i} + \hat{j} - 3\hat{k}$  and A (1, -2, 1), B(2, 0, -2)



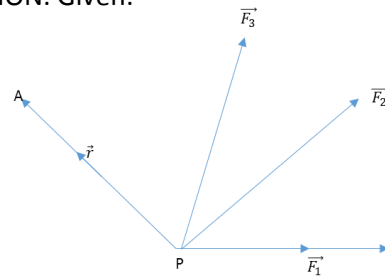
$$\underline{r} = \overline{AB} = (1 - 2)\hat{i} + (-2 - 0)\hat{j} + (1 + 2)\hat{k} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

Moment of force about B =  $\underline{r} \times \vec{F} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(6 - 3) - \hat{j}(3 - 6) + \hat{k}(-1 + 4) = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

**Q.14:** Find the moment about A(1, 1, 1) of each of the concurrent forces  $\hat{i} - 2\hat{j}$ ,  $3\hat{i} + 2\hat{j} - \hat{k}$ ,  $5\hat{j} + 2\hat{k}$ , where P(2, 0, 1) is their point of concurrency.

**SOLUTION:** Given:



Let  $\vec{F}_1 = \hat{i} - 2\hat{j}$ ,  $\vec{F}_2 = 3\hat{i} + 2\hat{j} - \hat{k}$ ,  $\vec{F}_3 = 5\hat{j} + 2\hat{k}$  and A(1, 1, 1), P(2, 0, 1)

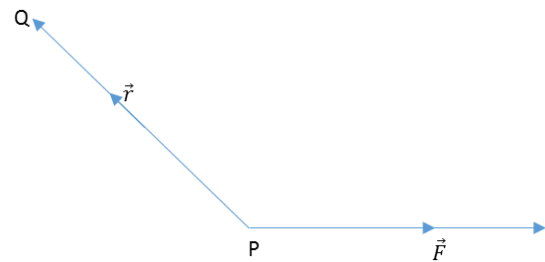
Total force:  $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \hat{i} - 2\hat{j} + 3\hat{i} + 2\hat{j} - \hat{k} + 5\hat{j} + 2\hat{k} = 4\hat{i} + 5\hat{j} + \hat{k}$

$$\underline{r} = \overline{AP} = (2 - 1)\hat{i} + (0 - 1)\hat{j} + (1 - 1)\hat{k} = \hat{i} - \hat{j} + 0\hat{k}$$

Moment of force about B =  $\underline{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix} = \hat{i}(-1 - 0) - \hat{j}(1 - 0) + \hat{k}(5 + 4) = -\hat{i} - \hat{j} + 9\hat{k}$

**Q.15:** A force  $\vec{F} = 7\hat{i} + 4\hat{j} - 3\hat{k}$  is applied at P(1, -2, 3). Find its moment about the point Q(2, 1, 1).

**SOLUTION:**  $\vec{F} = 7\hat{i} + 4\hat{j} - 3\hat{k}$



$$\underline{r} = \overline{QP} = (1 - 2)\hat{i} + (-2 - 1)\hat{j} + (3 - 1)\hat{k} = -\hat{i} - 3\hat{j} + 2\hat{k}$$

Moment of force about B =  $\underline{r} \times \vec{F} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix} = \hat{i}(9 - 8) - \hat{j}(3 - 14) + \hat{k}(-4 + 21) = \hat{i} + 11\hat{j} + 17\hat{k}$$

