MATHEMATICS 12 INTERMEDIATE PART II

Chapter 6.

CONSEQUENCE

A project of:https://NewsonGoogle.com/

Contents

Cone:

A solid figure generated by a straight line passing through a fixed point and revolving about a fixed line is called cone.

Axis of Cone:

The fixed line passing through vertex and perpendicular to the center of base is called axis of cone.

Apex:

The fixed point of cone is called vertex or apex of cone.

Napes of Cone:

The cone has two parts called napes. The part above the vertex is upper nape. The part below the vertex is lower nape.

Types of Cone:

Cones are named according to shape of their base. If the base of cone is circle then it is called circular cone. If the base of cone is ellipse, then it called elliptic cone.

" A cone having its axis perpendicular circular base is called Right circular cone."

Conic Section:

Some standard conic sections are Circle, Ellipse parabola and Hyperbola**.**

We first study properties of a circle other conics will be taken up later.

Circle:

A set of all points in a plane which are equidistance from a fixed point is called circle. The fixed point is celled center and fixed distance is called radius of the circle.

Points Circle:

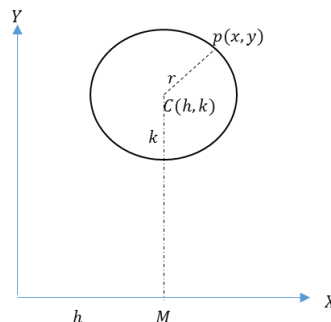If the plane passes through vertex of cone, the instruction is a single point or if $r = 0$

Equation of circle:

Let $P(x, y)$ be any point on the circle (h, k) from figure.

$$radius \ |CP| = r$$

Using distance formula

$$r^2 = (x - h)^2 + (y - k)^2 \ \ hence \ required.$$

$$Circle \ is \ (x - h)^2 + (y - k)^2 = r^2$$

Note:

If

$P(x, y)$ be any point on the circle having centre at orgin $O(o, o)$ and radius r.

$\because |OP| = r$

Using distance $(o, o) \ and \ radius \ r$.

$\because |OP| = r$

Using distance formula

$$r = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$r = \sqrt{x^2 + y^2}$$

Squaring both sides

$r^2 = x^2 + y^2$

$\Rightarrow \ \ x^2 + y^2 = r^2$

$\therefore$ A cicle having radius zero is called point circle.

So,

$$x^2 + y^2 = r^2 \ \ put \ r = 0$$

$\Rightarrow \ \ x^2 + y^2 = $

$\ \ \ 0 \ \ which \ is \ equation \ of \ point \ circle.$

Parametric Equations of Circle:

Let

$P(x, y)$ be any point on the circle having centre at origin and radius.

Draw $\perp ar \ PM \ f$ rom point on $x - axis$.

$In \ \Delta POM$

$|OM| = x, \ \ |PM| = y$

$|OP| = r, and$

$m\angle MOP = \theta \ \ \ so$

$cos\theta = \frac{|PM|}{|OP|} = \frac{x}{r}$

$\Rightarrow \ \ x = rcos\theta$

$\ \ \ \ sin\theta = \frac{|PM|}{|OP|} = \frac{y}{r}$

$\Rightarrow \ \ y = rsin\theta$

are called

$parametric \ eqs. \ of \ circle \ where \ \theta \ is \ parameter.$

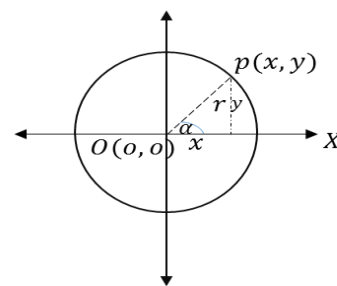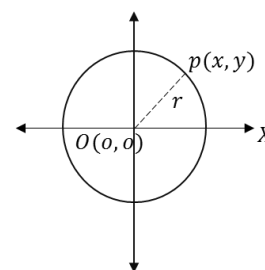**General form of an Equation of a Circle:**

Theorem:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0 \rightarrow (i)$

Represents a circle $i \ , f \ , and \ c \ being \ constants$.

By$(i) x^2 + y^2 + 2gx + 2fy = -c$

Adding $g^2 \ and \ f^2$ on both sides

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2)$$
$$= g^2 + f^2 - c$$

$\Rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$
$\qquad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$
$\Rightarrow [x-(-g)]^2 + [y-(-f)]^2 =$
$\qquad \left(\sqrt{g^2 + f^2 - c}\right)^2$
$\qquad\qquad it\ is\ of\ the\ form$
$\qquad (x-h)^2 + (y-k)^2 = r^2$
$\qquad where\ centre = (-g, -f)$
Radius $= \sqrt{g^2 + f^2 - c}$
The equation $x^2 + y^2 + 2gx + 2fy + c = 0$
$\qquad is\ called$ General equation of circle
Every equation of 2nd degree in x and y represents
an eq. of circle if (i) coefficients of x^2 and y^2 is 1
$\qquad$ (ii) *it does not involve product xy.*

1. A circle passing three non- collinear points.
2. A circle passing through two points and having its center on a given line.
3. A circle passing through two points and equation of tangent at one of these points is known.
4. A circle passing through two points and touching a given line.

Exercise6.1

Question # 1. In each of the following find an equation of the circle with

(a) Centre at $(5, -2)$ and radius 4.
Solution.
Centre: $C(h, k) = (5, -2)$
$\Rightarrow h = 5, k = -2$
Radius: $r = 4$
Required equation of circle is
$(x-h)^2 + (y-k)^2 = r^2$
$(x-5)^2 + (y+2)^2 = (4)^2$
$x^2 + 25 - 10x + y^2 + 4 + 4y = 16$
$x^2 + y^2 - 10x + 4y + 13 = 0$
(b) Centre at $\left(\sqrt{2}, -3\sqrt{3}\right)$ and radius$2\sqrt{2}$.
Solution.
Centre: $C(h, k) = \left(\sqrt{2}, -3\sqrt{3}\right)$
$\Rightarrow h = \sqrt{2},\ k = -3\sqrt{3}$
Radius: $r = 2\sqrt{2}$
Required equation of circle is
$(x-h)^2 + (y-k)^2 = r^2$
$\left(x - \sqrt{2}\right)^2 + \left(y + 3\sqrt{3}\right)^2 = \left(2\sqrt{2}\right)^2$
$x^2 + 2 - 2\sqrt{2}x + y^2 + 9(3) + 6\sqrt{3}y = 4(2)$
$x^2 + y^2 - 2\sqrt{2}x + 6\sqrt{3}y + 21 = 0$
(c) End of a diameter at $(-3,\ 2)$
$\qquad$ **and$(5, -6)$.**
Solution.

Let $A(5, -6)$ *and* $B(5, -6)$
Centre of circle= Mid-point $\overline{AB}$
$= C\left(\frac{-3+5}{2}, \frac{2-6}{2}\right) = C\left(\frac{2}{2}, -\frac{4}{2}\right) = C(1, -2)$
Centre: $C(h, k) = (1, -2)$
$\Rightarrow h = 1, k = -2$
Radius of circle$= |AC| =$
$\sqrt{(-3-1)^2 + (2+2)^2}$
$= \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$
Radius: $r = 4\sqrt{2}$
Required equation of circle is
$(x-h)^2 + (y-k)^2 = r^2$
$(x+1)^2 + (y+2)^2 = \left(4\sqrt{2}\right)^2$
$x^2 + 1 + 2x + y^2 + 4 + 4y = 32$
$x^2 + y^2 + 2x + 4y - 27 = 0$

Question # 2. Find the centre and radius of the circle with the given equation.
$\qquad$ **(a) $x^2 + y^2 + 12x - 10y = 0$**
Sol:
$\qquad$ Comparing it with
$x^2 + y^2 + 2gx + 2fy + c = 0$
$\Rightarrow 2g = 12 \Rightarrow g = 6$ also $\Rightarrow 2f = -10 \Rightarrow f = -5$
and $c = 0$
Centre=$(-g, -f) = (-6, 5)$
Radius$= \sqrt{g^2 + f^2 - c}$
$\qquad = \sqrt{(-6)^2 + (-5)^2 - 0} = \sqrt{36 + 25} = \sqrt{61}$

$\qquad$ **(b) $5x^2 + 5y^2 + 14x + 12y - 10 = 0$**
Sol: (b)
$$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0$$
$\qquad$ Comparing it with
$x^2 + y^2 + 2gx + 2fy + c = 0$
$\Rightarrow 2g = \frac{14}{5} \Rightarrow g = \frac{7}{5}$ also $\Rightarrow 2f = \frac{12}{5} \Rightarrow f = \frac{6}{5}$ and
$c = -2$
Centre=$(-g, -f) = (-\frac{7}{5}, -\frac{6}{5})$
Radius$= \sqrt{(-\frac{7}{5})^2 + (-\frac{6}{5})^2 + 2}$
$\qquad = \sqrt{\frac{49}{25} + \frac{36}{25} + 2} = \sqrt{\frac{49+36+50}{25}} = \sqrt{\frac{135}{25}} = \sqrt{\frac{27}{5}}$
$\qquad\qquad = \frac{3\sqrt{3}}{5}$
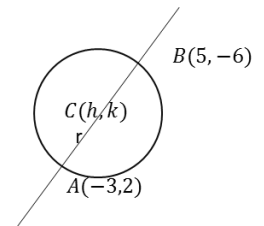$\qquad$ **(c) $x^2 + y^2 - 6x + 4y + 13 = 0$**
Sol: (c)
$\qquad$ Comparing it with
$x^2 + y^2 + 2gx + 2fy + c = 0$
$\Rightarrow 2g = -6 \Rightarrow g = -3$ also $\Rightarrow 2f = 4 \Rightarrow f = 2$
and $c = 13$
Centre=$(-g, -f) = (3, -2)$
Radius$= \sqrt{(3)^2 + (-2)^2 - 13}$

$$= \sqrt{9+4-13} = \sqrt{0} = \sqrt{61}$$

Which represent a point circle.

Q#3 Write an equation of the circle that passes through given points.

   (a) $A(4,5), B(-4,-3)$ and $C(8,-3)$

Sol: (a)

$A(4,5), B(-4,-3)$ and $C(8,-3)$

Let the required equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At $A(4,5)$

$(4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$

$16 + 25 + 8g + 10f + c = 0$

$8g + 10f + c + 41 = 0 \to(1)$

At $B(-4,-3)$

$(-4)^2 + (-3)^2 + 2g(-4) + 2f(-3) + c = 0$

$16 + 9 - 8g - 6f + c = 0$

$-8g - 6f + c + 25 = 0 \to(2)$

At $C(8,-3)$

$(8)^2 + (-3)^2 + 2g(8) + 2f(-3) + c = 0$

$64 + 9 + 16g - 6f + c = 0$

$16g - 6f + c + 73 = 0 \to(3)$

     Now Eq $(1) - Eq(2)$

$\quad 8g + 10f + c + 41 = 0$

$\quad \mp8g \mp 6f \pm c \pm 25 = 0$

$\overline{\quad 16g + 16f + 16 = 0 \quad}$

$\Rightarrow\ g + f + 1 = 0 \to(4)$

     Now Eq $(3) - Eq(2)$

$\quad 16g - 6f + c + 73 = 0$

$\quad \mp8g \mp 6f \pm c \pm 25 = 0$

$\overline{\quad 24g + 48 = 0 \quad}$

$\Rightarrow\ 24g = -48 \Rightarrow g = -2$

Put in (4)

$g + f + 1 = 0 \Rightarrow -2 + f + 1 = 0 \Rightarrow f = 1$

Put $f = 1$ and $g = -2$ in eq (2)

$-8g - 6f + c + 25 = 0 \Rightarrow -8(-2) - 6(1) + c + 25 = 0$

$\Rightarrow\ 16 - 6 + c + 25 = 0 \Rightarrow c + 35 = 0 \Rightarrow c = -35$

Hence, required equation of circle is

$x^2 + y^2 + 2(-2)x + 2(1)y + (-35) = 0$

$\Rightarrow\ x^2 + y^2 - 4x + 2y - 35 = 0$

   (b) $A(-7,7), B(5,-1)$ and $C(10,0)$

Sol: (b)

$A(-7,7), B(5,-1)$ and $C(10,0)$

Let the required equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At $A(-7,7)$

$(-7)^2 + (7)^2 + 2g(-7) + 2f(7) + c = 0$

$49 + 49 - 14g + 14f + c = 0$

$-14g + 14f + c + 98 = 0 \dots..(1)$

At $B(5,-1)$

$(5)^2 + (-1)^2 + 2g(5) + 2f(-1) + c = 0$

$25 + 1 + 10g - 2f + c = 0$

$10g - 2f + c + 26 = 0 \dots..(2)$

At $C(10,0)$

$(10)^2 + (0)^2 + 2g(10) + 2f(0) + c = 0$

$100 + 0 + 20g + 0f + c = 0$

$20g + c + 100 = 0 \dots..(3)$

Now Eq (1)-Eq(2)

$\quad -14g + 14f + c + 98 = 0$

$\quad \pm10g \mp 2f \pm c \pm 26 = 0$

$\overline{\quad -24g + 16f + 72 = 0 \quad}$

$\Rightarrow\ -3g + 2f + 9 = 0 \dots..(4)$

Now Eq (3)-Eq(2)

$\quad 20g + +0f + c + 100 = 0$

$\quad \pm10g \mp 2f \pm c \pm 26 = 0$

$\overline{\quad 10g + 2f + 74 = 0 \quad}\dots.(5)$

Now Eq (3)-Eq(2)

$\quad -3g + 2f + 9 = 0$

$\quad \pm10g \pm 2f \pm 74 = 0$

$\overline{\quad -13g - 65 = 0 \quad}\dots.(5)$

$\Rightarrow -13g = 65 \Rightarrow g = -5$

Put in (4)

$-3g + 2f + 9 = 0 \Rightarrow -3(-5) + 2f + 9 = 0 \Rightarrow 15 + 2f + 9 = 0 \Rightarrow f = -12$

Put $g = -5$ in eq (3)

$20g + c + 100 = 0 \Rightarrow 20(-5) + c + 100 = 0 = 0$

$\Rightarrow\ -100 + c + 100 = 0 \Rightarrow c = 0$

Hence, required equation of circle is

$x^2 + y^2 + 2(-5)x + 2(-12)y + (0) = 0$

$\Rightarrow\ x^2 + y^2 - 10x - 24y = 0$

   (c) $A(a,0), B(0,b)$ and $C(0,0)$

Sol: (c)

$A(a,0), B(0,b)$ and $C(0,0)$

Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At $A(a,0)$

$(a)^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$

$a^2 + 0 + 2ag + c = 0$

$a^2 + 2ag + c = 0 \dots..(1)$
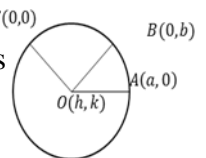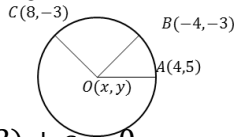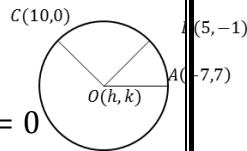
At $B(0,b)$

$(0)^2 + (b)^2 + 2g(0) + 2f(b) + c = 0$

$b^2 + 2bf + c = 0 \dots..(2)$

At $C(0,0)$

$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$

$c = 0$ put in eq (1)

$a^2 + 2ag + c = 0 \Rightarrow a^2 + 2ag + 0 = 0$

$g = -\frac{a^2}{2a} = -\frac{a}{2}$

Eq (2) $\Rightarrow b^2 + 2bf + 0 = 0 \Rightarrow f = -\frac{b^2}{2b} = -\frac{b}{2}$
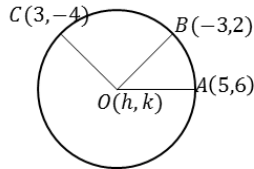
Hence, required equation of circle is

$x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 = 0$

$\Rightarrow x^2 + y^2 - ax - by = 0$

d) $A(5,6), B(-3,2), C(3,-4)$

Solution:

Clearly from fig.

$|AO|^2 = |BO|^2$

$(h-5)^2 + (k-6)^2 = (h+3)^2 + (k-2)^2$

$h^2 + 25 - 10h + k^2 + 36 - 12k$
$\qquad = h^2 + 9 + 6k + k^2 + 4 - 4k$

$-10h - 12k + 61 = 6h - 4k + 13$

$\Rightarrow -10h - 6h - 12k + 4k + 61 - 13 = 0$

$\Rightarrow -16h - 8k + 48 = 0$

$\Rightarrow 2h + k - 6 = 0 (\div by -8) \to (i)$

Also $|AO|^2 = |CO|^2$

$(h-5)^2 + (k-6)^2 = (h-3)^2 + (k+4)^2$

$h^2 + 25 - 10h + k^2 + 36 - 12k$
$\qquad = h^2 + 9 - 6h + k^2 + 16 + 8k$

$-10h + 6h - 12k - 8k + 61 - 25 = 0$

$\qquad -4h - 20k + 36 = 0$

$\Rightarrow -2h - 10k + 18 = 0 \to (ii)$

by $(i) + (ii)$

$\Rightarrow$
$$\begin{array}{r} 2h + k - 6 = 0 \\ -2h - 10k + 18 = 0 \\ \hline -9k + 12 = 0 \end{array}$$

$\Rightarrow -9k = -12$

$\Rightarrow k = \frac{4}{3}$ put in (i)

$$2h + \frac{4}{3} - 6 = 0$$
$$6h + 4 - 18 = 0$$

$\Rightarrow 6h - 14 = 0$

$\Rightarrow 6h = 14$

$\Rightarrow h = \frac{7}{3}$

As $r = |AO| = \sqrt{(h-5)^2 + (k-6)^2}$

$$r = \sqrt{\left(\frac{7}{3} - 5\right)^2 + \left(\frac{4}{3} - 6\right)^2}$$

$$r = \sqrt{\left(-\frac{8}{3}\right)^2 + \left(-\frac{14}{3}\right)^2}$$

$$r = \sqrt{\frac{64 + 196}{9}}$$

$\Rightarrow r = \sqrt{\frac{260}{9}}$

$\Rightarrow$ Now eq of circle is $(x-h)^2 + (y-k)^2 = r^2$

$\Rightarrow \left(x - \frac{7}{3}\right)^2 + \left(y - \frac{4}{3}\right)^2 = \frac{260}{9}$

$\Rightarrow \left(\frac{3x-7}{3}\right)^2 + \left(\frac{3y-4}{3}\right)^2 = \frac{260}{9}$

$\Rightarrow \frac{9x^2 + 49 - 42x}{9} + \frac{9y^2 + 16 - 24y}{9} = \frac{260}{9}$

$\Rightarrow 9x^2 + 49 - 42x + 9y^2 + 16 - 24y - 260 = 0$

$\Rightarrow 9x^2 + 9y^2 - 42x - 24y - 195 = 0$

$\qquad 3x^2 + 3y^2 - 14x - 8y - 65 = 0 (\div by\ 3)$

$\Rightarrow 3(x^2 + y^2) - 14x - 8y - 65 = 0$

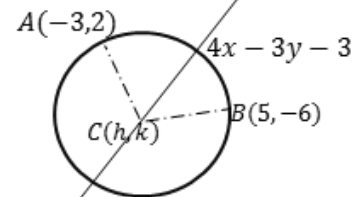**Question#4 Find equation of circle passing through**

(a) $A(3,-1), B(0,1)$ and having centre at $4x - 3y - 3 = 0$

Sol: (a)

$$A(3,-1), B(0,1)$$

Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At $A(3,-1)$

$(3)^2 + (-1)^2 + 2g(3) + 2f(-1) + c = 0$

$9 + 1 + 6g - 2f + c = 0$

$6g - 2f + c + 10 = 0 \ldots..(1)$

At $B(0,b)$

$(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0$

$1 + 2f + c = 0 \ldots.(2)$

As centre $(-g, -f)$ lies on the the line

$4x - 3y - 3 = 0$

$\Rightarrow 4(-g) - 3(-f) - 3 = 0$

$\Rightarrow -4g + 3f - 3 = 0 \ldots..(3)$

Now Eq (1)-Eq(2)

$$\begin{array}{r} 6g - 2f + c + 10 = 0 \\ 0g \pm 2f \pm c \pm 1 = 0 \\ \hline 6g - 4f + 9 = 0 \ldots.(4) \end{array}$$

Multiply Eq(3) by 3 and Eq(4) by 2 and adding

$$\begin{array}{r} -12g + 9f - 9 = 0 \\ 12g - 8f + 18 = 0 \\ \hline f + 9 = 0 \end{array}$$

$\Rightarrow f = -9$ put in Eq (4)

$-4g + 3f - 3 = 0 \Rightarrow -4g + 3(-9) - 3 = 0 \Rightarrow$

$-4g - 27 - 3 = 0$

$\Rightarrow g = -\dfrac{15}{2}$

Put in $f = -9$ in eq (2)

$1 + 2f + c = 0 \Rightarrow 1 + 2(-9) + c = 0$

$1 - 18 + c = 0$

$c = 17$

Hence, required equation of circle is

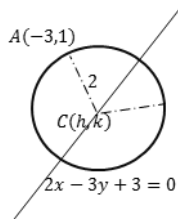$x^2 + y^2 + 2\left(-\dfrac{15}{2}\right)x + 2(-9)y + (17) = 0$

$\Rightarrow x^2 + y^2 - 15x - 18y + 17 = 0$

(b) Find an equation of circle passing through $A(-3, 1)$ with radius 2 and centre at $2x - 3y + 3 = 0$

Sol: $A(-3, 1)$ and radius 2

Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At $A(-3, 1)$

$(-3)^2 + (1)^2 + 2g(-3) + 2f(1) + c = 0$

$9 + 1 - 6g + 2f + c = 0$

$-6g + 2f + c + 10 = 0 \ldots..(1)$

As centre $(-g, -f)$ lies on the line $2x - 3y + 3 = 0$.

$\Rightarrow -2g + 3f + 3 = 0 \ldots..(2)$

Since radius is 2

$\Rightarrow \sqrt{g^2 + f^2 - c} = r$

$\Rightarrow \sqrt{g^2 + f^2 - c} = 2$

$\Rightarrow g^2 + f^2 - c = 4$

$\Rightarrow g^2 + f^2 - c - 4 = 0 \ldots..(3)$

Adding Eq(1) and Eq(3)

$\quad -6g + 2f + c + 10 = 0$

$\quad \underline{g^2 + f^2 - c - 4 = 0}$

$\quad \overline{-6g + 2f + g^2 + f^2 + 6 = 0} \ldots..(4)$

Eq(2) $\Rightarrow -2g = -3f - 3 \Rightarrow g = \dfrac{3f+3}{2}$

Put in Eq(4)

$-6g + 2f + g^2 + f^2 + 6 = 0$

$\Rightarrow -6\left(\dfrac{3f+3}{2}\right) + 2f + (\dfrac{3f+3}{2})^2 + f^2 + 6 = 0$

$\Rightarrow -3(3f + 3) + 2f + \dfrac{9f^2 + 9 + 18f}{4} + f^2 + 6 = 0$

$\Rightarrow -12(3f + 3) + 8f + 9f^2 + 9 + 18f + 4f^2 + 24 = 0$

$\Rightarrow -36f + 36 + 8f + 9f^2 + 9 + 18f + 4f^2 + 24 = 0$

$\Rightarrow 13f^2 - 10f - 3 = 0$

$\Rightarrow 13f^2 - 13f + 3f - 3 = 0$

$\Rightarrow 13f(f - 1) + 3(f - 1) = 0$

$\Rightarrow (f - 1)(13f + 3) = 0$

$\Rightarrow f = 1, -\dfrac{3}{13}$

When $f = 1$

$g = \dfrac{3f+3}{2} \Rightarrow g = \dfrac{3(1)+3}{2} = \dfrac{6}{2} = 3$

From Eq(3)

$\Rightarrow (3)^2 + (1)^2 - c - 4 = 0$

$\Rightarrow 9 + 1 - c - 4 = 0$

$\Rightarrow c = 6$

Hence required equation when $f = 1, g = 3$ and $c = 6$ is

$x^2 + y^2 + 2(3)x + 2(1)y + 6 = 0$

$\Rightarrow x^2 + y^2 + 6x + 2y + 6 = 0$

When $f = -\dfrac{3}{13}$

$g = \dfrac{3f+3}{2} \Rightarrow g = \dfrac{3(-\frac{3}{13})+3}{2} = \dfrac{\frac{-9+39}{13}}{2} = \dfrac{15}{13}$

From Eq(3)

$\Rightarrow \left(\dfrac{15}{13}\right)^2 + \left(-\dfrac{3}{13}\right)^2 - c - 4 = 0$

$\Rightarrow 225 + 9 - 169c - 676 = 0$

$\Rightarrow 169c = -442$

$\Rightarrow c = -\dfrac{442}{169} = -\dfrac{34}{13}$

Hence required equation when $f = -\dfrac{3}{13}, g = \dfrac{15}{13}$ and $c = -\dfrac{34}{13}$ is

$x^2 + y^2 + 2\left(\dfrac{15}{13}\right)x + 2\left(-\dfrac{3}{13}\right)y + \left(-\dfrac{34}{13}\right) = 0$

$\Rightarrow 13x^2 + 13y^2 + 30x - 6y - 34 = 0$

(c)

$A(5, 1)$ **and tangent to the line** $2x - y - 10 = 0$

at $B(3, -4)$

Solution: it is clear
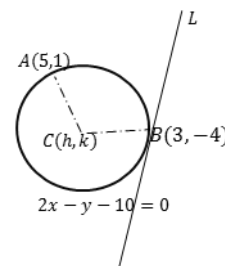
$|AC|^2 = |BC|^2$

$(h - 5)^2 + (k - 1)^2 = (h - 3)^2 + (k + 4)^2$

$h^2 + 25 - 10h + k^2 + 1 - 2k = h^2 + 9 - 6h + k^2 + 16 + 8k$

$-10h + 2k + 26 = -6h + 8k + 25$

$\Rightarrow -4h - 10k + 1 = 0 \rightarrow (i)$

slope of $2x - y - 10 = 0 = -\dfrac{a}{b} = \dfrac{-2}{-1} = 2$

slope of $BC = \dfrac{k + 4}{h - 3}$

$\because$ both lines are $\perp$ ar so

$\Rightarrow (2)\left(\dfrac{k+4}{h-3}\right) = -= 1$

$\Rightarrow 2k + 8 = -h + 3$

$\Rightarrow h + 22k + 5 = 0$

$\Rightarrow 5k + 10k + 25 = 0 \rightarrow (ii)$

By $(i) + (ii)$

$\Rightarrow$

$$5h + 10k + 25 = 0$$
$$\underline{-4h - 10k + 1 = 0}$$
$$h + 26 = 0$$

$\Rightarrow$ $h = -26$ put in (ii)

$$-26 + 2k + 5 = 0$$

$\Rightarrow$ $-26 + 2k + 5 = 0$

$\Rightarrow$ $2k - 21 = 0$

$\Rightarrow$ $k = \frac{21}{2}$

$$r = |AC| = \sqrt{(h-5)^2 + (k-1)^2}$$

$\Rightarrow$ $r = |AC| = \sqrt{(-26-5)^2 + \left(\frac{21}{2} - 1\right)^2}$

$\Rightarrow$ $= \sqrt{(-31)^2 + \left(\frac{19}{2}\right)^2}$

$\Rightarrow$ $= \sqrt{961 + \frac{361}{4}} = \sqrt{\frac{3844+361}{4}} = \sqrt{\frac{4205}{2}}$

$\Rightarrow$ Now $Eq.$ of $circle$ is

$$(x - h)^2 + (y - k)^2 = r^2$$

$\Rightarrow$ $(x + 26)^2 + \left(y - \frac{21}{2}\right)^2 = \left(\frac{\sqrt{4205}}{2}\right)^2$

$\Rightarrow$ $x^2 + 52x + 676 + y^2 - 21y + \frac{441}{4} = \frac{4205}{4}$

$\Rightarrow$ $x^2 + y^2 + 52x - 21y + \frac{441}{4} - \frac{4205}{4} = 0$

$\Rightarrow$ $x^2 + y^2 + 52x - 21y + \left(-\frac{1014}{4}\right) = 0$
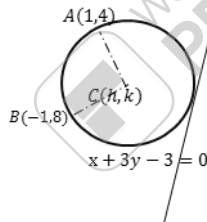
$\Rightarrow$ $x^2 + y^2 + 52x - 21y - 265 = 0$

(d)

$A(1, 4), B(-1, 8)$ and $tangent$ to the $line$
$x + 3y - 3) = 0$

Solution

It is clear from fig.

$|AC|^2 = |BC|^2$

$(h - 1)^2 + (k - 4)^2 (h + 1)^2 + (k - 8)^2$

$h^2 + 1 - 2h + k^2 + 16 - 8k = h^2 + 1 + 2h + k^2 + 64 - 16x$

$-2h - 8k + 16 = 2h + 1 - 16k + 64$

$$-4h + 8k - 48 = 0$$
$$h = 2k + 12 = 0$$
$$h = 2k - 12 \to (i)$$

Now $|AC| = |CL|$ $where$ L is $tangent$ $line.$

$$\sqrt{(h-1)^2 (k-4)^2} = \frac{|h + 3k - 3|}{\sqrt{(1)^2 + (3)^2}}$$

$$\sqrt{h^2 - 2h + 1 + k^2 - 8k + 16} = \frac{|h + 3k - 3|}{\sqrt{10}}$$

Squaring both sides

$$h^2 - 2h + k^2 - 8k + 17$$
$$= \frac{h^2 + 9k^2 + 9 + 6hk - 18k - 6h}{10}$$

$\Rightarrow$ $10h^2 - 20h + 10k^2 - 80k + 170 = h^2 + 9k^2 + 9 + 6hk - 18k - 6h$

$\Rightarrow$ $9h^2 + k^2 - 14h - 62k + 161 - 6hk = 0 \to (ii)$

Put (i) and (ii)

$9(2k - 12)^2 + k^2 - 14(2k - 12) - 62k + 161 - 6(2k - 12)k = 0$

$9(4k^2 + 144 - 48k) + k^2 - 28k + 168 - 62k + 161 - 12k^2 + 72k = 0$

$\Rightarrow$ $36k^2 + 1296 - 432k + k^2 - 28k + 168 - 62k + 161 - 12k^2 + 72k = 0$

$\Rightarrow$ $25k^2 - 450k + 1625 = 0$

$\Rightarrow$ $k^2 - 18k + 65 = 0$ $(\div$ by $25)$

$\Rightarrow$ $k^2 - 13k - 5k + 65 = 0$

$\Rightarrow$ $k(k - 13) - 5(k - 13) = 0$

$\Rightarrow$ $(k - 5)(k - 13) = 0$

$\Rightarrow$ $k - 5 = 0$ $k - 13 = 0$

$\Rightarrow$ $k = 5;$ $k = 13$

$when$ $k = 13$ $then$ $h = 2(13) - 12 = 14$

$when$ $k = 5$ $then$ $h = 2(5) - 12 = -2$

So

$$C_1(-2, 5) and C_2(14, 13)$$

Now $r_1 = |AC_1| = \sqrt{(1+2)^2 + (4-5)^2} = \sqrt{10}$

$r_2 = |AC_2| = \sqrt{(1-14)^2 + (4-13)^2} = \sqrt{250}$

Now $eq.$ of $circle$ is

$$(x + 2)^2 + (y - 5)^2 = \left(\sqrt{10}\right)^2$$

$\Rightarrow$ $x^2 + 4 + 4x + y^2 + 25 - 10y - 10 = 0$

$\Rightarrow$ $x^2 + y^2 + 4x \pm 10y + 19 = 0$

Also eq. of circle is

$$(x - 14)^2 + (y - 13)^3 = \left(\sqrt{250}\right)^2$$

$\Rightarrow$ $x^2 - 28x + 196 + y^2 + 169 - 26y - 250 = 0$

$\Rightarrow$ $x^2 + y^2 - 28x - 26y + 115 = 0$

Q#5) Find equation of circle of radius 'a' And lying in the second quadrant such that it is tangent to both axis.

Sol: Since circle is lying in second quadrant with radius "a" and tangent to the both axes, therefore

Centre: $(-a, a)$

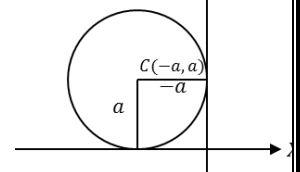Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

At $(-a, a)$

$(x + a)^2 + (y - a)^2 = a^2$

$\Rightarrow$ $x^2 + a^2 + 2ax + y^2 + a^2 - 2ay = a^2$

$\Rightarrow$ $x^2 + y^2 + 2ax - 2ay + a^2 = 0$

Which is required.

Q#6) Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangent to the circle $x^2 + y^2 + 6x - 4y = 0$

Sol:

$x^2 + y^2 + 6x - 4y = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 6 \Rightarrow g = 3$

Also $\Rightarrow 2f = -4 \Rightarrow f = -2$ and $c = 0$

Centre$=(-g, -f) = (-3, 2)$

Radius$= \sqrt{(-3)^2 + (2)^2 - 0}$

$= \sqrt{9 + 4} = \sqrt{13}$

Now distance from the centre and the line

$3x - 2y = 0$ is the radius

$r = \dfrac{|3(-3) - 2(2)|}{\sqrt{(3)^2 + (-2)^2}}$

$r = \dfrac{|13|}{\sqrt{13}} = \sqrt{13}$.....(1)

Now distance from the centre and the line

$2x + 3y - 13 = 0$ is the radius

$r = \dfrac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$

$r = \dfrac{|13|}{\sqrt{13}} = \sqrt{13} \rightarrow (2)$

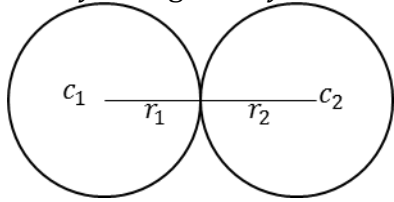Hence, from (1) and (2) it s clear that both lines are tangent to the given circle.

Q#7) Show that $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y - 9 = 0$ touch externally.

Sol:

Let C_1: $x^2 + y^2 + 2x - 2y - 7 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 2 \Rightarrow g = 1$ also $\Rightarrow 2f = -2 \Rightarrow f = -1$

and $c = -7$

Centre$= c_1(-g, -f) = (-1, 1)$

Radius$= \sqrt{(-1)^2 + (1)^2 + 7}$

$r_1 = \sqrt{1 + 1 + 7} = \sqrt{9} = 3$

Let C_2: $x^2 + y^2 - 6x + 4y - 9 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -6 \Rightarrow g = -3$ also $\Rightarrow 2f = 4 \Rightarrow f = 2$

and $c = 9$

Centre$= c_2(-g, -f) = (3, -2)$

Radius$= \sqrt{(3)^2 + (-2)^2 - 9}$

$r_2 = \sqrt{9 + 4 - 9} = \sqrt{4} = 2$

Two circles touching externally if

$|c_1 c_2| = r_1 + r_2$

$\Rightarrow \sqrt{(-1 - 3)^2 + (1 + 2)^2} = 3 + 2$

$\Rightarrow \sqrt{16 + 9} = 5$

$\Rightarrow \sqrt{25} = 5$

$\Rightarrow 5 = 5$

$\Rightarrow$ Both circles touches externally.

Q#8) Show that $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touch internally.

Sol:

Let C_1: $x^2 + y^2 + 2x - 8 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 2 \Rightarrow g = 1$

also $\Rightarrow 2f = 0 \Rightarrow f = 0$ and $c = -8$

Centre$= c_1(-g, -f) = (-1, 0)$

Radius$= \sqrt{(-1)^2 + (0)^2 + 8}$

$r_1 = \sqrt{1 + 8} = \sqrt{9} = 3$

Let C_2: $x^2 + y^2 - 6x + 6y - 46 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -6 \Rightarrow g = -3$ also $\Rightarrow 2f = 6 \Rightarrow f = 3$

and $c = -46$

Centre$= c_2(-g, -f) = (3, -3)$

Radius$= \sqrt{(3)^2 + (-3)^2 + 46}$

$r_2 = \sqrt{9 + 9 + 46} = \sqrt{64} = 8$

Two circles touching internally if

$|c_1 c_2| = |r_1 - r_2|$

$\Rightarrow \sqrt{(-1 - 3)^2 + (0 + 3)^2} = |3 - 8|$

$\Rightarrow \sqrt{16 + 9} = |-5|$

$\Rightarrow \sqrt{25} = 5$

$\Rightarrow 5 = 5$

$\Rightarrow$ Both circles touches internally.
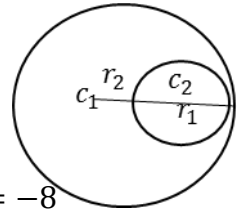
Question No.9

Find Equation of circles of line radius 2 and tangent to line $x - y - 4 = 0$ at $(1, -3)$

Solution:

$\because |AC| = 2$
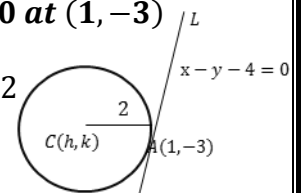
$\Rightarrow |AC|^2 = 4$

$\Rightarrow (h - 1)^2 + (k + 3)^2 = 4$

$h^2 + 1 - 2h + k^2 + 9 + 6k = 4$

$\Rightarrow h^2 + k62 - 2h + 6k + 6 = 0 \rightarrow (i)$

$\Rightarrow$ Slope of $x - y - 4 = 0 = \dfrac{-a}{b} = -\dfrac{1}{-1} = 1$

Slope of $AC = \dfrac{k+3}{h-1}$

$\because$ both lines are $\perp$ ar so

$\left(\dfrac{k+3}{h-1}\right)(1) = (-1)$

$\Rightarrow k + 3 = -h + 1$

$\Rightarrow h = -k - 2 = 0$ put in (i)

$(-k - 2)^2 + k^2 - 2(-k - 2) + 6k + 6 = 0$

$K^2 + 4 + 4k + k^2 + 2k + 4 + 6k + 6 = 0$

$$2k^2 + 12k + 14 = 0$$
$$k^2 + 6k + 7 = 0 \quad (\div \text{ by } 2)$$
$$\Rightarrow \quad k = \frac{-6 \pm \sqrt{36-28}}{2} = \frac{-6 \pm \sqrt{8}}{2}$$
$$\Rightarrow \quad k = \frac{-6 \pm 2\sqrt{2}}{2} = \frac{2(-3 \pm \sqrt{2})}{2}$$
$$\Rightarrow \quad k = -3 + \sqrt{2} \quad or \quad k = -3 - \sqrt{2}$$
When $k = -3 + \sqrt{2}$ then
$$h = -(-3 + \sqrt{2}) - 2 = 3 - \sqrt{2} - 2$$
$$h = 1 - \sqrt{2}$$
when $h = 1 - \sqrt{2}$, $k = -3 + \sqrt{2}$, $r = 2$
$Eq. of$ circle is
$$\left(x - 1 + \sqrt{2}\right)^2 + \left(y + 3 - \sqrt{2}\right)^2 = 4$$
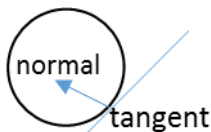
Tangent and Normal:

Tangent to a circle:
"A line which touches the circle without cutting it, is called tangent line"

Normal to a circle:
"A Line $\perp$
ar to the tangent line at a point of tandency
 is called Normal line.
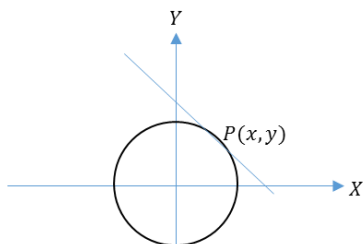$$for\ circle\ x^2 + y^2 = r^2$$

Equation of tangent:
Given equation of circle is
$$x^2 + y^2 = r^2 \to (i)$$
$\because$ point $P(x, y)$ lies on the circle so (i) becomes.
As
$$x_1^2 + y_1^2 + 7 = 0 \quad (\div\ by\ 2)$$

Diff $(i) w.r.t\ x, we\ get$
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$x + y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$
$\because m = slope\ of\ tangent\ line\ at\ P(x_1, y_1)$
So $m = \frac{dy}{dx}\Big|_p = \frac{-x_1}{y_1}$
Using $y - y_1 = m(x - x_1)$
$$\Rightarrow \quad y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - y_1 = \frac{-x_1}{y_1}(x - x_1)$$
$$\Rightarrow \quad yy_1 - y_1^2 = -xx_1 + x_1^2$$
$$\Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2$$
$$\Rightarrow \quad xx_1 + yy_1 = r^2$$
$(\because x_1 + y_1 = r^2) by\ (ii)$
 which is equation of tangent line.

Equation of Normal
$$\because slope\ of\ normal\ line = \frac{y_1}{x_1}$$
so eq. of normal line at $P(x, y) is$
$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$
$$\Rightarrow \quad x_1 y - x_1 y_1 = xy_1 - x_1 y_1$$
$$\Rightarrow \quad x_1 y = xy_1$$
$$\Rightarrow \quad \frac{y}{y_1} = \frac{x}{x_1}$$
Which is eq. of normal line.

For circle
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
Equation of tangent:
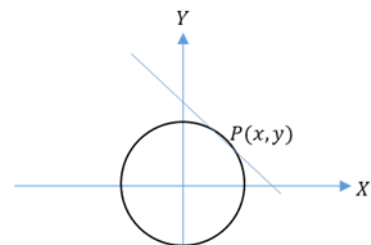
Given equation of circle is
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$\because P(x, y) lies\ on\ the\ cicle\ so\ eq(i) become\ as$
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_{1_c} = 0 \to (ii)$$
$Diff(i) w.r.t\ x\ we\ get$
$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(o)$$
$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$
$$\Rightarrow \quad x + y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$
$$\Rightarrow \quad x + y\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$
$$\Rightarrow \quad (y + f)\frac{dy}{dx} = -(x + g)$$
$$or\frac{dy}{dx} = -\left(\frac{x+g}{y+f}\right)$$
$m = slope\ of\ tangent\ line\ at\ point\ P(x_1, y_1)$

$$m = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

Using $y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - y_1 = -\left(\frac{x_1+g}{y_1+f}\right)(x - x_1)$

$\Rightarrow \quad y - y_1(y_1 + f) = -(x - x_1)(x_1 + g)$

$\Rightarrow \quad (y - y_1)(y_1 + f) = -(x - x_1)(x_1 + g)$

$\Rightarrow \quad yy_1 - y_1^2 + fy - fy_1 = -xx_1 + x_1^2 - xg + x_1 g$

$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

adding $gx_1 + fy_1 + c$ on both sides

$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c$
$\quad = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y - y_1) + c = 0$

$\because x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \ \ by (ii)$

Equation of Normal

Now slope of normal line is $\frac{y_1 + f}{x_1 + g}$

Equation of normal line is $y - y_1 = \left(\frac{y_1 - f}{x_1 + g}\right)(x - x_1)$

$\Rightarrow \quad \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$

Theorem: the point

$P(x_1, y_1)$ **lies outside**, **on or inside the circle**,
$x^2 + y^2 = r^2 \ according \ as \ x_1^2 + y_1^2 \geq 0$

Proof:

Given eq. of circle is $\quad x^2 + y^2 = r^2$
centre $= (0,0), radius = r$

Now point

$P(x_1, y_1)$ will lie outside, on inside the circle
$x^2 + y^2 = r^2 \ if$

$$|CP| \underset{<}{\overset{\geq}{=}} r$$

$\Rightarrow \quad \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} \underset{<}{\overset{\geq}{=}} r$

Squaring both sides

$\Rightarrow \quad x_1^2 + y_1^2 \underset{<}{\overset{\geq}{=}} r^2$

$\Rightarrow \quad x_1^2 + y_1^2 - r^2 \underset{<}{\overset{\geq}{=}} 0$

Hence proved.

theorem:

the point $p(x_1, y_1)$ **lies outside**, **on or inside the circle**

$x^2 + y^2 + 2gx + 2fy + c = 0 \ accorrding \ as$

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \underset{<}{\overset{\geq}{=}} 0$

Proof:

Given eq. of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

centre $= (-g, -f), radius = \sqrt{g^2 + f^2 - c}$

Now point

$P(x_1, y_1)$ will lie outside on or inside the
circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$$if \ |CP| \underset{<}{\overset{\geq}{=}} r$$

$\Rightarrow \quad \sqrt{(x_1 - (-g))^2 + (y_1 - (-f))^2} \underset{<}{\overset{\geq}{=}} \sqrt{g^2 + f^2 - c}$

Squaring both sides

$\Rightarrow \quad (x_1 + g)^2 + (y_1 + f)^2 \underset{<}{\overset{\geq}{=}} g^2 + f^2 - c$

$\Rightarrow \quad x_1^2 + g^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c \underset{<}{\overset{\geq}{=}} 0$

$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \underset{<}{\overset{\geq}{=}} 0 \ \ hence \ prove$

Theorem:

The line $y = mx + c$ **intersects the circle** $x^2 + y^2 = a^2$ **in at the most two points.**

Proof:
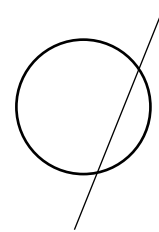
Given *equation of circle is*
$$x^2 + y^2 = a^2 \to (i)$$
also given equation of line is
$$y = mx + c \to (ii)$$

Put *value of y in* (i)

$\Rightarrow \quad x^2 + (mx + c)^2 = a^2$

$\Rightarrow \quad x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$

$\Rightarrow \quad \therefore (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0 \to (iii)$

this being quadratic in x, gives two value of x

Thus the line *intersects the circle* $x^2 + y^2 = a^2$

At the most two points.

Nature of points:

For nature of points, we examine the discriminant $b^2 - 4ac$ of eq. *The points ar*

(a) Real and distinct if
$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

(b) Real and coincident if
$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

(c) Imaginary if
$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) < 0$$

Condition of tangency

for condition of tangency;
$$b^2 - 4ac = 0$$

$\Rightarrow \quad (2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$

$\Rightarrow \quad 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2) = 0$

Or$\frac{m^2}{c^2} - c^2 + a^2 - m^2c^2 + m^2a^2 = 0$

$\Rightarrow \quad a^2(1 + m^2) = c^2$

$\qquad or\ c = \pm a\sqrt{1 + m^2}$

Required condition of tangency

$\qquad by\ (ii)\ y = mx \pm a\sqrt{1 + m^2}$

which is equation of tangent to circle

$x^2 + y^2 = a^2$

Theorem:

Two tangent can be drawn to a circle from any point

$P(x_1, y_1).$ *The tangents are real and distinct,* **Coincident**

or imaginary according as the point lies outside, on or inside, on or inside the circle.

Proof:

Consider *eq. of circle* $x^2 + y^2 = a^2 \rightarrow (i)$

$\qquad \because y = mx + a\sqrt{1 + m^2} \rightarrow (ii)$

Is*eq. of either tangent to the circle* $x^2 + y^2 = a^2$

let the tangent line passes through point $P(x_1, y_1)$

Then equation (ii)*becomes*

$$y_1 = mx_1 + a\sqrt{1 + m^2}$$

$$y_1 - mx_1 = a\sqrt{1 + m^2}$$

$$squring\ both\ sides$$

$$(y_1 - mx_1)^2 = \left(a\sqrt{1 + m^2}\right)^2$$

$\Rightarrow \quad y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2(1 + m^2)$

$\Rightarrow \quad x_1^2m^2 - a^2m^2 - 2mx_1y_1 + y_1^2 - a^2 = 0$

$\Rightarrow \quad (x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 - a^2 = 0 \rightarrow (iii)$

This being quadratic in m. it will give two valves of m.

Thus two tangent can be drawn to a circle from any point $P(x_1, y_1)$.

Nature of tangent:

For nature of tangent, we examine the discriminant of $eq(iii) i.e$

(a) Real and distinct if $b^2 - 4ac > 0$

(b) Real and coincident if $b^2 - 4ac = 0$

(c) Imaginary if $b^2 - 4ac < 0$

So $(-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) \gtrless 0$

$\Rightarrow \quad 4x_1^2y_1^2 - 4(x_1^2y_1^2 - a^2x_1^2 - a^2y_1^2 + a^4) \gtrless 0$

$\Rightarrow \quad x_1^2y_1^2 - x_1^2y_1^2 + a^2x_1^2 + a^2y_1^2 - a^4 \gtrless 0$

$\Rightarrow \quad a^2(x_1^2 + y_1^2) - a^4 \gtrless 0$

$\Rightarrow \quad x_1^2 + y_1^2 - a^2 \gtrless 0$

Which is condition for a point

$P(x_1, y_1)$*lying out side on or inside the circle*

$$x^2 + y^2 = a^2$$

Theorem:

Find length of tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c$$

$$= 0\ drawn\ from\ point$$

$P(x, y)$*lying outside the circle*.

Proof
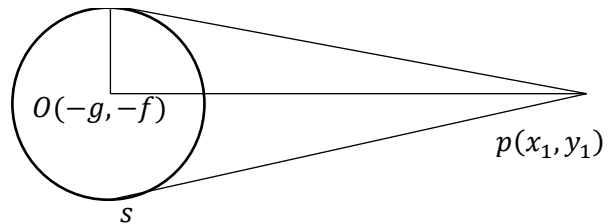
For $x^2 + y^2 + 2gx + 2fy + c = 0$

$$Centre = 0(-g, -f)$$

Radius $= |TO| = \sqrt{g^2 + f^2 - c}$

$\Rightarrow \quad |TO|^2 = g^2 + f^2 - c$

Let

PT *and* PS *be two tangent drwn from point* $P(x_1, y_1)$*to the given circle.*

from figure

$$in\ rt\ \Delta\ PTO$$

$\Rightarrow \quad |PO|^2 = |PT|^2 + |TO|^2$

$\Rightarrow \quad |PT|^2 = |PO|^2 - |TO|^2 \rightarrow (i)$

$\qquad \because |PC| = \sqrt{\left(x_1 - (-g)\right)^2 + \left(y_1 - (-f)\right)^2}$

$\Rightarrow \quad |PC|^2 = (x_1 + g)^2 + (y_1 + f)^2$

$\quad Now\ (i)$

$$|PT|^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$x_1^2 + g^2 + 2gx_1 + y_1^2 + f^2 + 2fy_1 - g^2 - f^2 + c$

$$|PT|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$|PT|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

which is required lenght of tangent.

Exercise 6.2

Question # 1. Write down equation of the tangent and normal to the circle.

i). $x^2 + y^2 = 25$ **at** $P(4, 3)$ **and at**

$(5\cos\theta, 5\sin\theta)$

Solution.

Circle: $x^2 + y^2 = 25$

$x^2 + y^2 - 25 = 0$

Comparing it with
$x^2 + y^2 + 2gx + 2fy + c = 0$
$\Rightarrow 2g = 0 \Rightarrow g = 0$ also $\Rightarrow 2f = 0 \Rightarrow f = 0$ and
$c = -25$
Since equation of tangent at $P(x_1, y_2)$ is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
At $P(x_1, y_2) = P(4, 3)$
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
$\Rightarrow x(4) + y(3) + 0(x + 4) + 0(y + 3) - 25 = 0$
$\Rightarrow 4x + 3y - 25 = 0$
$\Rightarrow 4x + 3y = 25$
The equation of Normal at $P(x_1, y_2)$ is
$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$
At $P(x_1, y_2) = P(4, 3)$
$(y - 3)(4 + g) = (x - 4)(3 + f)$
$\Rightarrow (y - 3)(4 + 0) = (x - 4)(3 + 0)$
$\Rightarrow (y - 3)4 = (x - 4)3$
$\Rightarrow 4y - 12 = 3x - 12$
$\Rightarrow 3x - 4y = 0$
Now, we take $P(x_1, y_2) = P(5cos\theta, 5cos\theta)$
Since equation of tangent at $P(x_1, y_2)$ is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
At $P(x_1, y_2) = P(5cos\theta, 5sin\theta)$
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
$\Rightarrow x(5cos\theta) + y(5sin\theta) + 0(x + 5cos\theta)$
$\qquad\qquad + 0(y + 5sin\theta) - 25 = 0$
$\Rightarrow 5xcos\theta + 5ysin\theta - 25 = 0$
$\Rightarrow xcos\theta + ysin\theta = 5$
The equation of Normal at $P(x_1, y_2)$ is
$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$
At $P(x_1, y_2) = (5cos\theta, 5sin\theta)$
$(y - 3)(4 + g) = (x - 4)(3 + f)$
$\Rightarrow (y - 5sin\theta)(5cos\theta + 0)$
$\qquad\qquad = (x - 5cos\theta)(5sin\theta + 0)$
$\Rightarrow (y - 5sin\theta)5cos\theta = (x - 5cos\theta)5sin\theta$
$\Rightarrow (y - 5sin\theta)cos\theta = (x - 5cos\theta)sin\theta$
$\Rightarrow ycos\theta - 5sin\theta cos\theta = xsin\theta - 5sin\theta cos\theta$
$\Rightarrow xsin\theta - ycos\theta = 0$

(ii). $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ at $P(1, \frac{10}{3})$
Solution. (ii)

Circle: $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ $x^2 + y^2 - 25 = 0$
$\Rightarrow x^2 + y^2 + \frac{5}{3}x - \frac{13}{3}y + \frac{2}{3} = 0$
Comparing it with
$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = \frac{5}{3} \Rightarrow g = \frac{5}{6}$ also $\Rightarrow 2f = \frac{-13}{3} \Rightarrow f = -\frac{13}{6}$
and $c = \frac{2}{3}$
Since equation of tangent at $P(x_1, y_2)$ is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
At $P(x_1, y_2) = P(1, \frac{10}{3})$
$x(1) + y\left(\frac{10}{3}\right) + g(x + 1) + f\left(y + \frac{10}{3}\right) + c = 0$
$x(1) + y\left(\frac{10}{3}\right) + \frac{5}{6}(x + 1) - \frac{13}{6}\left(y + \frac{10}{3}\right) + \frac{2}{3}$
$\qquad\qquad = 0$
Multiply by 18
$\Rightarrow 18x + 60y + 15(x + 1) - 13(3y + 10) - 12$
$\qquad\qquad = 0$
$\Rightarrow 18x + 60y + 15x + 15 - 39y - 130 - 12$
$\qquad\qquad = 0$
$\Rightarrow 33x + 21y - 103 = 0$
The equation of Normal at $P(x_1, y_2)$ is
$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$
At $P(x_1, y_2) = P(1, \frac{10}{3})$
$\left(y - \frac{10}{3}\right)\left(1 + \frac{5}{6}\right) = (x - 1)\left(\frac{10}{3} - \frac{13}{6}\right)$
$\Rightarrow \left(\frac{3y - 10}{3}\right)\left(\frac{6 + 5}{6}\right) = (x - 1)\left(\frac{20 - 13}{6}\right)$
$\Rightarrow (3y - 10)11 = (x - 1)21$
$\Rightarrow 33y - 110 = 21x - 21$
$\Rightarrow 21x - 33y + 89 = 0$

Q#2) Write down the equation of tangent and normal to the circle $4x^2 + 4y^2 - 16x + 24y - 117 = 0$ at the point on the circle whose abscissa is -4.
Sol:
Circle: $4x^2 + 4y^2 - 16x + 24y - 117 = 0$ ....(1)
At $x = -4$
$4(-4)^2 + 4y^2 - 16(-4) + 24y - 117 = 0$
$\Rightarrow 64 + 4y^2 + 64 + 24y - 117 = 0$
$\Rightarrow 4y^2 + 24y + 11 = 0$
$\Rightarrow 4y^2 + 22y + 2y + 11 = 0$
$\Rightarrow 2y(2y + 11) + 1(2y + 11) = 0$
$\Rightarrow (2y + 11)(2y + 1) = 0$
$\Rightarrow y = -\frac{1}{2}, -\frac{11}{2}$
So, points are $P_1\left(-4, -\frac{11}{2}\right), P_2\left(-4, -\frac{1}{2}\right)$
$take\ derivative\ of\ (i)\ w.r.t\ x$
$$8x + 8y\frac{dy}{dx} - 16 + 24\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx}(8y + 24) = -8x + 16$$
$$or\ \frac{dy}{dx} = \frac{-8x + 16}{8y + 24} = \frac{8(-x + 2)}{8(y + 3)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x+2}{y+3}$$

Slope of tangent $at\ P_1 = \frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = \frac{-(-4)+2}{-\frac{11}{2}+3}$

$$\frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = \frac{6}{\frac{-11+6}{2}}$$

$$or\ \frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = -\frac{12}{2}$$

Equation of tangent

$$\Rightarrow y + \frac{11}{2} = -\frac{12}{5}(x+4)$$

$$\Rightarrow 5y + \frac{55}{2} = -12x - 48$$

$$\Rightarrow 10y + \frac{55}{2} = -12x - 48$$

$$24x + 10y = -96 - 55$$

$$or\ 24x + 10y + 151 = 0$$

Slope of normal $= \frac{5}{12}$

Equation of normal

$$y + \frac{11}{2} = \frac{5}{12}(x+4)$$

$\Rightarrow\ 12y + 66 = 5x + 20$

$\Rightarrow\ 5x - 12y - 46 = 0$

$\Rightarrow\ $ Now slope of tangent at P_2

$$= \frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{-(-4)+2}{-\frac{1}{2}+3}$$

$$\frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{6}{\frac{-1+6}{2}}$$

$$or\ \frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{12}{5}$$

Equation of tangent

$$\Rightarrow y + \frac{1}{2} = \frac{12}{5}(x+4)$$

$$\Rightarrow 5y + \frac{5}{2} = 12x + 48$$

$$\Rightarrow 5y + \frac{5}{2} = 12x + 48$$

$$\Rightarrow 10y + 5 = 24x + 96$$

$$or\ 24x - 10y = 5 - 96$$

$$24x - 10y = -91$$

$$or\ 24x - 10y + 91 = 0$$

$slope\ of\ normal = -\frac{5}{12}$

$equation\ of\ normal$

$$y + \frac{1}{2} = -\frac{5}{12}(x+4)$$

$\Rightarrow\ 12y + 6 = -5x - 20$

$\Rightarrow\ 5x + !2y = -26$

$or\ 5x + 12y + 26 = 0$

Q#3) Check the position of the point $(5,6)$ with respect to the circle

(i). $x^2 + y^2 = 81$

Sol: (i).

$x^2 + y^2 = 81$

$\Rightarrow x^2 + y^2 - 81 = 0$

At $(5,6)$

$L.H.S = (5)^2 + (6)^2 - 81$

$\qquad = 25 + 36 - 81$

$\qquad = -20 < 0$

Thus, the point $(5,6)$ lies inside the circle.

(ii). $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

Sol: (ii).

$2x^2 + 2y^2 + 12x - 8y + 1 = 0$

At $(5,6)$

$L.H.S = 2(5)^2 + 2(6)^2 + 12(5) - 8(6)$

$\qquad\qquad + 1$

$\qquad\qquad = 50 + 72 + 60 - 48 + 1$

$\qquad = 135 > 0$

Thus, the point $(5,6)$ lies outside the circle.

Q#4) Find the length of the tangent drawn from the point $(-5,4)$ to the circle $5x^2 + 5y^2 - 10x + 15y - 131 = 0$

Sol:

Circle: $5x^2 + 5y^2 - 10x + 15y - 131 = 0$

$\Rightarrow x^2 + y^2 - 2x + 3y - \frac{131}{5} = 0$

At point $P(-5,4)$, we calculate length of tangent from P

length of tangent from P=

$$\sqrt{(-5)^2 + (4)^2 - 2(-5) + 3(4) - \frac{131}{5}}$$

$$= \sqrt{25 + 16 + 10 + 12 - \frac{131}{5}}$$

$$= \sqrt{63 - \frac{131}{5}} = \sqrt{\frac{315 - 131}{5}} = \sqrt{\frac{184}{5}}$$

Q#5) Find the length of the chord cut off from the line $2x + 3y = 13$ by the circle $x^2 + y^2 = 26$.

Sol:

Circle: $x^2 + y^2 = 26$....(1)

And $2x + 3y = 13 \Rightarrow 2x = 13 - 3y$

$\Rightarrow x = \frac{13-3y}{2}$ put in Eq(1)

$x^2 + y^2 = 26 \Rightarrow \left(\frac{13-3y}{2}\right)^2 + y^2 = 26$

$$\Rightarrow \frac{169 + 9y^2 - 78y}{4} + y^2 = 26$$

$$\Rightarrow 169 + 9y^2 - 78y + 4y^2 = 104$$

$\Rightarrow 13y^2 - 78y + 65 = 0$

$\Rightarrow y^2 - 6y + 5 = 0$

$\Rightarrow y^2 - 5y - y + 5 = 0$

$\Rightarrow y(y - 5) - 1(y - 5) = 0$

$\Rightarrow (y - 5)(y - 1) = 0$

$\Rightarrow y = 5, 1$ put in given st. line

$x = \frac{13-3y}{2} \Rightarrow x = \frac{13-3(5)}{2} = \frac{13-15}{2} = -\frac{2}{2} = -1$

and $x = \frac{13-3y}{2} \Rightarrow x = \frac{13-3(1)}{2} = \frac{13-3}{2} = -\frac{10}{2} = -5$

Hence, the points are $A(-1, 5)$ and $B(5, 1)$

Length of chord $\overline{AB} = \sqrt{(-1 - 5)^2 + (5 - 1)^2}$
$= \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

Q#6) Find the coordinates of the points of intersection of the line $x + 2y = 6$ with the circle $x^2 + y^2 - 2x - 2y - 39 = 0$

Sol:

Circle: $x^2 + y^2 - 2x - 2y - 39 = 0$...(1)

And $x + 2y = 6 \Rightarrow x = 6 - 2y$

put in Eq(1)

$x^2 + y^2 - 2x - 2y - 39 = 0$

$\Rightarrow (6 - 2y)^2 + y^2 - 2(6 - 2y) - 2y - 39 = 0$

$\Rightarrow 36 + 4y^2 - 24y + y^2 - 12 + 4y - 2y - 39 = 0$

$\Rightarrow 5y^2 - 22y - 15 = 0$

$\Rightarrow 5y^2 - 25y + 3y - 15 = 0$

$\Rightarrow 5y(y - 5) + 3(y - 5) = 0$

$\Rightarrow (y - 5)(5y + 3) = 0$

$\Rightarrow y = 5, -\frac{3}{5}$ put in given st. line

$x = 6 - 2y \Rightarrow x = 6 - 2(5) = 6 - 10 = -4$

and $x = 6 - 2y \Rightarrow x = 6 - 2\left(-\frac{3}{5}\right) = \frac{30+6}{5} = \frac{36}{5}$

Thus, the points of intersection $P_1(-4, 5)$ and $P_2\left(\frac{36}{5}, -\frac{3}{5}\right)$.

Q#7) Find equation of the tangents to the circle $x^2 + y^2 = 2$

(i). Parallel to the line $x - 2y + 1 = 0$

Sol: (i)

Circle: $x^2 + y^2 = 2$

$\Rightarrow x^2 + y^2 = \left(\sqrt{2}\right)^2$

Comparing it with $x^2 + y^2 = r^2$

$\Rightarrow r = \sqrt{2}$

Given the line $l_1: x - 2y + 1 = 0$

Slope $= m_1 = -\frac{1}{-2} = \frac{1}{2}$

Which is slope of tangent parallel to l_1.

Let required equation of the tangent is

$y = m_1 x \pm r \sqrt{1 + m_1^2}$

$= \left(\frac{1}{2}\right)x \pm \sqrt{2}\sqrt{1 + \left(\frac{1}{2}\right)^2}$

$= \frac{x}{2} \pm \sqrt{2}\sqrt{1 + \frac{1}{4}} = \frac{x}{2} \pm \sqrt{2\left(\frac{5}{4}\right)}$

$y = \frac{x}{2} \pm \sqrt{\frac{4}{2}}$ Multiply by 2

$\Rightarrow 2y = x \pm \sqrt{2}\sqrt{5}$

$\Rightarrow 2y = x \pm \sqrt{10}$

Hence $x - 2y + \sqrt{10} = 0$ and $x - 2y - \sqrt{10}$ are req.

Tangent.

(ii).perpendicular to the line $3x + 2y = 6$

Sol: (ii)

Circle: $x^2 + y^2 = 2$

$\Rightarrow x^2 + y^2 = \left(\sqrt{2}\right)^2$

Comparing it with $x^2 + y^2 = r^2$

$\Rightarrow r = \sqrt{2}$

Given the line $l_2: 3x + 2y - 6 = 0$

Slope $= m_2 = -\frac{3}{2} = \frac{-3}{2}$

Which is slope of tangent parallel to $l_1 = \frac{2}{3}$.

Let required equation of the tangent is

$y = m_2 x \pm r \sqrt{1 + m_1^2}$

$= \left(\frac{2}{3}\right)x \pm \sqrt{2}\sqrt{1 + \left(\frac{2}{3}\right)^2}$

$= \frac{2x}{3} \pm \sqrt{2}\sqrt{1 + \frac{4}{9}} = \frac{x}{2} \pm \sqrt{2\left(\frac{13}{9}\right)}$

$y = \frac{2x}{3} \pm \frac{\sqrt{26}}{3}$ Multiply by 3

$\Rightarrow 3y = 2x \pm \sqrt{26}$

$\Rightarrow 2x - 3y \pm \sqrt{26} = 0$

Hence $2x - 3y - \sqrt{26} = 0$ and $2x - 3y + \sqrt{26} = 0$

Are required tangent.

Question No.8 find equation of tangent drawn from

(i) $(0, 5)$ to $x^2 + y^2 = 16$

Also find points of contact.

Solution:

Eq. of circle is $x^2 + y^2 = 16$

$\Rightarrow x^2 + y^2 = (4)^2$

$a = 4$

Eq. of tangent is

$$y = mx + a\sqrt{1+m^2} \to (i)$$

At $(0,5)$ and $a = 4$

$$5 = m(0) + 4\sqrt{1+m^2}$$
$$5 = 4\sqrt{1+m^2}$$

⇨ $25 = 16(1 + m^2)$

⇨ $25 = 16 + 16m^2$

⇨ $16m^2 = 9$

⇨ Or $n^2 = \dfrac{9}{16}$

⇨ $m = \pm\dfrac{3}{4}$

$$Now\ (i) for\ m = \pm\dfrac{3}{4}\ and\ a = 4$$

$$y = \pm\dfrac{3}{4}x + 4\sqrt{1 + \dfrac{9}{16}}$$

$$y = \pm\dfrac{3}{4} + 4\sqrt{\dfrac{25}{16}}$$

$$y = \pm\dfrac{3}{4}x + 4\left(\dfrac{5}{4}\right)$$

Or $y = \pm\dfrac{3}{4}x + 5$

⇨ $4y = \pm3x + 20$

Or $\pm3x - 4y + 20 = 0$

Hence $3x - 4y + 20$ and $-3x - 4y + 20 = 0$ or

  $3x + 4y - 20 = 0$ are required tangent.

Points of contact:

Remember to find points of contact, solve of tangent and circle:

We solve $3x - 4y + 20 = 0$

And $x^2 + y^2 = 16$ for points of contact.

Now $x^2 + y^2 = 16 \to (ii)$

$$3x - 4y + 20 = 0$$

⇨ $x = \dfrac{4y-20}{3}$ put in $)ii)$

⇨ $\left(\dfrac{4y-20}{3}\right)^2 + y^2 = 16$

⇨ $\dfrac{16y^2+400-160y}{9} + y^2 = 16$

⇨ $16y^2 + 400 - 160y + 9y62 = 144$

⇨ $25y^2 - 8y - 80y + 256 = 0$

⇨ $5y(5y - 16) - 16(5y - 16) = 0$

⇨ $(5y - 16)(5y - 16) = 0$

⇨ $(5y - 16)^2 = 0$

⇨ $5y - 16 = 0$

⇨ $5y = 16$

⇨ $y = \dfrac{16}{5}$

Now $x = \dfrac{4\left(\frac{16}{5}\right)-20}{3}$

$$= \dfrac{64 - 100}{15} = -\dfrac{36}{15}$$

$$x = -\dfrac{12}{5}$$

So point of contact is $\left(-\dfrac{12}{5}, \dfrac{16}{5}\right)$

Now we slove $3x + 4y - 20 = 0$

And $x^2 + y^2 = 16$

$$3x + 4y - 20 = 0$$

⇨ $x = \dfrac{20-4y}{3}n$ put in (ii)

$$\left(\dfrac{20-4y}{3}\right)^2 + y^2 = 16$$

$$\dfrac{400 + 16y^2 - 160y}{9} + y^2 = 16$$

⇨ $400 + 16y^2 - 160y + 9y^2 = 144$

⇨ $25y^2 - 160y + 256 = 0$

⇨ $(5y - 16)^2 = 0$

⇨ $5y - 16 = 0$

⇨ $y = \dfrac{16}{5}$

So $x = \dfrac{20-4\left(\frac{16}{5}\right)}{3} = \dfrac{20-\frac{64}{5}}{3}$

$$x = \dfrac{100 - 64}{15} = \dfrac{36}{15} = \dfrac{12}{5}$$

Also $\left(\dfrac{12}{5}, \dfrac{16}{5}\right)$ is point of contact.

(ii)

$$(-1, 2)\ to\ x^2 + y^2 + 4x + 2y = 0$$

Also find points of contact.

Solution:

$$x^2 + y^2 + 4x + 2y = 0 \to (i)$$

Compare with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$2g = 4$$
$$g = 2$$
$$2f = 2 \Rightarrow f = 1, c = 0$$
$$r = a = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (1)^2 - 0}$$
$$\Rightarrow r = a = \sqrt{5}$$

Centre $(-g, -f) = (-2, -1)$

Let

m be slope of tangent drawn form point $(-1,2$

    to given circle so eq. of tangent os

$$y - y_1 = m(x - x_1)$$

⇨ $y - 2 = m(x + 1)$

⇨ $y - 2 = mx + m$

⇨ $mx - y + m + 2 = 0 \to (i)$

Distance of line from

Centre $(-2, -1) = r$

⇨ $\dfrac{|m(-2)-(-1)+m+2|}{\sqrt{m^2+1}} = \sqrt{5}$

⇨ $\dfrac{|-m+3|}{\sqrt{m^2+1}} = \sqrt{5}$

⇨ $\dfrac{m^2-6m+9}{m^2+1} = 5$ $(squaring)$

⇨ $5m^2 + 5 = m^2 - 6m + 9$

$\Rightarrow 5m^2 + 5 - m^2 + 6m - 9 = 0$

Or $4m^2 + 6m - 4 = 0$

$$4m^2 + 6m - 4 = 0$$
$$4m^2 + 8m - 2m - 4 = 0$$
$$4m(m + 2) - 2(m + 2) = 0$$
$$(m + 2)(4m - 2) = 0$$
$$m + 2 = 0 \quad or \quad m = \frac{2}{4} = \frac{1}{2}$$

$\Rightarrow m = -2 \ or \ m = \frac{1}{2}$

When $m = -2$ so (i)

$$-2x - y - 2 + 2 = 0$$

$\Rightarrow -2x - y = 0$

$\Rightarrow 2x + y = 0$

When $m = \frac{1}{2}$ so (i)

$$\frac{1}{2}x - y + \frac{1}{2} + 2 = 0$$

$\Rightarrow x - 2y + 1 + 4 = 0$

Or $x - 2y + 5 = 0$

Hence req. tangent are

$$2x + y = 0 \ and \ x - 2y + 5 = 0$$

Points of contact:

$for \ 2x + y = 0 \Rightarrow y = -2x \ put \ in \ (i)$

$$x^2 + (-2x)^2 + 4x + 2(-2x) = 0$$
$$x^2 + 4x^2 + 4x - 4 = 0$$

$\Rightarrow 5x^2 = 0 \Rightarrow x = 0 \ so \ y = -(0) \Rightarrow y = 0$

Thus $(0,0) is \ the \ point \ of \ contact.$

For $x - 2y + 5 = 0 \quad \Rightarrow x = 2y - 5 \ put \ in \ (i)$

$$(2y - 5)^2 + y^2 + 4(2y - 5) + 2y = 0$$
$$4y^2 - 20y + 25 + y^2 + 8y - 20 + 2y = 0$$
$$5y^2 - 10y + 5 = 0$$

$\Rightarrow y^2 - 2y + 1 = 0 \Rightarrow (y - 1)^2 = 0$

$$y - 1 = 0$$

$\Rightarrow y - 1 = 0 \Rightarrow y = 1$

So $x = 2(1) - 5 \Rightarrow x = -3$

Thus $(-3,1) is \ point \ of \ contact.$

(iii) $(-7, -2) to \ (x + 1)^2 + (y - 2)^2 =$
$26 \quad also$

Find points of contact.

Solution:

$$(x + 1)^2 + (y - 2)^2 = 26 \to (i)$$

$\Rightarrow \left(x - (-1)\right)^2 + (y - 2)^2 = \left(\sqrt{26}\right)^2$

Compare with

$$(x - h)^2 + (y - k)^2 = r^2$$
$$centre \ (h, k) = (-1,2)$$
$$r = \sqrt{26}$$

Suppose $eq. of \ tangent$

$$y - y_1 = m(x - x_1)$$

$\Rightarrow y + 2 = m(x + 7) at \ (-7, -2)$

$$y + 2 = mx + 7m$$

Or $mx - y + 7m - 2 = 0 \to (ii)$

Distance of tangent from $(-1,2) = r$

$\Rightarrow \frac{|m(-1) - 2 + 7m - 2|}{\sqrt{m^2 + (-1)^2}} = \sqrt{26}$

$\Rightarrow \frac{|6m - 4|}{\sqrt{m^2 + 1}} = \sqrt{26}$

$\Rightarrow \frac{26m^2 + 16 - 48m}{m^2 + 1} = 26 \qquad$ (squaring)

$\Rightarrow 36m^2 + 16 - 48m = 26m^2 + 26$

$\Rightarrow 36m^2 - 26m^2 - 48m + 16 - 26 = 0$

Or $10m^2 - 48m - 10 = 0$

$$10m^2 - 50m + 2m - 10 = 0$$
$$10m(m - 5) + 2(m - 5) = 0$$
$$(10m + 2)(m - 5) = 0$$
$$m - 5 = 0 \quad 10m + 2 = 0$$
$$m = 5 \quad ; 10m = -2$$
$$\Rightarrow m = -\frac{2}{10} = -\frac{1}{5}$$

$when$ m=5   so (ii)

$$5x - y + 7(5) - 2 = 0$$

$\Rightarrow 5x - y + 35 - 2 = 0$

$\Rightarrow 5x - y + 33 = 0$

When $m = -\frac{1}{5}$ so (ii)

$$-\frac{1}{5}x - y + 7\left(-\frac{1}{5}\right) - 2 = 0$$
$$-\frac{1}{5}x - y - \frac{7}{5} - 2 = 0$$
$$\Rightarrow -x - 5y - 7 - 10 = 0$$
$$\Rightarrow -x - 5y - 17 = 0$$
$$or \ x + 5y + 17 = 0$$

$hence \ 5x - y + 33 = 0 \ and \ x + 5y + 17 = 0$

are required tangents.

Points of contact:

For $x + 5y + 17 = 0$

$$\Rightarrow x = -5y - 17 \ put \ in \ (i)$$
$$(-5y - 17 + 1)^2 + (y - 2)^2 = 26$$
$$\Rightarrow (-5y - 16y)^2 + (y - 2)^2 = 26$$
$$25y^2 + 160y + 256 + y^2 + 4 - 4y - 26 = 0$$
$$26y^2 + 156y + 234 = 0$$

$\Rightarrow y^2 + 6y + 9 = 0 \quad (\div \ by \ 26)$

Or $(y + 3)^2 = 0$

$\Rightarrow y + 3 = 0 \Rightarrow y = -33$

So $x = -5(-3) - 17$

$$x = 15 - 17 = -2$$

So $(-2, -3)$ is point of contact

For $5x - y + 33 = 0$

$\Rightarrow y = 5x + 33 \ put \ in \ (i)$

$$(x + 1)^2 + (5x + 33 - 2)^2 = 26$$
$$x^2 + 1 + 2x + (5x + 31)^2 = 26$$
$$x^2 + 1 + 2x + 25x^2 + 310x + 961 - 26 = 0$$
$$26x^2 + 312x + 936 = 0$$

$\Rightarrow x^2 + 12x + 36 = 0 \quad (\div \ by \ 26)$

Or $(x + 6)^2 = 0 \Rightarrow x + 6 = 0$

$$x = -6 \ \ so \ y = 5(-6) + 33$$
$$\Rightarrow y = -30 + 33 = 3$$
$$so \ (-6,3) is \ point \ of \ contact.$$

Q#9) Find an equation of the chord of contact of the tangent drawn from $(4, 5)$ to the circle $2x^2 + 2y^2 - 8x + 12y + 21 = 0$.

Sol:

Circle: $2x^2 + 2y^2 - 8x + 12y + 21 = 0$

$\Rightarrow x^2 + y^2 - 4x + 6y + \frac{21}{2} = 0.$

Comparing it with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -4 \Rightarrow g = -2$ also $\Rightarrow 2f = 6 \Rightarrow f = 3$

and $c = \frac{21}{2}$

Let $A(x_1, y_1) \ and \ B(x_2, y_2)$ be the points of contact of two tangents. Equation of the tangent at $A(x_1, y_1)$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow xx_1 + yy_1 - 2(x + x_1) + 3(y + y_1) + \frac{21}{2}$
$$= 0$$

Since it passes through $(4, 5)$

$\Rightarrow 4x_1 + 5y_1 - 2(4 + x_1) + 3(5 + y_1) + \frac{21}{2} = 0$

$\Rightarrow 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$

$\Rightarrow 2x_1 + 8y_1 + 7 + \frac{21}{2} = 0$

$\Rightarrow 4x_1 + 16y_1 + 14 + 21 = 0$

$\Rightarrow 4x_1 + 16y_1 + 14 + 21 = 0$

$\Rightarrow 4x_1 + 16y_1 + 35 = 0 \rightarrow 1)$

In similar fashion, we equation of the tangent at $B(x_2, y_2)$

$\Rightarrow 4x_2 + 16y_2 + 35 = 0 \rightarrow (2)$

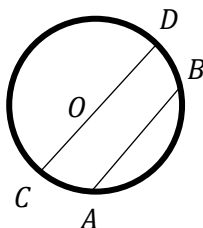Equation (1) and (2) represent that $A(x_1, y_1) \ and \ B(x_2, y_2)$ lie on

$\Rightarrow 4x + 16y + 35 = 0$

Which is the required equation of line.

Analytic Proofs of Important Properties of a circle:

"Chord of a Circle:

A line segment where end points lie on a circle is called chord of a circle. In figure AB is chord.

Diameter of a circle:

A chord passing through Centre of a circle. In figure , if O is Centre the CD is diameter of circle.

Theorem 1

Length of a diameter of the circle $x^2 + y^2 = a^2 \ is \ 2a$

Proof:

Given equation of circle is $x^2 + y^2 = a^2$

Centre $= O(0,0)$

Radius $= r = a$

Let

$A(a, 0) and \ B(-a, o) be \ ends \ of \ diameter \ AB \ then$

Length of diameter

$$= |AB| = \sqrt{(a - (-a)^2 + (0 - 0)^2}$$
$$|AB| = \sqrt{(2a)^2}$$
$$= 2a$$

Hence Proved.

Theorem:

Perpendicular dropped from the Centre of a circle on a chord bisects the chord.

Proof:

Consider $eq. of \ circle \ \ x^2 + y^2 = a^2 \rightarrow (i)$

Let

$A(x_1, y_1) and \ B(x_2, y_2) be \ ends \ of \ chord \ AB.$

$\because A(x_1, y_1) and \ B(x_2, y_2) lie \ on \ the \ circle \ (i) so$
$$x_1^2 + y_1^2 = a^2 \rightarrow (ii) and$$
$$x_2^2 + y_2^2 = a^2 \rightarrow (iii)$$

Let we draw a $\perp$

$ar \ OC \ from \ centre \ O \ on \ the \ chord \ AB$

$$Slope \ of \ chord \ AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope \ of \ OC = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$
$$\because \overline{OC} \perp \overline{AB}$$

$\because line \ passing \ through \ (x_1, y_1) and \ having \ slope \ m$

Eq. of OC is

$$y - 0 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)(x - 0)$$
$$\Rightarrow \ y(y_2 - y_1) = -x(x_2 - x_1)$$
$$or \ (x_2 - x_1)x + (y_2 - y_1)y = 0 \rightarrow (iv)$$

Midpoint of chord AB is

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ Now OC will bisect the chord AB

$if \ \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) satisfies. its \ equation$

$i.e; put \ x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2} \ in \ (iv)$

We have

$$(x_2 - x_1)\left(\frac{x_2 + x_1}{2}\right) + (y_2 - y_1)\left(\frac{y_1 + y_2}{2}\right) = 0$$

⇒ $(x_2 - x_1)(x_2 + x_1) + (y_2 - y_1)(y_2 + y_1) = 0$

⇒ $x_2^2 - x_1^2 + y_2^2 - y_1^2 = 0$

⇒ $x_2^2 + y_2^2 = x_1^2 + y_1^2$

⇒ $a^2 = a^2$ by (ii) and (iii) Hence ⊥

   ar dropped

from centre of a circle on a chord bisect the chord

Theorem3.

The perpendicular bisector of any chord of a circle passes through the Centre of the circle.

Proof:

Consider equation of circle is

$$x^2 + y^2 = a^2 \rightarrow (i)$$

Let

$A(x_1, y_1)$ and $B(x_2, y_2)$ be ends of chord AB.

∵ the points A and B lie on the circle (i) then

$$x_1^2 + y_1^2 = a^2 \rightarrow (ii)$$ and

$$x_2^2 + y_2^2 = a^2 \rightarrow (iii)$$

let C be midpoint of chord AB so

Coordinates of C are $C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Slope of chord AB=$\frac{y_2-y_1}{x_2-x_1}$
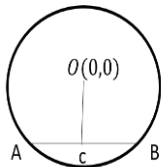
Slope of ⊥ ar bisector $AB = -\left(\frac{x_2-x_1}{y_2-y_1}\right)$

Using $y - y_1 = m(x - x_1)$

⇒ $y - \left(\frac{y_1+y_2}{2}\right) = -\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(x - \frac{x_1+x_2}{2}\right) \rightarrow (iv)$

This perpendicular bisector will pass through

Centre of (0,0) satisfies eq. (iv) for this

$$put\ x = 0, y = o\ in\ (iv)$$

$0 - \left(\frac{y_1+y_2}{2}\right) = -\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(0 - \frac{x_1+x_2}{2}\right)$

⇒ $-\left(\frac{y_1+y_2}{2}\right) = \left(\frac{x_2-x_1}{y_2-y_1}\right)\left(\frac{x_1+x_2}{2}\right)$

⇒ $-(y_2 + y_1)(y_2 - y_1) = (x_2 - x_1)(x_2 + x_1)$

⇒ $-(y_2^2 - y_1^2) = x_2^2 - x_1^2$

⇒ $x_1^2 + y_1^2 = x_2^2 + y_2^2$

$$a^2 = a^2\ \ (by\ (ii) and\ (iii)$$

thus

⊥ ar bisector of any chord of a circle passes

   Through Centre of the circle.

   Theorem:4

   The line joining the Centre of a circle to the mid-point of a chord is perpendicular to the chord.

   Proof:

   Consider eq. of circle.

$$x^2 + y^2 = a^2 \rightarrow (i)$$

 Let

$A(x_1, y_1)$ and $B(x_2, y_2)$ be ends of chord AB.

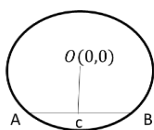∵ the points A and B lie on the circle (i) then

$$x_1^2 + y_1^2 = a^2 \rightarrow (ii)$$ and

$$x_2^2 + y_2^2 = a^2 \rightarrow (iii)$$

let C be midpoint of chord AB so

Coordinates of C are $C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Slope of chord AB=$\frac{y_2-y_1}{x_2-x_1}$

Here OC is line joining Centre O of circle to midpoint C of chord AB then

Slope of OC=$\frac{\frac{y_1+y_2}{2}-o}{\frac{x_1+x_2}{2}-0} = \frac{y_1+y_2}{x_1+x_2}$

$(slope\ of\ OC)(slope\ of\ chord\ AB)$

$$= \left(\frac{y_1+y_2}{x_1+x_2}\right)\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

$$= \frac{(y_2+y_1)(y_2-y_1)}{(x_2+x_1)(x_2-x_1)}$$

$$= \frac{y_2^2 - y_1^2}{x_2^2 - x_1^2}$$

By $(iii) - (ii)$

$$x_2^2 - x_1^2 + y_2^2 - y_1^2 = a^2 - a^2$$

⇒ $y_2^2 - y_1^2 = -(x_2^2 - x_1^2)$

 Now

$(slope\ of\ OC)(slope\ of\ chord\ AB)$

$$= -\frac{(x_2^2 - x_1^2)}{x_2^2 - x_1^2} = -1$$

⇒ $OC \perp AB$

   Hence line joining the Centre of a circle to the midpoint of chord is perpendicular to chord.

Congruent Chords:

"two chords are congruent if they are equal in length."

Theorem:5

Congruent chords of a circle are equidistance from the Centre.

Proof:

Consider eq. of circle

Consider eq. of circle $x^2 + y^2 = a^2 \rightarrow (i)$

Let AB and CD be two congruent chords.

Let coordinates of the points be

$A(x_1, y_1), B(x_2, y_2),$

$C(x_3, y_3)$ and D. Sine these points lie on the circle (i) so

$$x_1^2 + y_1^2 = a^2 \rightarrow (ii)\ \ x_2^2 + y_2^2 = a^2 \rightarrow (iii)$$

$$x_3^2 + y_3^2 = a^2 \rightarrow (iv)\ \ x_4^2 + y_4^2 = a^2 \rightarrow (v)$$

For *congruent chords*

So $|AB| = |CD|$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$$

Squaring both sides

$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2$

$x_2^2 + x_1^2 - 2x_1x_2 + y_2^2 + y_1^2 - 2y_1y_2$
$= x_4^2 + x_3^2 - 2x_3^2 - 2x_3x_4 + y_4^2 + y_3^2 - 2y_3y_4$

from $(ii), (iii), (iv), and (v)$

$\Rightarrow a^2 + a^2 - 2(x_1x_2 + y_1y_2) = a^2 + a^2 - 2(x_3x_4 + y_3y_4)$

$\Rightarrow -2(x_1x_2 + y_1y_2) = -2(x_3x_4 + y_3y_4)$

$\quad or \; x_1x_2 + y_1y_2 = x_3x_4 + y_3y_4 \rightarrow (vi)$

Let us draw $\perp$

ars OM and ON from centre O on the chords AB and CD resp. Now by property of circle, M and N will be midpoints of chords AB And CD respectively Now

Coordinates of M are $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Also coordinates of N are $N\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

We are to prove that
$$|OM| = |ON|$$
$\Rightarrow |OM|^2 = |ON|^2$

$$|OM|^2 = \left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{y_1+y_2}{2}\right)^2$$

$$= \frac{x_1^2 + x_2^2 + 2x_1x_2}{4} + \frac{y_1^2 + y_2^2 + 2y_1y_2}{4}$$

$$= \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)}{4}$$

$$= \frac{a^2 + a^2 + 2(x_1x_2 + y_1y_2)}{4} \; by \; (ii) and \; (iii)$$

$$|OM|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

Also

$$|0N|^2 = \left(\frac{x_3+x_4}{2}\right)^2 + \left(\frac{y_3+y_4}{2}\right)^2$$

$$= \frac{x_3^2 + x_4^2 + 2x_3x_4}{4} + \frac{y_3^2 + y_4^2 + 2y_3y_4}{4}$$

$$= \frac{x_3^2 + x_4^2 + y_3^2 + y_4^2 + 2(x_3x_4 + y_3y_4)}{4}$$

$$= \frac{a^2 + a^2 + 2(x_3x_4 + y_3y_4)}{4}$$

$$|ON|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

By $eq \; (vi)$

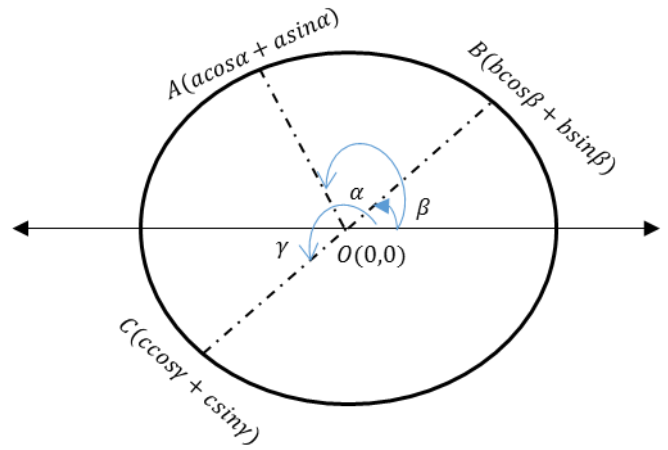$$|ON|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

Hence we conclude that $|OM|^2 = |ON|^2$
Thus congruent chords of a circle are equidistant from Centre.

Remember from fig.
Ab is minor arc ACB is major are O(o,o) is Centre of circle OA,OB,OC are radii,
$x^2 + y^2 = a^2$ and $|OA| = |OB| = |OC| = a$

From fig $|OA| = |OB| = |OC| = a$
Let $m\angle XOA = \alpha, m\angle XOB = B, m\angle XOC = r$
Coordinates of points are $A(a\cos\alpha, a\sin\alpha)$,
$\quad B(a\cos\beta, a\sin\beta)$ and $C(a\cos\gamma, a\sin\gamma)$

Theorem 6:
Show that the measure of the central angle of a minor arc is double the measure of the angle in the corresponding major arc.
Proof:
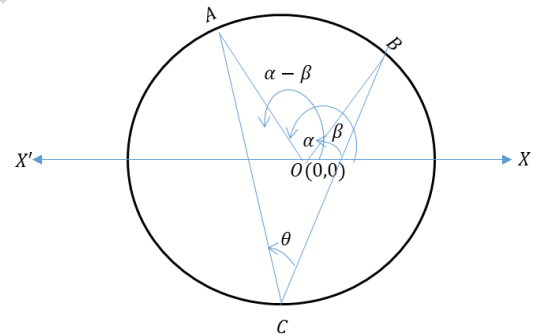
Consider eq. of circle
$$x^2 + y^2 = a^2 \rightarrow (i)$$
$$where \; a \; is \; radius$$
Let AB be minor arc and ACB be major arc. Let $m\angle XOA = \alpha, m\angle XOB = \beta, and \; m\angle XOC = \gamma$
Then $m\angle BOA = \alpha - \beta$ (angle subtended by minor arc AB)
Let $m\angle BCA = \theta =$ (angle subtended by major arc ACB)
We are to prove that $\alpha - \beta = 2\theta$ now
$$A(a\cos\alpha, a\sin\alpha)$$
$$B(a\cos\beta, a\sin\beta)$$
$$C(a\cos\gamma, a\sin\gamma)$$
Let $m_1 = slope \; of \; AC$
$m_2 = slope \; of \; BC$
$$\because m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = slope\ of\ AC = \frac{a\sin\gamma - a\sin\alpha}{a\cos\gamma - a\cos\alpha}$$

$$m_1 = \frac{a(\sin\gamma - \sin\alpha)}{a(\cos\gamma - \cos\alpha)}$$

$$m_1 = \frac{2\cos\frac{\gamma + \alpha}{2}\sin\frac{\gamma - \alpha}{2}}{-2\sin\frac{\gamma + \alpha}{2}\sin\frac{\gamma - \alpha}{2}}$$

$$m_1 = -\cot\left(\frac{\gamma + \alpha}{2}\right)$$

$$\because \tan\left(\frac{\pi}{2} + \theta\right) = -cot\theta$$

$$m_1 = Tan\left(\frac{\pi}{2} + \frac{\gamma + \alpha}{2}\right)$$

$$m_1 = Tan\left(\frac{\pi + \gamma + \alpha}{2}\right)$$

Similarly,

$$m_2 = slope\ of\ BC = Tan\left(\frac{\pi + \gamma + \beta}{2}\right)$$

$$\because \theta\ is\ angle\ from\ BC\ to\ AC\ so,$$

$$Tan\theta = \frac{Tan\left(\frac{\pi + \gamma + \alpha}{2}\right) - Tan\left(\frac{\pi + \gamma + \beta}{2}\right)}{1 + Tan\left(\frac{\pi + \gamma + \alpha}{2}\right)Tan\left(\frac{\pi + \gamma + \beta}{2}\right)}$$

$$\because \frac{Tan\alpha - \tan\beta}{1 + tan\alpha Tan\beta} = Tan(\alpha - \beta)$$

$$\Rightarrow\ Tan\theta = Tan\left(\frac{\pi + \gamma + \alpha}{2} - \frac{\pi + \gamma + \beta}{2}\right)$$

$$\Rightarrow\ Tan\left(\frac{\pi + \gamma + \alpha - \pi - \gamma - \beta}{2}\right)$$

$$\Rightarrow\ Tan\theta = Tan\left(\frac{\alpha - \beta}{2}\right)$$
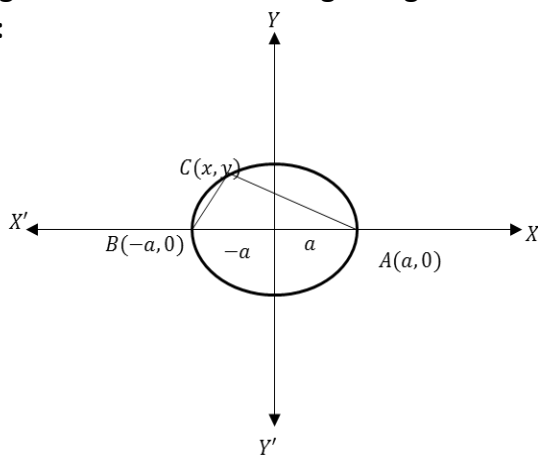
$$Or \theta = \frac{\alpha - \beta}{2}$$

$$\Rightarrow\ \alpha - \beta = 2\theta$$

Hence prove.

Theorem 7.

An angle in a semi-circle is a right angle.

Proof:

Consider eq. of circle $x^2 + y^2 = a^2$
with centre $O(0,0)$ and radius a. let $A(a,0)$ and
$B(-a,0)$ be ends of its diameter. let $C(x,y)$ be any
Point on the circle. We are to prove that
$$AC \perp BC$$

i.e;
$$(slope\ of\ AC)(slope\ of BC) = -1$$
Now
$$slope\ of\ AC = \frac{y - o}{x - a} = \frac{y}{x - a}$$

$$slope\ of\ BC = \frac{y - 0}{x - (-a)} = \frac{y}{x + a}$$

So
$$(slope\ of\ AC)(Slope\ of\ BC) = \left(\frac{y}{x - a}\right)\left(\frac{y}{x + a}\right)$$

$$= \frac{y^2}{x^2 - a^2}$$

$$= -\frac{y^2}{y^2} = -1$$

$$hence\ proved$$

Theorem 8:

The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Proof:

Consider eq. of circle
$x^2 + y2 = a^2 \to (i)$
Let the point $P(x_1, y_1)$ be point of tangency.
$\because eq.\ of\ tangent\ to\ the\ circle\ (i)at\ P(x_1, y_1)is$
$$xx_1 + yy_1 = a^2$$
$$\Rightarrow\ yy_1 = -xx_1 + a^2$$
$$\Rightarrow\ y = \left(-\frac{x_1}{y_1}\right)x + \frac{a^2}{y_1}$$
$$compare\ with\ y = mx + c$$
Slope of tangent line$= -\frac{x_1}{y_1}$
Slope of radial segment $OP = \frac{y_1}{x_1}$

$$(slope\ of\ tangent)(slope\ of\ OP)$$
$$= \left(-\frac{x_1}{y_!}\right)\left(\frac{y_1}{x_1}\right) = -1$$
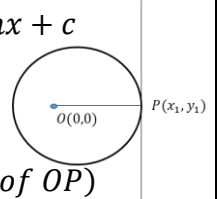
This show that tangent line to a circle at any point of the circle is perpendicular to the radial segment at that point.

Theorem 9:

The perpendicular at the outer end of a radial segment is tangent to the circle.

Proof:

Consider eq. of circle $x^2 + y^2 = a^2 \to (i)$
Let
$P(x_1, y_1)$ be outer end of radial segment OP.
As $P(x_1, y_1)$ lies on the circle $(i)so$
$$x_1^2 + y_1^2 = a^2 \to (ii)$$

Slope of radial segment $OP = \frac{y_1 - o}{x_1 - 0}$

$$= \frac{y_1}{x_1}$$

Slope of line $\perp$
ar to radial segment and passing
through $P(x_1, y_1) is$

$$y - y_1 = \frac{-x_1}{y_1}(x - x_1) \quad (\because y - y_1$$
$$= m(x - x_1))$$

$\Rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2$
$\Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$
$\Rightarrow xx_1 + yy_1 = a^2 \text{ by } (ii)$

Which represent eq. of tangent to the circle
at point $P(x_1, y_1)$

Exercise 6.3

Q#1) Prove that normal lines of a circle pass through the Centre of circle.

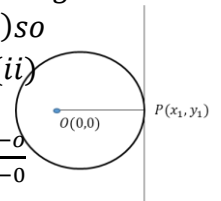**Sol:** Consider a circle $x^2 + y^2 = r^2 \to (1)$
Centre$=O(0, 0)$
Radius$=r$
Let $P(x_1, y_2)$ be any point on the circle
$\Rightarrow x_1^2 + y_2^2 = r^2$
Differentiate (1) w.r.t x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Slope of tangent at $P = \frac{dy}{dx}|_P = -\frac{x_1}{y_1}$

Slope of Normal at $P = m = \frac{y_1}{x_1}$

Equation of Normal through $P(x_1, y_1)$ is
$(y - y_1) = m(x - x_1)$
$\Rightarrow (y - y_1) = \frac{y_1}{x_1}(x - x_1)$
$\Rightarrow x_1 y - x_1 y_1 = xy_1 - x_1 y_1$
$\Rightarrow x_1 y = xy_1$
At $O(0, 0)$
Put $x = 0$ and $y = 0$
$\Rightarrow x_1 y = xy_1 \Rightarrow x_1(0) = (0)y_1$
$\Rightarrow 0 = 0$
Hence normal line pass through Centre.

Q#2) Prove that the straight line drawn from the Centre of a circle perpendicular to a tangent passes through the point of tangency.
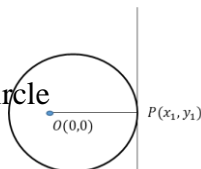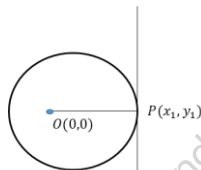
Sol: Consider a circle $x^2 + y^2 = r^2 \to (1)$
Centre$=O(0, 0)$
Radius$=r$
Let $P(x_1, y_2)$ be any point on the circle
$\Rightarrow x_1^2 + y_2^2 = r^2$
Differentiate (1) w.r.t x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Slope of tangent at $P = \frac{dy}{dx}|_P = -\frac{x_1}{y_1}$

Let l be the line through "O" and $\perp$ to the tangent
at P, so

Slope of line $l = m = \frac{y_1}{x_1}$

Equation of l $O(0, 0)$ is
$(y - y_1) = m(x - x_1)$
$\Rightarrow (y - 0) = \frac{y_1}{x_1}(x - 0)$
$\Rightarrow x_1 y = xy_1$
At $P(x_1, y_1)$
Put $x = x_1$ and $y = y_1$
$\Rightarrow x_1 y = xy_1 \Rightarrow x_1(y_1) = (x_1)y_1$
$\Rightarrow x_1 y_1 = x_1 y_1$

Thus, the $\perp$ ar line "l" passes through the point of
tangency $P(x_1, y_1)$.

Q#3) Prove that the midpoint of the hypotenuse of a right triangle is the circumcenter of the triangle.

Sol:
$$let \ x^2 + y^2 = r^2 \to (i)$$
Is eq. of circle with Centre
$O(0,0) and P(a, b) any$
Point lies on (i)
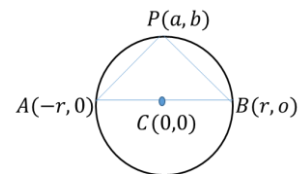
So,
$$a^2 + b^2 = r^2 \to (ii)$$
$$|OA| = \sqrt{(-r - 0)^2 + (0 - 0)^2} = r$$
$$|OB| = \sqrt{(r - o)^2 + (0 - 0)^2} = r$$
$$|OP| = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$$
$$= \sqrt{r^2} = r \text{ by } (ii)$$
$\Rightarrow |OA| = |OB| =$
$|OP|$ and O is circumcentre
Hence Proved.

Q#4) Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.

Sol: Consider a circle $x^2 + y^2 = r^2$
$P(a, b) lies on it. so \ a^2 + b^2 = r^2$
$\Rightarrow b^2 = r^2 - a^2 \to (i)$
Now
$$|PQ| = \sqrt{(a - a)^2 + (b - 0)^2}$$

$$= b$$
$$|AQ| = |AO| + |OQ| = r + a$$
$$|BQ| = |OB| - |OQ| = r - a$$
$$|AQ|.|QB| = (r + a)(r - a) = r^2 - a^2$$
$$|AQ|.|QB| = b^2 \quad by\ (i)$$

$$|AQ|.|QB| = |PQ|^2$$
$\Rightarrow$ $|AQ|.|QB| = |PQ||PQ|$
Hence proved.

Parabola:
Set of all points which are equidistance from a fixed point and a fixed line.
- Fixed point is called focus of parabola.
- Fixed line is called Directrix of parabola.

Equation of parabola:

$$y^2 = 4ac$$

opening of parabola is

symmetric w.r.t x axis

Let AB be a fixed line called directrix drawn parallel to y- axis such that its equation is $x = -a\ or\ x + a = 0$
Let $P(x, y)$ be any point on the parabola. Let $M(-a, y)$ be point on the direcrrix AB.
By def.
$$\frac{|FP|}{|PM|} = 1 \Rightarrow |FP| = |PM|$$
$\Rightarrow$ $\sqrt{(x - a)^2 + (y - 0)^2} = \sqrt{x - (-a)^2 + (y - y)^2}$
$\Rightarrow$ $(x - a)^2 + y^2 = (x + a)^2 \quad (by\ squaring)$
$\Rightarrow$ $x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$
Or $y^2 = 4ax$
This is req. eq. of parabola.

Definitions:
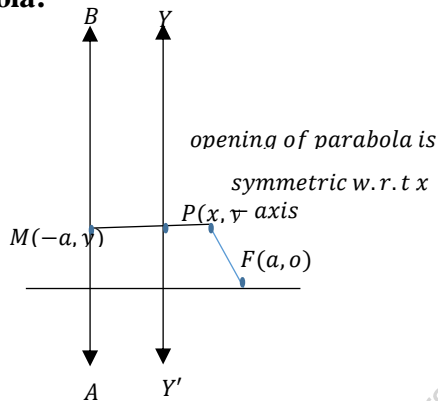1) The line through the focus and perpendicular to the $directrix$ is called **axis of parabola.**
2) The point where the axis meets the parabola is called **vertex** of parabola.
3) In parabola, fixed point not on the $directrix$ is called **focus of parabola.**

4) In parabola, fixed line is called ***directrix*** of parabola.
5) A line passing through vertex and perpendicular to axis of parabola **is called tangent at vertex of parabola.**
6) A line whose points lie on parabola is called chord of parabola.
7) A chord passing through focus of parabola is called **focal chord.**
8) A focal chord perpendicular to axis of parabola is called **Let rectum.**
9) If $|FP|$ *is distance between* focus F and point P on parabola and $|PM|$ *is* $\perp$ *ar dis.* of point p from *directrix* of parabola then $\frac{|FP|}{|PM|}$
   Is called **eccentricity. Denoted by e. for parabola $e = 1$**
10) If the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax\ for\ any\ t \in R$ then $x = at^2$, $y = 2at$ *are called **parametric equation of parabola.***
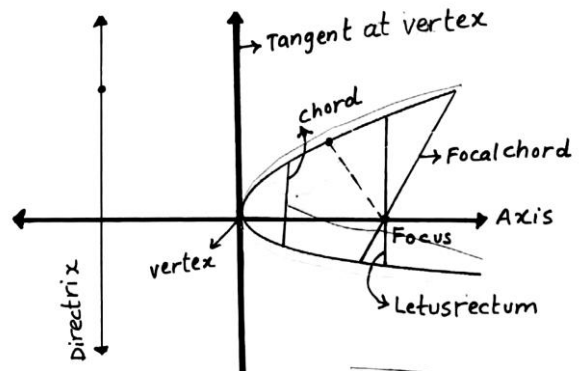
General form of equation of parabola.
let $P(x, y)$ be any point on the parabola having $F(h, k)$ as focus and M be point on the directrix $lx + my + c = 0$ *by def; eq of parabola is*
$$\sqrt{(x - h)^2 + (y - k)^2} = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$
$$or\ (x - h)^2 + (y - k)^2 = \frac{(lx + my + n)^2}{l^2 + m^2}$$

Remember that
Every second degree equation of the form
$$ax^2 + by^2 + 2gx + 2fy + c = 0$$
Will represented parabola if either $a = 0\ or\ b - o$ but
Both *are not zero simultaneously.*

Other standard parabolas:

1. If the focus lies on the $y-axis$ with coordinates $F(0, a)$ and directrix of the parabola is $y = -a$ then eq. of parabola is $x^2 = 4ay$.

The graph is shown as;

$$opening\ of\ parabola\ is\ upward\ and\ symmetric\ w.r.t\ y-axis.$$

2. If the focus is $F(o, -a)$ and directrix is the Line $y = a$ the eq. of parabola is $x^2 = -4ay$
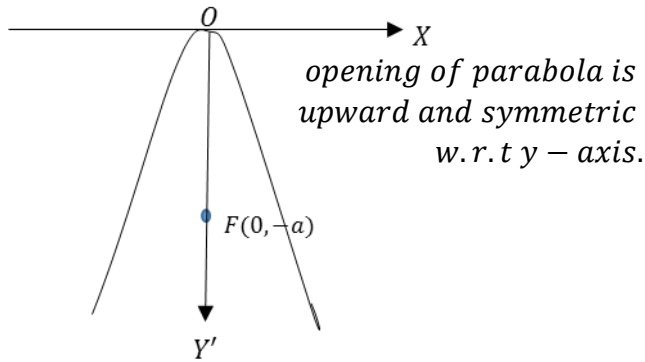
The graph is shown as,

$$opening\ of\ parabola\ is\ downward\ and\ symmetric\ w.r.t\ y-axis.$$
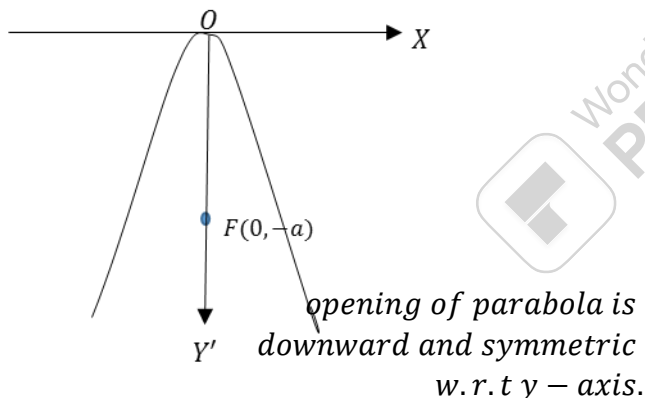
3. If the focus of parabola is $F(-a, o)$ and is directrix Is the line $x = a$, then eq. of parabola is $y^2 = -4ax$

The graph is shown as;

$$opening\ of\ parabola\ is\ to\ left\ and\ symmetric\ w.r.t\ x-axis.$$
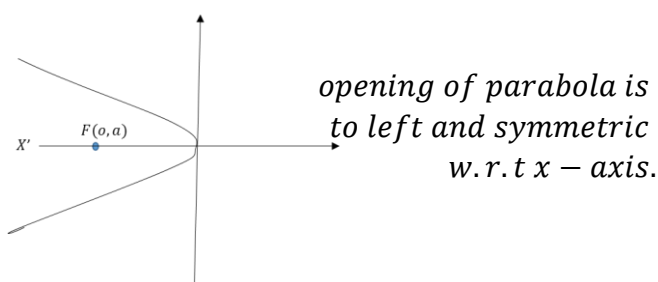
Summary of Standard Parabolas

Sr.No.	1	2	3	4
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Graph				

Prove that length of $letusrectum$ of parabola is 4a.
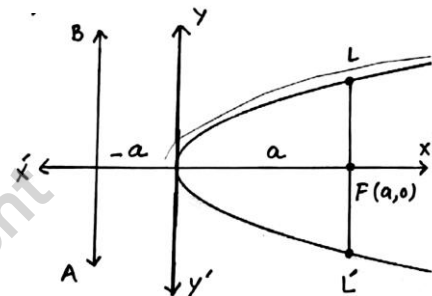
Consider eq. of parabola is $y^2 = 4ax \rightarrow (i)$
Let LL' be focal chord $\perp$
to axis of parabola is called letusrecum. let $x = a$ be eq. of letusretum. LL'

Put $x = a$ in (i)
$\Rightarrow y^2 = 4a(a) \Rightarrow y^2 = 4a^2$
$\quad or\ y = \pm 2a \Rightarrow y = 2a\ or\ y = -2a$
so coordinates of L and L' are $L(a, 2a)$ and $L'(a, -2a)$
so coordinates of L and L' are $L(a, 2a)$ and $L'(a, -2a)$
Now,
Length $of\ letusretum = |LL'| =$
$\sqrt{(a-a)^2 + (-2a-2a)^2} = \sqrt{(-4a)^2} = \sqrt{16a^2}$
$$= 4a$$

Hence proved.

Theorem:

The point of a parabola which is close to the focus is the vertex

Proof:

Consider eq. of parabola $y^2 = 4ax \rightarrow (i), a > 0$
Let
$F(a, o)$ be focus and $P(x, y)$ be any point on the Parabola, Now

$$|PF| = \sqrt{(x-a)^2 + (y-0)^2}$$
$$= \sqrt{x^2 + a^2 - 2ax + y^2}$$

$$\sqrt{x^2 + a^2 - 2ax + 4ax} \quad \because y^2 = 4ax$$
$$= \sqrt{x^2 + a^2 + 2ax}$$
$$= \sqrt{(x + a)^2}$$
$$|PF| = x + a \ , a > 0, x > 0$$

If $V(0,0)$ is vertex then
$$|VF| = \sqrt{(a,o)^2 + (0,0)^2} = \sqrt{a^2} = a$$
$$|VF| < |FP|$$
$$\Rightarrow distance \ b\backslash w \ focus, Vertex$$
$$< Distance \ b\backslash w \ Focus, point$$
hence the point of parabola which is closest to the
focus is vertex of parabola.

Reflecting property of parabola:
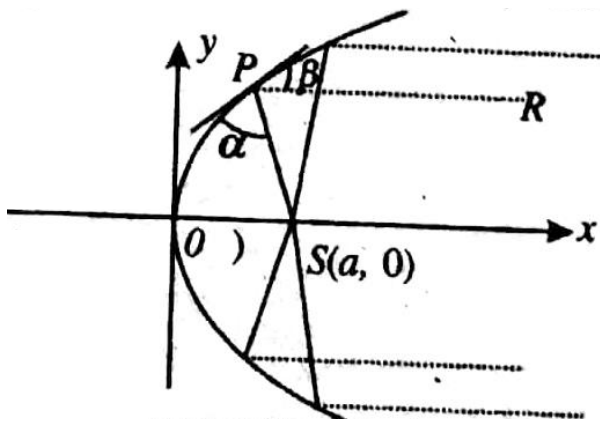
When a light source is placed at the focus of a
parabolic reflecting surface, then light ray
travelling from focus S to a point P on parabola
will be reflected in the direction PR which is
parabola to the axis of the parabola shown in
figure.

Exercise 6.4

Question.1.
**Find the focus, vertex and directrix of the
parabola sketch its graph**
(i). $y^2 = 8x$
Solution.
Given Parabola
$$y^2 = 8x$$
$$Here \quad 4a = 8$$
$$a = 2$$
$$Vertex: V(0,0)$$
The axis of the parabola is along $x - axis$ and
opening of parabola is to the right side.
$$Focus: F(a,0) = F(2,0)$$
$$Directrix: \ x + a = 0$$
$$x + 2 = 0 \Rightarrow x = -2$$
Sketch:
$$\because y^2 = 4ax$$

$\Rightarrow \ y = \pm\sqrt{8y} \ , D_f = [0, \infty)$

x	$y = \pm\sqrt{8x}$	Points
0	$y = \pm\sqrt{8(0)} = 0$	$(0,0)$
2	$y = \pm\sqrt{8(2)} = \pm 4$	$(2,4), (2,-4)$
8	$y = \pm\sqrt{8(8)} = \pm 8$	$(8,8), (8,-8)$
18	$y = \pm\sqrt{8(18)} = \pm\sqrt{144} = \pm 12$	$(18,12), (18,-12)$
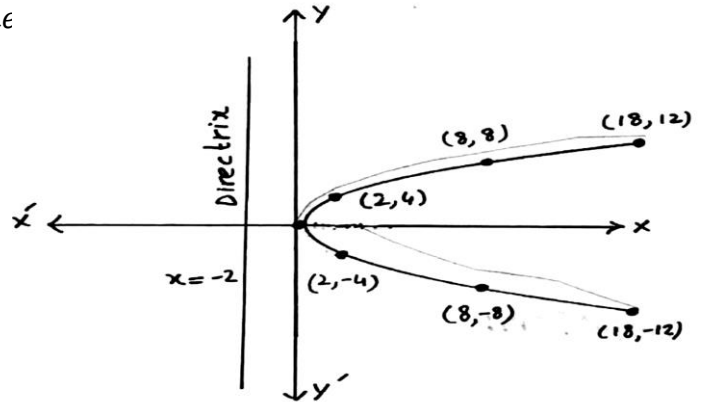
($\qquad$ 16y
Solution.
Given Parabola
$$x^2 = -16y$$
$$Here \quad 4a = 16$$
$$a = 4$$
$$Vertex: V(0,0)$$
The axis of the parabola is along $y - axis$ and
opening of parabola is to the downward side.
$$Focus: F(0, -a) = F(0, -4)$$
$$Directrix: \ y - a = 0$$
$$y - 4 = 0 \Rightarrow y = 4$$

Sketch:
$$\because x^2 = -16y$$
$\Rightarrow \ x = \pm\sqrt{-16y} \ , D_f = (-\infty, -1]$

y	$x = \pm\sqrt{-16y}$	Points
-1	$x = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$	$(4,-1), (-4,-1)$
0	$x = \pm\sqrt{-16(0)} = 0$	$(0,0)$
-4	$x = \pm\sqrt{-16(-4)} = \pm\sqrt{64} = \pm 8$	$(8,-4), (-8,-4)$
-9	$x = \pm\sqrt{-16(-9)} = \pm\sqrt{144} = \pm 12$	$(12,-9), (-12,-9)$
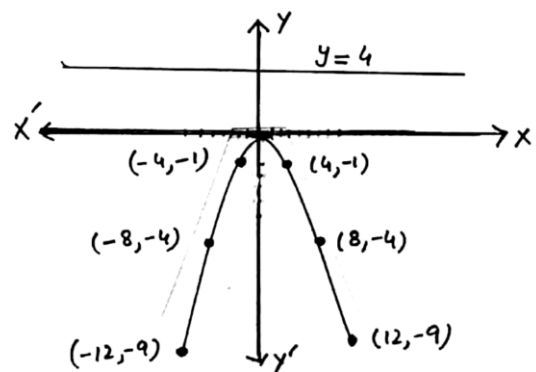
(iii). $x^2 = 5y$

Solution.

Given Parabola

$$x^2 = 5y$$
$$Here \quad 4a = 5$$
$$a = \frac{5}{4}$$
$$Vertex: V(0,0)$$

The axis of the parabola is along $y - axis$ and opening of parabola is to the upward side.

$$Focus: F(0, a) = F\left(0, \frac{5}{4}\right)$$
$$Directrix: \quad x + a = 0$$
$$x + \frac{5}{4} = 0.$$

Sketch:

$$x^2 = 5y \Rightarrow x = \pm\sqrt{5y}$$
$$D_f = [o, +\infty)$$

y	$x = \pm\sqrt{5y}$	Points
0	$x = \pm\sqrt{5(0)} = 0$	(0,0)
5	$x = \pm\sqrt{5(5)} = 5 \pm 5$	(5,5), (-5,5)
20	$x = \pm\sqrt{5(20)} = \pm\sqrt{100} = \pm 10$	(10,20), (-10,20)
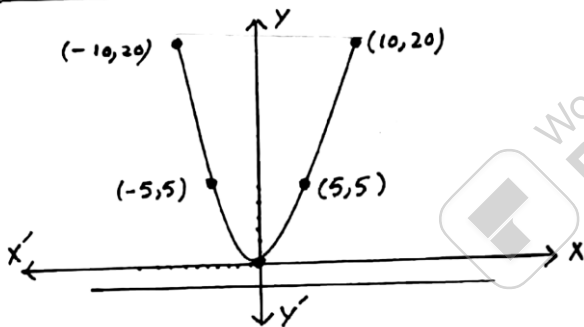
(iv). $y^2 = -12x$

Solution.

Given Parabola

$$y^2 = -12x$$
$$Here \quad 4a = 12$$
$$a = 3$$
$$Vertex: V(0,0)$$

The axis of the parabola is along $x - axis$ and opening of parabola is to the left side.

$$Focus: F(-a, 0) = F(-3, 0)$$
$$Directrix: \quad x - a = 0$$
$$x - 3 = 0.$$

Sketch:

$$\because y^2 = -12x$$
$$\Rightarrow y = \pm\sqrt{-12x}, D_f = (-\infty, o)$$

x	$y = \pm\sqrt{-12x}$	Points
0	$y = \pm\sqrt{-12(0)} = 0$	(0,0)
-3	$y = \pm\sqrt{-12(-3)} = \pm\sqrt{36} = \pm 6$	(-3,6), (-3,-6)
-12	$y = \pm\sqrt{(-12)(-12)} = \pm\sqrt{144} = \pm 12$	(-12,12), (-12,-12)
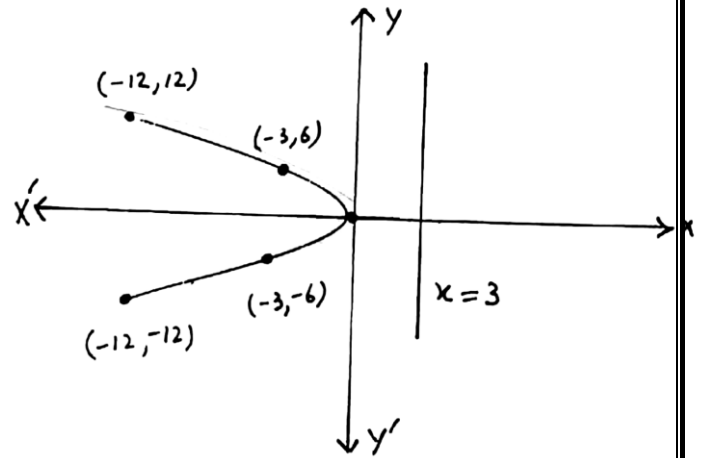
(v). $x^2 = 4(y - 1)$

Solution.

Given Parabola

$$x^2 = 4(y - 1) \rightarrow (i)$$

Put $X = x$ and $Y = y - 1$, we have

$$X^2 = 4Y \rightarrow (ii)$$
$$Here \quad 4a = 4$$
$$a = 1$$

The vertex of (ii) is $O(0,0)$ with axis of the parabola is along Y-axis and open upward.

$$Vextex: V(0,0)$$
$$\Rightarrow \quad X = 0 \quad and \quad Y = 0$$
$$\Rightarrow \quad x = 0 \quad and \quad y - 1 = 0$$
$$\Rightarrow \quad x = 0 \quad and \quad y = 1$$

Hence Vertex of (i) parabola is (0,1).

Now $Focus: (0, a) = (0, 1)$

$$X = 0 \quad and \quad Y = 1$$
$$x = 0 \quad and \quad y - 1 = 1$$
$$x = 0 \quad and \quad y = 2$$

Hence Focus of the parabola (i) is $F(0,2)$.

Directrix of the parabola (ii) is

$$Y + a = 0$$
$$Y + 1 = 0$$
$$y - 1 + 1 = 0$$
$$y = 0 \text{ is the directrix of the parabola } (i).$$

Sketch:

$$\because x^2 = 4(y - 1)$$
$$\Rightarrow x = \pm 2\sqrt{y - 1} \quad D_f = [1, \infty)$$

y	$x = \pm 2\sqrt{y-1}$	Points
1	$x = \pm 2\sqrt{1-1} = 0$	$(0,0)$
2	$x = \pm 2\sqrt{2-1} = \pm 2$	$(2,2), (-2,2)$
5	$x = \pm 2\sqrt{5-1} = \pm 2(2) = \pm 4$	$(4,5), (-4,5)$
10	$x = \pm 2\sqrt{10-1} = \pm 2(3) = \pm 6$	$(6,10), (-6,10)$

vii) $x - 1)^2 = 8(y + 2)$

x	$y = \pm \sqrt{-8(x-3)}$	Points
3	$y = \pm \sqrt{-8(3-3)} = 0$	$(0,0)$
1	$y = \pm \sqrt{-8(1-3)} = \pm\sqrt{16} = \pm 4$	$(1,4), (1,-4)$
-5	$y = \pm \sqrt{-8(-5-3)} = \pm\sqrt{64} = \pm 8$	$(-5,8), (-5,-8)$
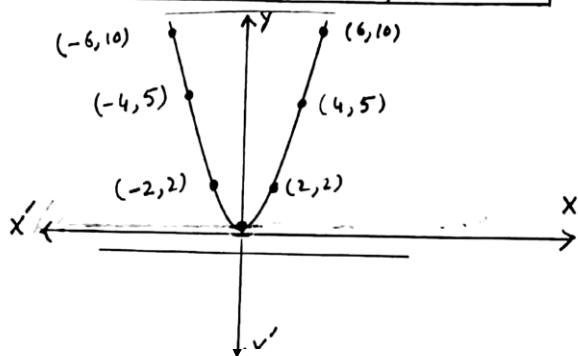
vi) $y^2 = \grave{}-8(x-3)$

⇨ $(y - 0)^2 = -8(x - 3)$

     Compare with $y^2 = -4ax$

         $Y = y - 0 = Y, \qquad X = x - 3$

             $4a = 8 \Rightarrow a = 2$

Focus: $F(X = -a, Y = 0)$

          $\because X = x - 3 \Rightarrow -a = -2 + 3$

        $or\ x = 1 \qquad (\because a = 2)$

       $Y = 0 \Rightarrow y = 0 \quad (\because Y = y)$

So $F(1,0)$

Vertex; $(X = 0, Y = 0)$

     $\because X = x - 3 \Rightarrow 0 = x - 3 \quad (\because X = 0)$

Or $x = 3$

Also $Y = y \Rightarrow 0 = y \quad (\because Y = 0)$

Or $y = 0$

So vertex $= V(3,0)$

 Directrix

           $X = a$

     $\Rightarrow x - 3 = a \Rightarrow x - 3 = 2$

        $or\ x = 2 + 3 \Rightarrow x = 5$

Sketch:

        $\because y^2 = -8(x - 3)$

⇨ $y = \pm\sqrt{-8(x-3)} \quad , D_f = (-\infty, 3)$
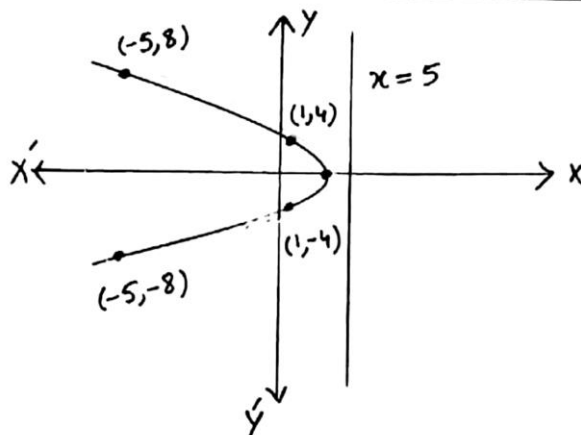
Compare with $X^2 = 4aY$

    $X = x - 1, \ Y = y + 2, \ 4a = 8 \Rightarrow a = 2$

Focus; $F(X = 0, Y = a)$

      $X = x - 1 \Rightarrow 0 = x - 1 \Rightarrow x = 1$

   $Y = y + 2 \Rightarrow a = y + 2 \Rightarrow 2 = y + 2$

            $\Rightarrow y = 0$

So $F(1,0)$

Vertex:

          $V(X = 0, Y = 0)$

     $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

     $Y = o \Rightarrow y + 2 = 0 \Rightarrow y = -2$

          $V(1, -2)$

Directrix:

          $Y = -a$

⇨ $y + 2 = -a \Rightarrow y + 2 = -2$

⇨ $y = -2 - 2 \Rightarrow y = -4$

Sketch:

        $(x - 1)^2 = 8(y + 2)$

⇨ $x - 1 = \pm\sqrt{8(y+2)}$

⇨ $x = 1 \pm \sqrt{8(y+2)} \ , D_f = [-2, +\infty)$

y	$x = 1 \pm \sqrt{8(y+2)}$	Points
-2	$x = 1 \pm \sqrt{8(-2+2)} = 1 \pm 0 = 1$	$(1, -2)$
0	$x = 1 \pm \sqrt{8(0+2)} = 1 \pm 4 = 5, -3$	$(5,0), (-3,0)$
6	$x = 1 \pm \sqrt{8(6+2)} = 1 \pm 8 = 9, -7$	$(9,6), (-7,6)$

y	$x = \pm \sqrt{\dfrac{y+1}{6}}$	Points
-1	$x = \pm \sqrt{\dfrac{-1+1}{6}} = 0$	$(0, -1)$
5	$x = \pm \sqrt{\dfrac{5+1}{6}} = \pm 1$	$(1,5), (-1,5)$
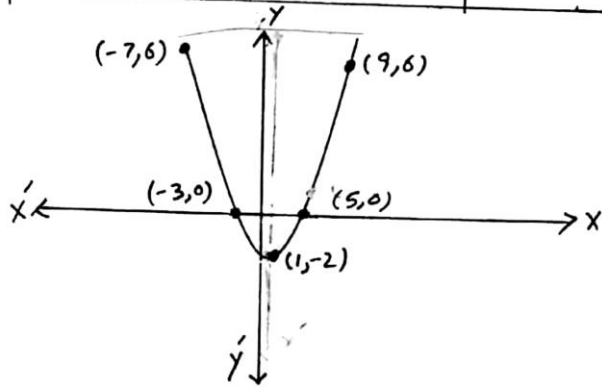
viii) $y = 6x^2 - 1$

$\Rightarrow 6x^2 = y + 1$

Or $(x - 0)^2 = \dfrac{1}{6}(y + 1)$

Compare with $X^2 = 4aY$

$X = x - o = x, \qquad Y = y + 1$

$$4a = \frac{1}{6} \Rightarrow a = \frac{1}{24}$$

Focus; $F(X = 0, \ Y = a)$

$X = 0 \Rightarrow y + 1 = \dfrac{1}{24} \qquad \left(\because a = \dfrac{1}{24}\right)$

$\Rightarrow y = \dfrac{1}{24} - 1 \Rightarrow y = -\dfrac{23}{24}$

$$so, \qquad F\left(0, -\frac{23}{24}\right)$$

Vertex:

$$V(X = 0, Y = 0)$$

$\Rightarrow X = 0 \Rightarrow x = 0$

$\Rightarrow Y = 0 \Rightarrow y + 1 = 0 \Rightarrow y = -1$

$$so \ V(0, -1)$$

Directrix:

$$Y = -a$$

$\Rightarrow y + 1 = -\dfrac{1}{24} \Rightarrow y + 1 + \dfrac{1}{24} = 0$

$\Rightarrow y + \dfrac{25}{24} = 0$

Sketch: $\because x^2 = \dfrac{1}{6}(y + 1)$

$\Rightarrow \pm\sqrt{\dfrac{y+1}{6}}, \qquad D_f = [-1, +\infty)$
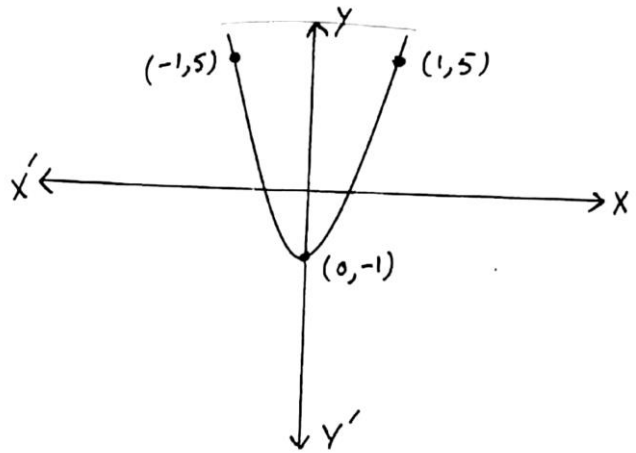
ix) $x + 8 - y^2 + 2y = 0$

$\Rightarrow x + 8 = y^2 - 2y$

$\Rightarrow x + 8 + 1 = y^2 - 2y + 1 \ (adding\ 1\ both\ sides)$

$\Rightarrow x + 9 = (y - 1)^2$

$\qquad or \ (y - 1)^2 = 1(x + 9)$

Compare with $Y^2 = 4aX$

$\Rightarrow Y = y - 1, X = x + 9, 4a = 1 \Rightarrow a = \dfrac{1}{4}$

Focus:

$$F(X = a, Y = 0)$$

$X = x + 9 \ \Rightarrow a = x + 9 \Rightarrow \dfrac{1}{4} = x + 9$

$$x = \frac{1}{4} - 9 \Rightarrow x = -\frac{35}{4}$$

$$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$$

$$so, F\left(-\frac{35}{4}, 1\right)$$

$Vertex$:

$$V(X = 0, Y = 0)$$

$$X = 0 \Rightarrow x + 9 = 0 \Rightarrow x = -9$$

$$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$$

$$s0 \quad V(-9, 1)$$

Directrix:

$$X = -a$$

$\Rightarrow X = -\dfrac{1}{4} \Rightarrow x + 9 + \dfrac{1}{4} = 0$

$\Rightarrow x + \dfrac{37}{4} = 0$

Sketch;

$$\because (y - 1)^2 = x + 9$$

$$\Rightarrow y - 1 = \pm\sqrt{x + 9}$$

$$or \ y = 1 \pm \sqrt{x + 9} \quad D_f = [-9, +\infty)$$

x	$y = 1 \pm \sqrt{x+9}$	Points
-9	$y = 1 \pm \sqrt{-9+9} = 1 \pm 0 = 1$	(-9, 1)
-8	$y = 1 \pm \sqrt{-8+9} = 1 \pm 1 = 2, 0$	(-8,0),(-8, 2)
-5	$y = 1 \pm \sqrt{-5+9} = 1 \pm \sqrt{4} = 1 \pm 2 = 3, -1$	(-5,3), (-5,-1)
0	$y = 1 \pm \sqrt{0+9} = 1 \pm 3 = 4, -2$	(0, 4), (0,-2)

x) $x^2 - 4x - 8y + 4 = 0$

$\Rightarrow x^2 - 4x + 4 = 8y$

$$\Rightarrow (x-2)^2 = 8(y-0)$$

Compare with $X^2 = 4aY$

$$X = x - 2, 4a = 8 \Rightarrow a = 2$$
$$Y = y - 0 = y$$

Focus:

$$F(X = 0, Y = a)$$
$$X = x - 2 \Rightarrow 0 = x - 2 \Rightarrow x = 2$$
$$Y = a \Rightarrow y = 2 \quad (\because a = 2)$$
$$Y = 0 \Rightarrow y = 0$$

So $V(2,0)$

Directrix:

$$Y = -a$$
$$\Rightarrow \quad Y = -2 \Rightarrow y = -2$$
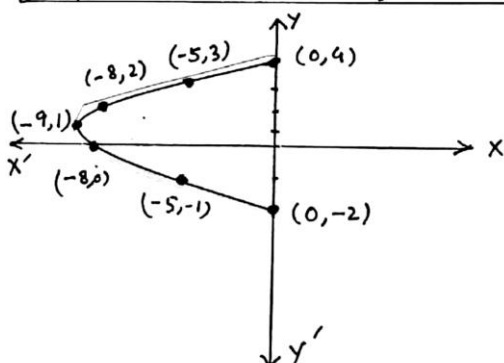$$\text{or } y + 2 = 0$$

Sketch:

$$(x-2)^2 = 8y$$
$$\Rightarrow \quad x - 2 = \pm\sqrt{8y}$$
$$\text{or } x = 2 \pm \sqrt{8y} \quad , D_f = [0, +\infty)$$

y	$x = 2 \pm \sqrt{8y}$	Points
0	$x = 2 \pm \sqrt{8(0)} = 2$	(2,0)
2	$x = 2 \pm \sqrt{8(2)} = 2 \pm \sqrt{16} = 6, -2$	(6,2),(-2,2)
8	$x = 2 \pm \sqrt{8(8)} = 2 \pm 8 = 10, -6$	(10,8),(-6,8)
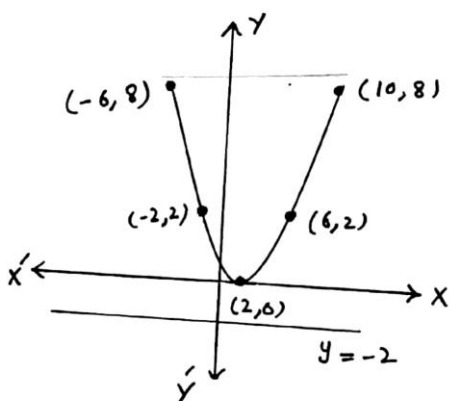
Question. 2
Write an equation of the parabola with given elements
(i). $Focus \, (-3, 1) \,; Directrix \, x = 3$
Solution.
$$Focus \, (-3, 1)$$
$$M; x - 3 = 0$$
Suppose $P(x, y)$ be any point on parabola then by definition

$$|PF|^2 = |PM|^2$$
$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x-3|}{\sqrt{1^2 + 0^2}}$$
$$\sqrt{x^2 + 6x + 9 + y^2 - 2y + 1} = |x - 3|$$

On squaring both sides
$$x^2 + 6x + 9 + y^2 - 2y + 1 = x^2 - 6x + 9$$
$$x^2 + 6x + 9 + y^2 - 2y + 1 - x^2 + 6x - 9 = 0$$
$$y^2 + 12x - 2y + 1$$

Which is required equation of the parabola.

(ii). $Focus \, (2, 5) \,; Directrix \, y = 1$
Solution.
$$Focus \, (2,5)$$
$$M; y - 1 = 0$$
P(x,y) be any point on parabola then by definition
$$|PF|^2 = |PM|^2 \; (\text{where } |PM| \text{ is}$$
$$\perp \text{ ar distance of } F$$
$$\text{from } y - 1 = 0)$$

$$\sqrt{(x-2)^2 + (y-5)^2} = \frac{|y-1|}{\sqrt{0^2 + 1^2}}$$
$$\sqrt{x^2 - 2x + 4 + y^2 - 10y + 25} = |y - 1|$$

On squaring both sides
$$x^2 - 2x + 4 + y^2 - 10y + 25 = y^2 - 2y + 1$$
$$\text{or } x^2 - 4x + 29 - 10y - 1 + 2y = 0$$
$$\text{or } x^2 - 4x - 8y + 28 = 0$$

Which is required equation of the parabola.

(iii). $Focus \, (-3, 1) \,; Directrix \, x - 2y - 3 = 0$
Solution.
$$Focus \, (-3, 1)$$
$$M; x - 2y - 3 = 0$$
$take \, point \, P(x, y)$ be any point on parabola then by definition
$$|PF|^2 = |PM|^2$$
$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x - 2y - 3|}{\sqrt{1^2 + (-2)^2}}$$
$$\sqrt{x^2 + 6x + 9 + y^2 - 2y + 1} = \frac{|x-3|}{\sqrt{5}}$$

On squaring both sides

$$x^2 + 6x + 9 + y^2 - 2y + 1 = \frac{x^2 - 6x + 9}{5}$$
$$5(x^2 + 6x + 9 + y^2 - 2y + 1) = x^2 - 6x + 9$$

$$5x^2 + 30x + 45 + 5y^2 - 10y + 5 - x^2 + 6x$$
$$- 9 = 0$$
$$4x^2 + y^2 + 36x - 22y + 4xy + 41$$

Which is required equation of the parabola.

Note:

"**When Focus and vertex have same abscissa** $(x - coordinates)$"

- *Ordinate vertex $<$ ordinate of focus*

If *ordinate of vertex $>$*
ordinate of focus
Then parabola opens downwards.

- **when Focus and vertex have same ordinate**
  $(y - coordinates)$"

Abscissa of Vertex
$< $ Absissa of Fcocus

Then parabola opens towards right

If

Abscissa of Vertex $>$ Abscissa of Fpcus

Then parabola opens towards left.

(vi). $Focus\ (-3, 1)$; $Vertex : (3, 2)$

Solution. Given that

$$Focus\ (-3, 1)$$
$$Vertex\ (3, 2) = V(h, k)$$

Focus and vertex have same ordinate and

Abscissa of Vertex $>$ Abscissa of Fpcus

Then parabola opens towards left.

$$i.e\ (y - k)^2 = -4a(x - h)$$
$$(y - 2)^2 = -4a(x - 3)$$

Now $a = $ distance between focus and vertex

$$a = |FV| = \sqrt{(3 - 1)^2 + (2 - 2)^2} = \sqrt{4 + 0} = 2$$

Putting in (i), we have

$$(y - 2)^2 = -4(2)(x - 3)$$
$$y^2 - 4y + 4 = -8(x - 3)$$
$$y^2 - 4y + 4 = -8x + 24$$
$$y^2 + 8x - 4y - 24 + 4 = 0$$
$$y^2 + 8x - 4y - 20 = 0$$

Which is required equation of the parabola.

(v). $Focus\ (2, 5)$; $Vertex\ (-1, 2)$

Solution.

$$Focus\ (-1, 0)$$
$$Vertex\ (-1, 2) = V(h, k)$$

Focus and vertex have same ordinate and

ordinate of vertex $>$ ordinate of focus

Then parabola opens downwards.

$$i.e\ (x - h)^2 = -4a(y - k)$$
$$\Rightarrow (x + 1)^2 = -a(y - 2) \rightarrow (i)$$
$$a = |FV| = \sqrt{(2 - 0)^2 + (-1 + 1)^2} = 2$$

Put $a = 2$ in (i)

$$(x + 1)^2 = -a(2)(y - 2)$$
$$\Rightarrow x^2 + 1 + 2x = -8y + 16$$
$$or\ x^2 + 2x + 1 + 8y - 15 = 0$$
$$(required\ parabola)$$

(vi). $Directrix\ x = -2$ $Focus\ (2, 2)$

Solution.

$$take\ any\ poit\ P(x, y)$$
$$F((2,2), M; x + 2 = 0$$

By definition.

$$|PF|^2 = |PM|^2$$
$$\Rightarrow \left(\sqrt{(x - 2)^2 + (y - 2)^2}\right)^2 = \left(\frac{|x+2|}{\sqrt{(1)^2 + (0)^2}}\right)^2$$
$$\Rightarrow x^2 + 4 - 4x + y^2 + 4 - 4y = \frac{x^2 + 4 + 4x}{1}$$
$$\Rightarrow x^2 - 4x + y^2 - 4y + 8 - x^2 - 4 - 4x = 0$$
$$\Rightarrow y^2 - 4y + 4 - 8x = 0$$
$$or\ (y - 2)^2 = 8x\ (req.\ parabola)$$

(vii). $Vertex\ (2, 2)$; $Directrix\ y = 3$

Solution.

$$Vertex\ (2, 2)$$
$$y - 3 = 0$$

Since axis of parabola is parallel to $Y -$
$axis (because\ direcrix\ is\ parallel\ to\ X - axis)$
And opening is downward.

So equation of parabola with vertex $(h, k) = (2,2)$

$$(x - h)^2 = -4a(y - k)^2$$
$$(x - 2)^2 = -4a(y - 2)^2 - - - (i)$$

Now $a = $
$Distance\ of\ vertex\ (2,2) from\ directrix$

$$a = \frac{|2 - 3|}{\sqrt{(0)^2 + (1)^2}} = \frac{|-1|}{\sqrt{0 + 1}} = 1$$

Putting in equation (i), we have

$$(x - 2)^2 = -4(1)(y - 2)$$
$$x^2 - 4x + 4 = -y + 8$$
$$x^2 - 4x + 4 + y - 8 = 0$$
$$x^2 - 4x + y - 4 = 0$$

Which is required equation of the parabola.

(viii). $rectrix\ y = 1$, Length of latusrectum is 8, Opens Downward.

Solution.

$$Directrix\ y = 1$$
$$Length\ of\ latusrectum\ is = 4a = 8 \Rightarrow a = 2$$

Parabola is open downward.
Consider Vertex $= (h, -1)$.
Equation of the parabola is

$$(x - h)^2 = -4a(y - k)$$
$$(x - h)^2 = -4(2)(y + 1)$$
$$x^2 - 2xh + h^2 = -8(y + 1)$$
$$x^2 - 2xh + h^2 = -8y - 8$$
$$x^2 - 2xh + h^2 + 8y + 8 = 0$$
$$x^2 - 2xh + 8y + h^2 + 8 = 0$$

Which is required equation of the parabola.

(ix). $Axis\ y = 0\ through\ (2, 1) and\ (11, -2)$.

Solution.

Let vertex is (h, k).
Since it lies on $x - $ axis so $k = 0$.
Now equation of the parabola with vertex $(h, 0)$.

$$(y - 0)^2 = 4a(x - h)$$
$$y^2 = 4a(x - h) - - - (i)$$

Since $(2,1)$ lies on the parabola (i),we have
$$(1)^2 = 4a(2 - h)$$
$$1 = 4a(2 - h) - - - (ii)$$

Also $(11, -2)$ lies on the parabola (i),we have
$$(-2)^2 = 4a(11 - h)$$
$$4 = 4a(11 - h) - - - (iii)$$

Dividing (i) and (ii), we have
$$\frac{1}{4} = \frac{4a(2 - h)}{4a(11 - h)}$$
$$\frac{1}{1} = \frac{4(2 - h)}{(11 - h)}$$
$$11 - h = 4(2 - h)$$
$$11 - h = 8 - 4h$$
$$-h + 4h = 8 - 11$$
$$3h = -3$$
$$h = -1.$$

Putting in (ii), we have
$$1 = 4a(2 - (-1))$$
$$1 = 4a(2 + 1)$$
$$1 = 4a(3)$$
$$1 = 12a$$
$$a = \frac{1}{12}$$

Putting in (i), we have
$$y^2 = 4\left(\frac{1}{12}\right)(x - (-1))$$
$$y^2 = \frac{1}{3}(x + 1)$$
$$3y^2 = (x + 1)$$
$$3y^2 - x - 1 = 0.$$

Which is required equation of the parabola.

(x). *Axis paralel to Y −*
axis , the points $(0, 3), (3, 4)$ ***and*** $(4, 11)$ ***lie on the***
graph.

Solution.

Let eq. of parabola is
$$(x - h)^2 = 4a(y - k) \to (i)$$

Put $(0,3)$ in (i)
$$\Rightarrow \quad h^2 = 4a(3 - k)$$
$$h^2 = 12a - 4ak \to (ii)$$

Put $(3,4)$ in (i)
$$\Rightarrow \quad (3 - h)^2 = 4a(4 - k)$$
$$9 + h^2 - 6h = 16a - 4ak \to (iii)$$

Put $(4,1)$ in (i)
$$\Rightarrow \quad (4 - k)^2 = 4a(11 - k)$$
$$16 + h^2 - 8h = 44a - 4ak \to (iv)$$

By $(iv) - (iii)$
$$16 - 8h + h^2 = 44a - 4ak \to (iv)$$

By $(iv) - (iii)$
$$16 - 8h + h^2 = 44a - 4ak$$

$$\frac{\pm 9 \mp 6h \pm h^2 = \pm 12a \mp 4ak}{7 - 2h = 28a \to (v)}$$

By $(iii) - (ii)$
$$9 - 2h + h^2 = 16a - 4ak$$
$$\frac{\pm h^2 = \pm 12a \mp 4ak}{9 - 6h = 4a \to (vi)}$$

Multiplying (v) by 3
$$21 - 6h = 84a \to (vii)$$

By $(vii) - (vi)$
$$21 - 6h = 84a$$
$$\frac{\pm 9 \mp 6h = -4a}{12 = 80a}$$

$\Rightarrow \quad a = \frac{12}{80} = \frac{3}{20}$

$\Rightarrow \quad$ Put $a = \frac{3}{20}$ in (vi)

$$9 - 6h = 4\left(\frac{3}{20}\right)$$

$\Rightarrow \quad 9 - 6h = \frac{3}{5}$

$\Rightarrow \quad 9 - \frac{3}{5} = 6h \ or \ 6h - \frac{42}{5}$

$\Rightarrow \quad h = \frac{7}{5} \ put \ in \ (ii)$

$\Rightarrow \quad \left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)k$

$\Rightarrow \quad \frac{49}{25} = \frac{18}{10} - \frac{49}{25}$

$\Rightarrow \quad \frac{3}{5}k = \frac{450 - 490}{250}$

$\Rightarrow \quad \frac{3}{5}k = \frac{-40}{250}$

$\Rightarrow \quad k = -\frac{4}{25} \times \frac{5}{3}$

$\Rightarrow \quad k = -4/15$

$\Rightarrow \quad$ Now (I) becomes as $\left(x - \frac{7}{5}\right)^2 =$
$$4\left(\frac{3}{20}\right)\left(y + \frac{4}{15}\right)$$

$\Rightarrow \quad \left(x - \frac{7}{5}\right)^2 = \frac{3}{5}\left(y + \frac{4}{15}\right)$

Question.3.

Find an equation of the parabola having its focus
at the origin and directrix parallel to the
(i). $x - axis$ **(ii).** $y - axis$

Solution.

(i). $Directrix; y = a, Focus = F(o, o)$
or $M; y - a = 0 \ P(x, y)$ any point
by definition of parabola
$$|PF|^2 = |PM|^2$$
$$\sqrt{(x - 0)^2 + (y - 0)^2} = \frac{|y - a|}{\sqrt{(0)^2 + (1)^2}}$$
$$\sqrt{x^2 + y^2} = \frac{|y - a|}{1}$$

On squaring both sides , we have
$$x^2 + y^2 = y^2 - 2ay + a^2$$
$$x^2 + y^2 - y^2 + 2ay - a^2 = 0.$$
$$x^2 + 2ay - a^2 = 0.$$

Which is required parabola.

Case II:

$\qquad Directrix ; y = -a \ (aslo \ ||to \ x - axis)$

Or $M; y + a = 0$, $F(o,o), P(x,y)$

By definition.

$$|PF|^2 = |PM|^2$$

$\Rightarrow \left(\sqrt{(x-0)^2 + (y-0)^2}\right)^2 = \left(\frac{|y+a|}{\sqrt{(1)^2 + (0)^2}}\right)^2$

$\Rightarrow \ x^2 + y^2 = y^2 + a^2 + 2ax$

$\Rightarrow \ x^2 - a^2 - 2ax = 0$

(ii). $y - axis$:

Case I: $Directrix ; x = a$, $F(o,o), P(x,y)$

$\qquad or \ M; x - a = 0$

$\qquad by \ def; |PF|^2 = |PM|^2$

$\sqrt{(x-0)^2 + (y-0)^2} = \frac{|x-a|}{\sqrt{(1)^2 + (0)^2}}$

$\sqrt{x^2 + y^2} = \frac{|x-a|}{1}$

On squaring both sides, we have

$$x^2 + y^2 = x^2 - 2ax + a^2$$
$$x^2 + y^2 - x^2 + 2ax - a^2 = 0.$$
$$y^2 + 2ax - a^2 = 0.$$

Which is required parabola.

Case II.

$\qquad Directrix ; x = -a \ (also \ ||to \ y - axis)$

$\qquad or M, x + a = 0, \qquad F(0,0), P(x,y)$

By definition.

$\Rightarrow \ |PF|^2 = |PM|^2$

$\Rightarrow \left(\sqrt{(x-0)^2(y-0)^2}\right)^2 = \left(\frac{|x+a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\Rightarrow \ x^2 + y^2 = x^2 + a^2 + 2ax$

$\qquad Or \ y^2 - a^2 - 2ax = 0$

Question.4

Show that an equation of the parabola with focus at $(aCos\alpha , a \ Sin\alpha)$ and directrix

$xCos\alpha + ySin\alpha + a$

$\qquad = 0 \ is \ (xSin\alpha - yCos\alpha)^2$

$\qquad = 4a(xCos\alpha + ySin\alpha)$

Solution.

$\qquad Focus \ (aCos\alpha , a \ Sin\alpha)$

$\qquad M; \ xCos\alpha + ySin\alpha + a = 0$

P(x,y) be any point on parabola then by definition

$$|PF|^2 = |PM|^2$$

$\sqrt{(x - aCos\alpha)^2 + (y - aSin\alpha)^2}$

$\qquad = \frac{|xCos\alpha + ySin\alpha + a|}{\sqrt{Cos^2\alpha + Sin^2\alpha}}$

$\sqrt{(x - aCos\alpha)^2 + (y - aSin\alpha)^2}$

$\qquad = \frac{|xCos\alpha + ySin\alpha + a|}{1}$

On squaring both sides

$(x - aCos\alpha)^2 + (y - aSin\alpha)^2$

$\qquad = |xCos\alpha + ySin\alpha + a|^2$

$x^2 + a^2Cos^2\alpha - 2axCos\alpha + y^2 + a^2Sin^2\alpha$
$\qquad - 2aySin\alpha$
$\qquad = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2$
$\qquad + 2axCos\alpha + 2aySin\alpha$
$\qquad + 2xySin\alpha Cos\alpha$

$x^2 - 2axCos\alpha + y^2 - 2aySin\alpha$
$\qquad + a^2(Cos^2\alpha + Sin^2\alpha)$
$\qquad = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2$
$\qquad + 2axCos\alpha + 2aySin\alpha$
$\qquad + 2xySin\alpha Cos\alpha$

$x^2 - 2axCos\alpha + y^2 - 2aySin\alpha + a^2$
$\qquad = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2$
$\qquad + 2axCos\alpha + 2aySin\alpha$
$\qquad + 2xySin\alpha Cos\alpha$

$x^2(1 - Cos^2\alpha) + y^2(1 - Sin^2\alpha) + a^2 - a^2$
$\qquad - 2xySin\alpha Cos\alpha$
$\qquad = 4axCos\alpha + 4aySin\alpha$

$x^2Sin^2\alpha + y^2Cos^2\alpha - 2xySin\alpha Cos\alpha$
$\qquad = 4a(xCos\alpha + ySin\alpha)$

$(xsin\alpha - ycos\alpha)^2 = 4a(xCos\alpha + ySin\alpha)$

Which is required equation of the parabola.

Hence Proved.

Question.5.

Show that ordinate at any point P of the parabola is a mean proportional between the length of the latus rectum and the abscissa of P.

Solution.

Consider the equation of the parabola is

$$y^2 = 4ax$$
$$y.y = 4a.x$$
$$\frac{4a}{y} = \frac{y}{x}$$
$$\frac{Latus \ ractum}{odinate} = \frac{ordinate}{abscissa}$$

Hence ordinate is mean proportional between latus rectum and abscissa.

Qustion.6.

A comet has a parabolic orbit with the earth at the focus. When the comet is $150,000 km$ from the earth, the line joining the comet and the earth makes an an $30 \ with \ the \ axis \ of \ the \ parabola$. How close will the comet come to the earth.

Solution.
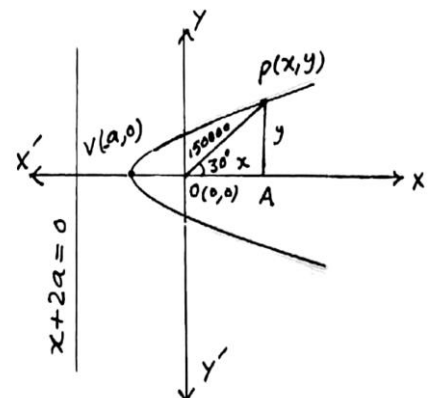
Suppose earth be at focus is origin and $V(-a, o)$ be vertex of parabola. Then directrix of parabola is

$x = 2a$

$\Rightarrow M; x + 2a = 0$

Let comet be at point $P(x, y)$ then by $def.$

$$|PF|^2 = |PM|^2$$

$\Rightarrow \left(\sqrt{(x-0)^2 + (y-0)^2}\right)^2 = \left(\frac{|x+2a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\Rightarrow x^2 + y^2 = a^2 \to (i)$

In ΔOAP, by pathgoras theorem

$$|OA|^2 + |AP|^2 = |OP|^2$$

$\Rightarrow x^2 + y^2 = (150000)^2 \to (ii)$

by (i) and (ii)

$$(x + 2a)^2 = \pm 150000$$

$\Rightarrow (x + 2a)^2 = \pm 150000 \to (iii)$

From right ΔOAP

$$cos30^0 = \frac{|OA|}{|AP|} \Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

$\Rightarrow x = \frac{\sqrt{3}}{2}(150000)$

By (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

$\Rightarrow 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000)$

$\Rightarrow 2a = \pm 150000 - \sqrt{3}(75000)$

$\Rightarrow 2a = 75000(\pm 2 - \sqrt{3})$

$\Rightarrow a = 37500(\pm 2 - \sqrt{3})$

$\because$ a is short distance and can't be $-ve$ so

$$a = 37500(2 - \sqrt{3})km$$

Question.7.

Find the equation of the parabola formed by the cables of a suspension bridge whose span is a m and the vertical height of the supporting towards is b m.

Solution.

$$P\left(\frac{a}{2}, b\right)$$

$$O(0,0)$$

$$a$$
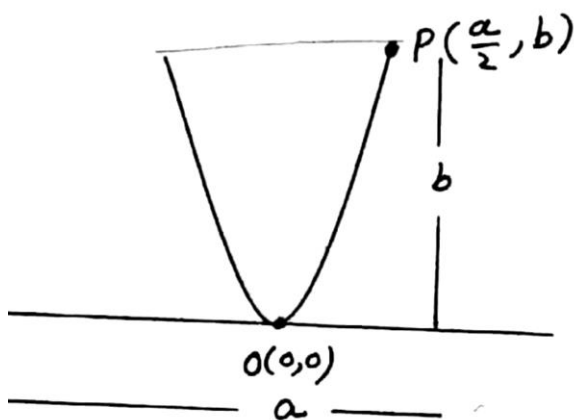
Consider the equation of the parabola with vertex O(0,0)

$$x^2 = 4a'y \to (i)$$

Since $P\left(\frac{a}{2}, b\right)$ lies on the parabola

$$\left(\frac{a}{2}\right)^2 = 4a'(b)$$

$$a' = \frac{a^2}{16b}$$

Putting this value in (i), we have

$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

$$x^2 = \frac{a^2}{4b}y$$

Which is required equation.

Question.8.

A parabolic arch has a 100m base and height 25m. Find the height of the arch at the point 30m from the center of the base.

Solution.
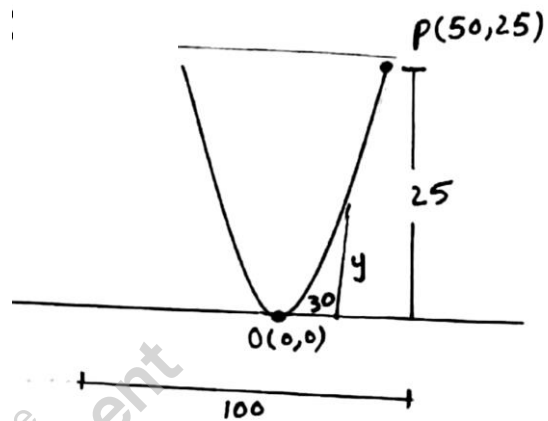
Consider the equation of the parabola with vertex O(0,0)

$$x^2 = 4ay ----(i)$$

Since $P(50, 25)$ lies on the parabola

$$(50)^2 = 4a(25)$$

$$2500 = 100a$$

$$a = 25$$

Putting in (i), we have

$$x^2 = 4(25)y = 100y$$

When $x = 30$ then

$$(30)^2 = 100y$$

$$y = \frac{900}{100} = 9$$

Hence the required height is 9m.

Question.9.

Show that tangent at any point P of a parabola marks equal angles which the line PF and line through P and parallel to $x - axis$.

Solution:

Let $eq.$ of parabola is $y^2 = 4ax \to (i)$

$$m_1 = 0(\because line \parallel to\ x - axis)$$

For m_2; take derivative of (i)

$$2y\frac{dy}{dx}\bigg|_{x_1, y_1} = \frac{2a}{y}$$

$\Rightarrow m_3 = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a} = \frac{y_1}{x_1 - a}$

$Tan\theta_1 = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)}$

$$Tan\theta_1 = \frac{2a}{y_1} \Rightarrow \theta_1 = Tan^{-1}\left(\frac{2a}{y_1}\right) \to (i)$$

$$Tan\theta_2 = \frac{m_3 - m_2}{1 + m_3 m_2}$$

$$= \frac{\left(\frac{v_1}{x_1 - a}\right) - \left(\frac{2a}{y_1}\right)}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)}$$

$$Tan\theta_2 = \frac{\frac{y_1^2 - 2ax_1 + 2a^2}{(x_1 - a)(y_1)}}{\frac{y_1 x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

$$put \ y_1^2 = 4ax_1 \quad (\because \ P \text{ is a parabola})$$

$$\Rightarrow Tan\theta_2 = \frac{\frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 - a)}}{\frac{y_1 x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

$$\Rightarrow Tan\theta_2 = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$$

$$\Rightarrow \theta_2 = Tan^{-1}\left(\frac{2a}{y_1}\right) \to (ii)$$

By (i) and (ii)

$$\theta_1 = \theta_2 \quad \text{Hence proved.}$$
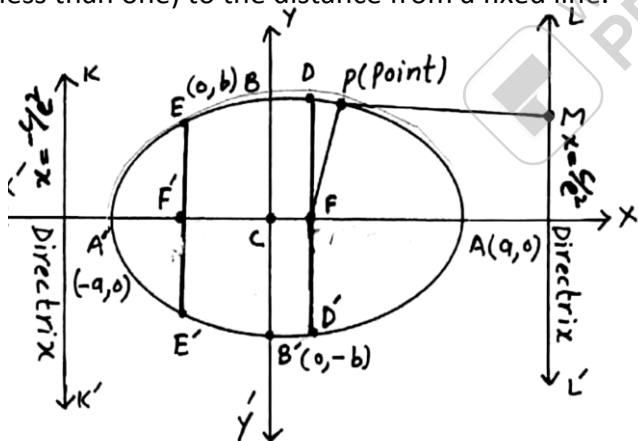
Ellipse and its Elements

Ellipse:

Set of all points in a plane, such that distance of each from a fixed point bears a constant ratio (less than one) to the distance from a fixed line.

Note:

- Fixed point is called Focus. Here F is Focus
- Fixed line LL' is directrix.
- Constant ratio is called eccentricity, denoted by e in Fig. $\frac{|FP|}{|PM|} = e$

$\Rightarrow |FP| = e|PM|$ where $|FP| < |PM|$

So in ellipse $e < 1$

Definition:

1) An ellipse has two diameters, called axes of symmetry. In fig. AA' and BB' are diameter of ellipse.

2) The largest diameter is called major axis. In fig. AA' is major axis and length of major axis $|AA'| \ is \ 2a$.

3) The smallest diameter is called minor axis. In fig. BB' is Minor axis. And length of minor axis and length of minor axis BB' is $2b$.

4) An ellipse has two directrix, called **directrices. In fig**. LL' and kk' are directrices.

5) Two fixed points are called foci (plural of focus) in fig.
F and F' are foci. Foci always lie on major axis.

6) The end points of major axis called vertices. In fig.
$A(a, o)$ and $A'(-a, 0)$ are vertices of ellipse.

7) The end points of minor axis are called co-vertices. In fig $B(o, b)$ and $B'(o, -b)$ are co-vertices of ellipse.

8) A straight line joining two points of ellipse is called chord of ellipse.

9) A straight and passing through focus is called focal chord of ellipse.

10) Focal chord perpendicular to mojor axis is called letusrectum. or focal chord parallel to minor axis is called letusrectum. In fig. DD' and EE' are letusrectra (plural of letusrectum) length of letusrectum is $\frac{2b^2}{a}$

11) The point of intersection of major and minor axis or foci is called Centre.
Or minor axis or foci is called Centre of ellipse. In figure C is Centre.

12) In fig. AC and CA' are semi-major asix.

13) In fig BC and CB' are semi minor axis.

14) If the point $a\cos\theta, b\sin\theta)$ lies on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $\theta \in \mathbb{R}$ then $x = a\cos\theta, y = b\sin\theta$ are called parametric equations of ellipse.

Standard Equation of Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Proof:

Let
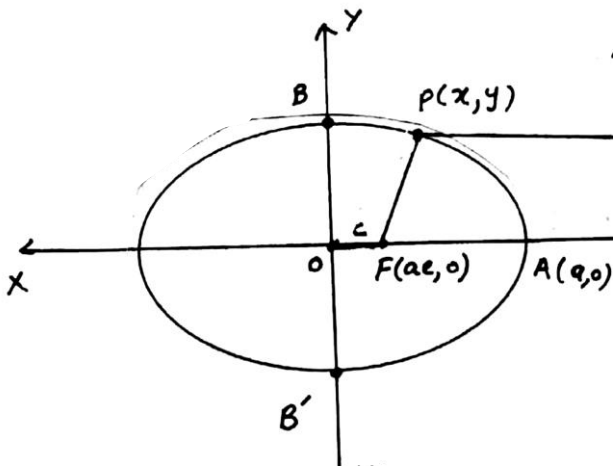
$P(x, y)$ be any point on the ellipse having focus

Having focus $F(ae, o)$ and directed $x = \frac{a}{e}$

$\Rightarrow \quad ex - a = 0$

Let $|PM|$ be $\perp$

ar distance of $P(x, y)$ from directrix

$$x = \frac{a}{e}$$

$\Rightarrow \quad ex - a = 0$

By def. $|FP|^2 = e|PM|^2$

$$\left(\sqrt{(x - ea)^2 + (y - o)^2}\right)^2$$

$$= e\left(\frac{|ex - a|}{\sqrt{(e)^2 + (0)^2}}\right)^2$$

$\Rightarrow \quad x^2 + e^2 a^2 - 2eax + y^2 = e\frac{(ex-a)^2}{e}$

$\Rightarrow \quad x^2 + e^2 a^2 - 2eax + y^2 = e^2 x^2 + a^2 - 2eax$

$\Rightarrow \quad x^2 - e^2 x^2 + y^2 = a^2 - e^2 a^2$

Or $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$$\div \text{ by } a^2(1 - e^2)$$

$\Rightarrow \quad \frac{x^2(1-e^2)}{a^2(1-e^2)} + \frac{y^2}{a^2(1-e^2)} = \frac{a^2(1-e^2)}{a^2(1-e^2)}$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a^2(1 - e^2) = b^2$

Note:

$$\because b^2 = a^2(1 - e^2)$$

$\Rightarrow \quad b^2 = a^2 - a^2 e^2$

$\Rightarrow \quad b^2 = a^2 - (ae)^2 \quad \because ae = c$

$\Rightarrow \quad b^2 = a^2 - c^2$

Or $c^2 = a^2 - b^2$

Prove that circle is special case of ellipse.

Proof:

$$\because \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$= 1 \text{ represents an ellipse having}$

Centre at origin. Let $b = a$

Then eq. of ellipse becomes as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Rightarrow x^2 + y^2 = a^2$$

Which represents eq. of Circle having Centre at origin.

$\Rightarrow$ **Circle is a special case of an ellipse.**

Note:

In an ellipse

$$b^2 = a^2(1 - e^2)$$
$$for \ a \ circle \ b = a$$

$\Rightarrow \quad a^2 = a^2(1 - e^2)$

$\Rightarrow \quad 1 = 1 - e^2$

$\Rightarrow \quad 1 = 1 - e^2 \Rightarrow e^2 = 1 - 1$

$$or \ e^2 = 0 \Rightarrow e = 0$$

So for a circle, eccentricity is $e = 0$

Summary of standard Ellipses

Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$
	$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x = \pm\frac{c}{e^2}$	$y = \pm\frac{c}{e^2}$
Major axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Centre	$(0, 0)$	$(0, 0)$
Eccentricity	$e = \frac{c}{a} < 1$	$e = \frac{c}{a} < 1$
Graph		

Note: In each ellipse

Length of major axis = $2a$, Length of minor axis = $2b$

Length of Latusrectum $\approx \frac{2b^2}{a}$, Foci lie on the major axis

Exercise No.6.5

Q1. Find an equation of the ellipse with given data and sketch its graph.

(i) Foci

$(\pm 3, 0) \ and \ minor \ axis \ of \ length \ 10.$

Solution:

Length of minor axis $= 10$

$$c = 3 \ and \ 2b = 10 \Rightarrow b = 5$$

Now $c^2 = a^2 - b^2 \Rightarrow a^2 = c^2 + b^2$

$\Rightarrow \quad a^2 = 9 + 25 \Rightarrow a^2 = 34$

Now $eq. \ of \ ellipse \ is$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{34} + \frac{y^2}{25} = 1$$

Sketch:

$$Centre = \ mid \ point \ of \ foci$$

$$F(3,0) \, and \, F'(-3,0)$$

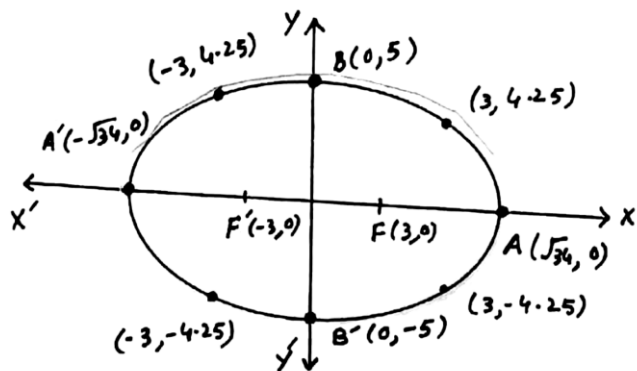Centre $= \left(\frac{3-3}{2}, \frac{0+0}{2}\right) = (0,0)$

Vertices $= (\pm a, 0) = (\pm\sqrt{34}, o)$

Co$-vertices \, (0, \pm b) = (0, \pm 5)$

$i.e \, B(0,5), B'(0,-5)$

Length of letusrect:

$$\frac{2b^2}{a} = \frac{2(25)}{\sqrt{34}} = \frac{50}{\sqrt{34}}$$

(ii)

$Foci(0,-1) \, and \, (0,-5), major \, axis \, of \, length \, 6.$

Solution:

$$centre = midpoint \, of \, foci$$

$$(h, k) = \left(\frac{0+0}{2}, \frac{-1-5}{2}\right) = (0,-3)$$

$$\because 2C = Distance \, b\backslash w \, foci$$

$$2C = \sqrt{(0-0) + (-5+1)^2} = \sqrt{16} = 4$$

$\Rightarrow \, 2C = 4 \Rightarrow C = 2 \Rightarrow C^2 = 4$

$$\because length \, of \, major \, axis = 2a = 6$$

$$\Rightarrow a = 3 \Rightarrow a^2 = 9$$

$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 = 9 - 4 = 5$

$\because foci \, have \, same \, abscissa, so \, major \, axis \, is \, along$

$$y - axis \, is \, along \, y - axis$$

Now eq. of ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \rightarrow (i)$$

$$\because h = 0, k = -3, a^2 = a, b^2 = 5$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y+3)^2}{a^2} = 1 \rightarrow (i)$$

$$\because h = 0, k = -3, a^2 = a, b^2 = 5$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

Sketch:

Vertices$(X = 0, Y = \pm a)$

$\Rightarrow \, X = x \Rightarrow x = 0, y = \pm a$

$\Rightarrow \, y + 3 = \pm 3 \Rightarrow y = -3 \pm 3 = 0, -6$

So vertex $are \, A(0.0) \, A'(0, -6)$

Co-vertices$(X = \pm b, Y = 0)$

$$\because X = x \rightarrow x = \pm b \Rightarrow x = \pm\sqrt{5}$$

$$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$

So $co - vertices \, are \, (\sqrt{5}, -3), (-\sqrt{5}, -3)$

Length of $letusrectum$:

$$\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$$
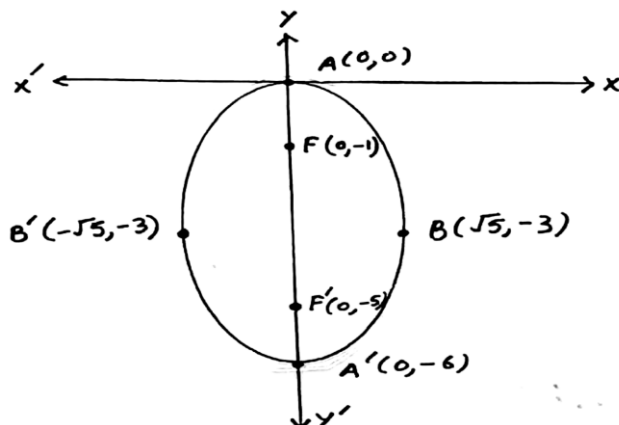
(iii) $Foci \, (\pm 3\sqrt{3}, 0) \, and \, Vertices \, (\pm 6, 0)$

Solution:

$$Foci(\pm c, o) = (\pm 3\sqrt{3}, 0)$$

$\Rightarrow \, C = 3\sqrt{3} \Rightarrow C^2 = 27$

$\Rightarrow \, Centre = Midpoint \, of \, foci$

$$= \left(\frac{3\sqrt{3} - 3\sqrt{3}}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$vertices = (\pm a, 0) = (\pm 6, 0)$$

$\Rightarrow \, a = 6 \Rightarrow a^2 = 36$

$$\because Foci = (\pm 3\sqrt{3}, 0))$$

$\Rightarrow \, same \, ordinates, so \, major \, axis \, is \, along \, x - axis$

So,

$$ellipse \, is \, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \, \frac{x^2}{36} + \frac{y^2}{9} = 1$

Sketch:

Vertices $= (\pm 6, 0) i.e \, A(6, 0) A'(-6, 0)$

Co-vertices $= (0, \pm b) = (0, \pm 3)$

$i.e \, B(o, 3), B'(0, -3)$

Length of $letusrectum = \frac{2b^2}{a} = \frac{2(9)}{6} = 3$

(iv) *Vertices* $(-1, 1)$ *and* $(5, 1)$; *Foci* $(4, 1)$ *and* $(0, 1)$

Solution:

$$\because Foci = F(4,1), F'(0,1)$$

Centre = *midpoint of foci*

$$(h, k) = \left(\frac{4+0}{2}, \frac{1+!}{2}\right) = (2,1)$$

$2C = Dist. \, b\backslash w \, vertics$

$$2a = \sqrt{(5+1)^2 + (1-1)^2} = 6$$

$\Rightarrow \ 2a = 6 \Rightarrow a = 3 \ or \ a^2 = 9$

$$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$$
$$\Rightarrow b^2 = 9 - 4 \Rightarrow b^2 = 5$$

$\because$ *foci have same ordinate so major axis is along*
x − axis.

$Eq.$ *Of ellipse is* $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$$\because h = 2, k = 1, a^2 = 9, b^2 = 5 \ so$$

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$

Sketch:

Vertices: $A(5,1), A'(-1,1)$
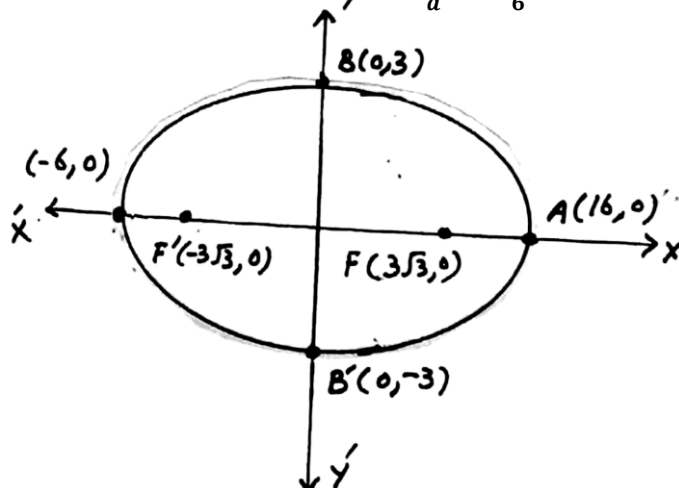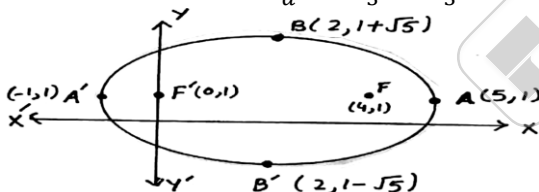
Co-vertices $(X = 0, Y = \pm b)$

$$\because X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y - 1 \Rightarrow y - 1 = \pm b \Rightarrow y = 1 \pm \sqrt{5}$$

Covertices are $B(2, 1+\sqrt{5}), B'(2, 1-\sqrt{5})$

Length of letusrectum $\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$

(v)

Foci $(\pm\sqrt{5}, o)$ *and passing through point* $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution:

foci $= (\pm c, o) = (\pm\sqrt{5}, 0)$

$\Rightarrow \ c = \sqrt{5} \Rightarrow c^2 = 5$

As $Foci : F(\sqrt{5}, 0), F'(-\sqrt{5}, 0)$

Centre = *midpoint of foci*

$$= \left(\frac{\sqrt{5} - \sqrt{5}}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$\because c^2 = a^2 - b^2 \Rightarrow a^2 - b^2 = 5$$
$$\Rightarrow a^2 = b^2 + 5 \to (i)$$

$\because$ *ellipse* (ii) *passes through* $\left(\frac{3}{2}, \sqrt{3}\right)$

$$so \ (ii) \Rightarrow \frac{\left(\frac{3}{2}\right)^2}{a^2} + \frac{\left(\sqrt{3}\right)^2}{b^2} = 1$$

$\Rightarrow \ \frac{9}{4a^2} + \frac{3}{b^2} = 1$

$\Rightarrow \ 9b^2 + 12a^2 = 4a^2b^2 \ (\times \ by \ 4a^2b^2)$

$\Rightarrow \ 9b^2 + 12(b^2 + 5) = 4(b^2 + 5)b^2 \ (\because a^2 = b^2 + 5)$

$\Rightarrow \ 9b^2 + 12b^2 + 60 = 4b^4 + 20b^2$

$\Rightarrow \ 21b^2 - 20b^2 + 60 = 4b^2$

$Or \ 4b^4 - b^2 - 60 = 0$

$$4b^2 - 16b^2 + 15b^2 - 60 = 0$$
$$4b^2(b^2 - 4) + 15(b^2 + 15) = 0$$
$$b^2 - 4, 4b^2 + 15 = 0$$

$\Rightarrow \ b^2 = 4, b^2 = -\frac{15}{4} (impossible)$

$$\because b^2 = 4 \Rightarrow a^2 = 4 + 5 \ (by(i)$$

$\Rightarrow \ a^2 = 9$

$So \ (ii)$ *becomes*

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Sketch:

Vertices $= (\pm a, 0) = (\pm 3, o)$

$\Rightarrow \ A(3,0), A'(-3,0)$

Co-vertices $= (0, \pm b) = (0, \pm 2)$

$\Rightarrow \ B(0,2), B'(o, -2)$

Length of leturectum : $\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$
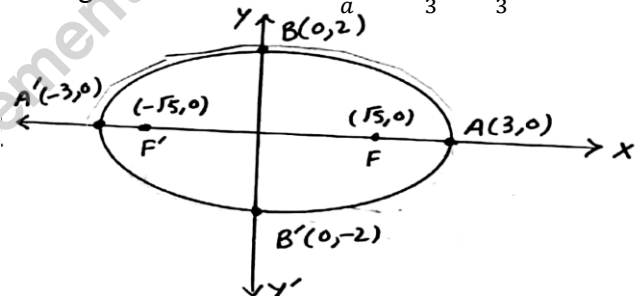
$vi)$ *vertics* $(o, \pm 5)$ *and eccentricity* $\left(\frac{3}{5}\right)$

Solution:

$$Vertices = (0, \pm a) = (0, \pm 5)$$

$\Rightarrow \ a = 5, a^2 = 25$

centre = midpont of vertices

$$= \left(\frac{0+0}{2}, \frac{5-5}{2}\right) = (0,0)$$

$$\because eccebtricity = l = \frac{3}{5}$$

$$\because c = ae \Rightarrow c = 5\left(\frac{3}{5}\right) = 3$$
$$\Rightarrow c^2 = 9$$

$$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 = 25 - 9$$
$$\Rightarrow b^2 = 16$$

$\because$ *vertices have same abscissa*
so major axis is y
$\qquad$ *− axis. and eq. of ellipse is*

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$\Rightarrow \ \frac{x^2}{16} + \frac{y^2}{25} = 1$

Sketch:

Vertices:$(o, \pm 5) \Rightarrow A(0,5) \, A'(0,-5)$

Covertices;$(\pm b, o) = (\pm 4, 0)$

$\Rightarrow B(4,0), B'(-4,0)$

Length $of \, letusrectum: \dfrac{2b^2}{a}$

$$= \dfrac{2(16)}{5} = \dfrac{32}{5}$$

(vii) Centre

$(o, o) \, , focus \, (0, -3) \, and \, Vertex(0, 4)$

Solution:

Centre $= (0,0), Focus = f'(0,-3)$

So other focus $F(0,3)$

$\Rightarrow foci = (0, \pm c) = (0, \pm 3)$

$\Rightarrow c = 3 \Rightarrow c^2 = 9$

Also vertex $= (0,4)$

Other vertex $= (0, -4)$

$\Rightarrow vertices = (0, \pm a) = (o, \pm 4)$

$\Rightarrow a = 4 \, or \, a^2 = 16$

$\Rightarrow c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$

$\qquad or \, b^2 = 16 - 9 = 7$

$\because foci \, have \, same \, abscissa \, so \, eq. \, of \, ellipse \, is$

$$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \qquad (\because major \, is \, y - axis)$$

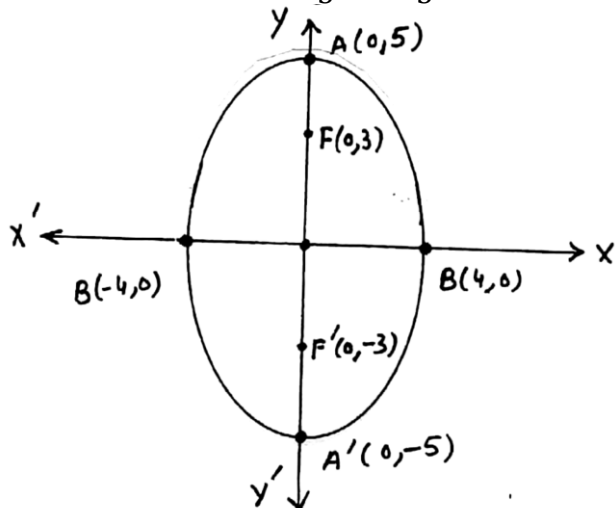$\Rightarrow \dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$

Sketch:

Vertices $= (0, \pm a) = (0, \pm 4)$

$\Rightarrow A(0,4), A'(0, -4)$

Co-vertices$= (\pm b, o) = (\pm \sqrt{7}, 0)$

$\Rightarrow B(\sqrt{7}, 0), B'(-\sqrt{7}, o)$

Length of $letusrectum = \dfrac{2b^2}{a} = \dfrac{2(7)}{4} = \dfrac{7}{2}$
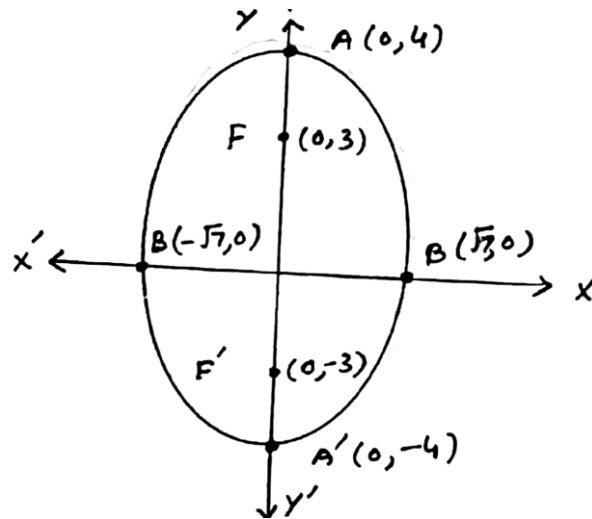
(viii) Centre $(2,2)$ **major axis parallel to** $y - axis$ **and length 8 units, minor axis parallel to** $x - axis$ **and of length 6 units.**

Solution:

Centre $= (h, k) = (2,2)$

Length of major axis $= 2a = 8$

$\Rightarrow a = 4 \Rightarrow a^2 = 16$

Length of minor axis $= 2b = 6$

$\Rightarrow b = 3 \Rightarrow b^2 = 9$

$\qquad \because c^2 = a^2 - b^2 = 16 - 9 = 7$

$\Rightarrow c = \sqrt{7}$

Equation of Ellipse is

$$\dfrac{(x - h)^2}{b^2} + \dfrac{(y - k)^2}{a^2} = 1$$

$$(\because major \, axis \, is \, || to \, y - axis.)$$

$$Put \, h = 2, k = 2, b^2 = 9, a^2 = 16$$

$\Rightarrow \dfrac{(x-2)^2}{9} + \dfrac{(y-2)^2}{16} = 1$

Sketch:

Vertices: $(X = 0, Y = \pm a)$

$\qquad \because X = x - 2 \Rightarrow x - 2 \Rightarrow x = 2$

$\qquad Y = y - 2 \Rightarrow y - 2 = \pm 2$

$\qquad \Rightarrow y - 2 = \pm 4 \Rightarrow y = 2 \pm 4$

$\qquad or \, y = 6, -2$

So $vertices \, area \, A(2,6), A'(2, -2)$

Covertices: $(X = \pm b, Y = 0)$

$\qquad \because X = x - 2 \Rightarrow x - 2 = \pm b$

$\qquad \Rightarrow x - 2 = \pm 3 \Rightarrow x = 2 \pm 3$

$\qquad \Rightarrow x = 5, -1$

$\qquad Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

$\qquad so \, converties \, are \, B(5,2), B'(-1,2)$

Foci: $(X = 0, Y = \pm c)$

$\qquad X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

$\qquad Y = y - 2 \Rightarrow y - 2 = \pm c$

$\Rightarrow y - 2 = \pm \sqrt{7} \Rightarrow y = 2 \pm \sqrt{7}$

$\qquad So Foci \, are : F(2, 2 + \sqrt{7})$

$$F'(2, 2 - \sqrt{7})$$

Length of $letusrectum$: $\frac{2b^2}{a} = \frac{2(9)}{4} = \frac{9}{2}$

(ix) Centre (0,0) ,symmetric with respect to both the axis and passing through the points $(2, 3)$ and $(6, 1)$

Solution:

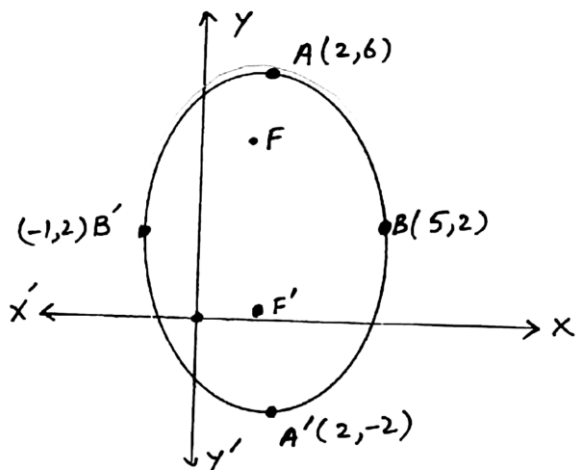Centre $= (0,0), symmetric\ w.r.t\ both\ axis$

Eq. of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ $(i) passsses\ through\ (2,3) so$

$\frac{(2)^2}{a^2} + \frac{(3)^2}{b^2} = 1 \Rightarrow \frac{4}{a^2} + \frac{9}{b^2} = 1$

$\Rightarrow 4b^2 + 9a^2 = a^2 b^2 \rightarrow (ii)$

Also $(i) passes\ through\ (6,1) so$

$\frac{(6)^2}{a^2} + \frac{(1)^2}{b^2} = 1 \Rightarrow \frac{36}{a^2} + \frac{1}{b^2} = 1$

$\Rightarrow 36b^2 + a^2 = a^2 b^2 \rightarrow (iii$

$\Rightarrow$ By$(ii)\ and\ (iii)$

$36b^2 + a^2 = 4b^2 + 9a^2$

$\Rightarrow 32b^2 = 8a^2$

$\Rightarrow a^2 = 4b^2 \rightarrow (iv)\ put\ in\ (ii)$

$4b^2 + 9(4b^2) = (4b^2)b^2$

$\Rightarrow 40b^2 = 4b^4$

$or\ 10 = b^2 (\div\ by\ b^2)$

$\Rightarrow b = \sqrt{10}$

$(iv) \Rightarrow a^2 = 4(10) \Rightarrow a^2 = 40$

$or\ a = \sqrt{40} = 2\sqrt{10}$

$req.ellipse\ (i) becomes\ as$

$\frac{x^2}{40} + \frac{y^2}{10} = 1$

Sketch:

$vertices: (\pm a, 0) = (\pm 2\sqrt{10}, 0)$

$A(2\sqrt{10}, o), A'(-2\sqrt{10}, 0)$

$covertices: (0, \pm b) = (0, \pm\sqrt{10})$

$B(0, \sqrt{10}), B'(o, -\sqrt{10})$

Length of $letusrectum = \frac{2b^2}{a}$

$= \frac{2(10)}{2\sqrt{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}$
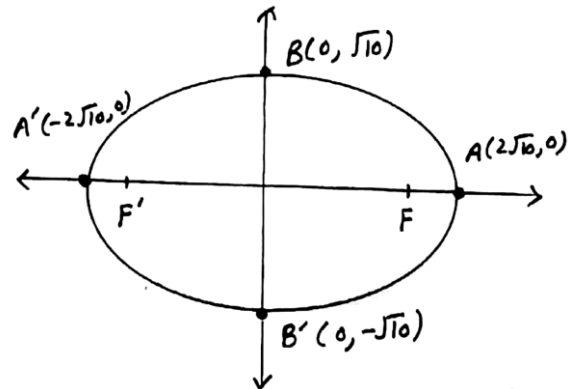
(x) Centre (0,0) Major axis horizontal, the points $(3, 1)$ and $(4, 0)$ lie on the graph.

Solution:

Centre $= (0,0)$

Major $axis\ horizonal\ (i.e\ x-axis) Eq.of\ ellipse\ is$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (i)$

$\because$ $(i)\ passes\ through\ (3,1) so\ \frac{(3)^2}{a^2} + \frac{(1)^2}{b^2} = 1$

$\Rightarrow \frac{9}{a^2} + \frac{1}{b^2} = 1$

$\Rightarrow 9b^2 + a^2 = a^2 b^2 \rightarrow (ii)$

Also $(i) passes\ through\ (4,0) so$

$\frac{(4)^2}{a^2} + \frac{(0)^2}{b^2} = \frac{9}{a^2} + \frac{1}{b^2} = 1$

$\Rightarrow 9b^2 + a^2 = a^2 b^2 \rightarrow (ii)$

Also $(i) passes\ through\ (4,0) so$

$\frac{(4)^2}{a^2} + \frac{(0)^2}{b^2} = 1 \Rightarrow \frac{16}{a^2} = 1$

$\Rightarrow a^2 = 16 \Rightarrow a = 4$

Now $(ii) \Rightarrow 9b^2 + 16 = 16b^2$

$\Rightarrow 16 = 16b^2 - 9b^2$

$\Rightarrow 16 = 7b^2 \Rightarrow b^2 = \frac{16}{7}$

$\Rightarrow or\ b = \frac{4}{\sqrt{7}}$

$so\ req.ellipse\ (i) becomes\ \frac{x^2}{16} + \frac{y^2}{\frac{16}{7}} = 1$

Sketch:

$Vertices: A(4,0), A'(-4,0)$
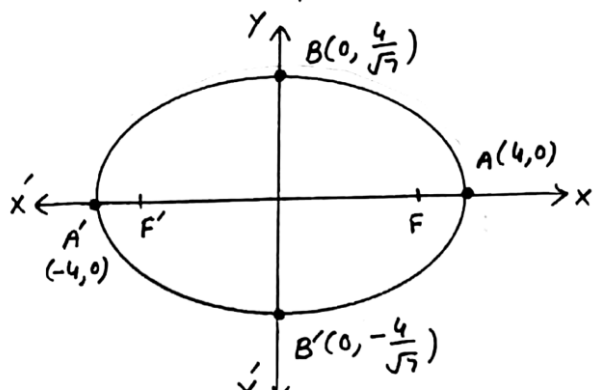
$Convertices:$

$B\left(0, \frac{4}{\sqrt{7}}\right), B'\left(0, -\frac{4}{\sqrt{7}}\right)$

Length of $latusrectum = \frac{2b^2}{a} = \frac{2\left(\frac{16}{7}\right)}{4} = \frac{8}{7}$

Question No.2

Find the Centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

(i) $x^2 + 4y^2 = 16$

Solution:

$$x^2 + 4y^2 = 16$$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1 \ (\div \ by \ 16)$

$$compare \ with \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow a^2 = 16 \Rightarrow a = 4$

$\qquad b^2 = 4 \Rightarrow b = 2$

$\qquad \because c^2 = a^2 - b^2 = 16 - 4 = 12$

$\Rightarrow c = \sqrt{12} \Rightarrow c = 2\sqrt{3}$

Centre: $(0,0)$

Foci$= (\pm c, 0) = (\pm 2\sqrt{3}, 0)$

Eccentricity: e=$\frac{c}{a} = \frac{2\sqrt{3}}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$

$\qquad$ *Vertices*: $(\pm a, 0) = (\pm 4, 0)$

Directrices: $x = \pm \frac{a}{e}$

$$\Rightarrow x = \pm \frac{4}{\frac{\sqrt{3}}{2}} \Rightarrow x = \pm \frac{8}{\sqrt{3}}$$

(ii) $9x^2 + y^2 = 18$

solution:

$$9x^2 + y^2 = 18$$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{18} = 1 \ (\div \ by \ 18)$

$$compare \ with \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow a^2 = 18 \Rightarrow a = \sqrt{18} = 3\sqrt{2}$

$\qquad b^2 = 2 \Rightarrow b = \sqrt{2}$

$\qquad \because c^2 = a^2 - b^2 = 18 - 2 = 16$

$\Rightarrow c = 4$

Centre: $(0,0)$

Foci$= (o, \pm c) = (o, \pm 4)$

Eccentricity: e=$\frac{c}{a} = \frac{4}{3\sqrt{2}} \Rightarrow e = \frac{2 \times 2}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$

$\qquad$ *Vertices*: $(0, \pm a) = (0, \pm 3\sqrt{2})$

Directrices: $y = \pm \frac{a}{e}$

$$\Rightarrow y = \pm \frac{\frac{3\sqrt{2}}{2\sqrt{2}}}{\frac{2\sqrt{2}}{3}} \Rightarrow y = \pm \frac{9}{2}$$

(iii) $25x^2 + 9y^2 = 225$

Solution:

$$25x^2 + 9y^2 = 225$$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1 \ (\div \ by \ 225)$

$$compare \ with \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow a^2 = 5 \Rightarrow a = \sqrt{5}$

$\qquad b^2 = 9 \Rightarrow b = 3$

$\qquad \because c^2 = a^2 - b^2 = 25 - 9 = 16$

$\Rightarrow c = 4$

Centre: $(0,0)$

Foci$= (o, \pm c) = (o, \pm 4)$

Eccentricity: e=$\frac{c}{a} = \frac{4}{5}$

$\qquad$ *Vertices*: $(0, \pm a) = (0, \pm 5)$

Directrices: $y = \pm \frac{a}{e}$

$$\Rightarrow y = \pm \frac{5}{\frac{4}{5}} \Rightarrow y = \pm \frac{25}{4}$$

(iv) $\frac{(2x-1)^2}{16} + \frac{(y+2)^2}{16} = 1$

Solution:

$$\frac{(2x - 1)^2}{16} + \frac{(y + 2)^2}{16} = 1$$

$\Rightarrow \frac{\left[2\left(x - \frac{1}{2}\right)\right]^2}{16} + \frac{(y+2)^2}{16} = 1 \rightarrow (i)$

$$compare \ with \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$\Rightarrow X = x - \frac{1}{2}, Y = y + 2$

$\Rightarrow a^2 = 16 \Rightarrow a = 4$

$\qquad b^2 = 4 \Rightarrow b = 2$

$\qquad \because c^2 = a^2 - b^2 = 16 - 2 = 12$

$\Rightarrow c = \sqrt{12} = 2\sqrt{3}$

Centre: $(X = 0, Y = 0)$

$\qquad \because X = x - \frac{1}{2} \Rightarrow x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$

$\Rightarrow$ *so centre* $\left(\frac{1}{2}, -2\right)$

Foci$=$ *Major axis is along y $-$ axis*

Foci $(X = 0, Y = \pm c)$

$\qquad \because X = x - \frac{1}{2} \Rightarrow x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$

$\qquad Y = y + 2 \Rightarrow y + 2 \Rightarrow y + 2 = \pm c$

$\Rightarrow y + 2 = \pm 2\sqrt{3}$

$\Rightarrow y = -2 \pm 2\sqrt{3}$

Hence $foci \left(\frac{1}{2}, -2 \pm 2\sqrt{3}\right)$

Eccentricity: $e = \frac{c}{a} = \frac{2\sqrt{3}}{4} \Rightarrow e = \frac{3}{\sqrt{2}}$

Vertices: $(X = 0, Y = \pm a)$

$\quad \because X = x - \frac{1}{2} \Rightarrow x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$

$\quad Y = y + 2 \Rightarrow y + 2 = \pm a \Rightarrow y + 2 = \pm 4$

$\Rightarrow Y = y + 2 \Rightarrow y + 2 = \pm a \Rightarrow y + 2 = \pm 4$

$\Rightarrow y = -2 \pm 4 \Rightarrow y = 2, -6$

Hence vertices are $\left(\frac{1}{2}, 2\right), \left(\frac{1}{2}, -6\right)$

Directrices: $y = \pm \frac{a}{e} \Rightarrow y + 2 = \pm \frac{4}{\frac{\sqrt{3}}{2}}$

$\qquad y + 2 = \pm \frac{8}{\sqrt{3}} \Rightarrow y = -2 \pm \frac{8}{\sqrt{3}}$

(ii) $x^2 + 16x + 4y + 76 = 0$

Solution:

$\qquad x^2 + 16x + 4y^2 - 16y + 76 = 0$

$\Rightarrow x^2 + 16x + 4(y^2 - 4y) = -76$

$\Rightarrow x^2 + 2(8)(x) + (8)^2 + 4(y^2 - 4y + 4 - 4) = -76 + (8)^2$

$\Rightarrow (x+8)^2 + 4(y-2)^2 - 16 = -12$

$\Rightarrow (x+8)^2 + 4((y-2)^2 - 4) = -76 + 64$

$\Rightarrow (x+8)^2 + 4(y-2)^2 - 16 = -12$

$\Rightarrow (x+8)^2 + 4(y-2)^2 = 4$

$\Rightarrow \frac{(x+8)^2}{4} + \frac{(y-2)^2}{1} = 1$

$\Rightarrow$ compare with $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

$\Rightarrow X = x + 8, \quad Y = y - 2 \qquad a^2 = 4$

$\Rightarrow a = 2 \because c^2 = a^2 - b^2 \Rightarrow c^2 = 4 - 1 = 3$

$\Rightarrow c = \sqrt{3}$

$\Rightarrow b^2 = 1 \Rightarrow b = 1$

Centre: $(X = 0, Y = 0)$

$\quad \because X = x + 8 \Rightarrow x + 8 = 0 \Rightarrow x = -8$

$\quad Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

$\Rightarrow$ so centre $(-8, 2)$

Foci= Major axis is along $x - axis$

Foci $(X = \pm c, Y = 0)$

$\quad \because X = x + 8 \Rightarrow x + 8 = \pm c$

$\quad x + 8 = \pm\sqrt{3} \Rightarrow x = -8 \pm \sqrt{3}$

$\quad Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$

Hence $foci \left(-8 \pm \sqrt{3}, 2\right)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{3}}{2}$

Vertices: $(X = \pm a, Y = 0)$

$\quad \because X = x + 8 \Rightarrow x + 8 = \pm a$

$\Rightarrow x + 8 = \pm 2 \Rightarrow x = -8 \pm 2$

$\qquad \Rightarrow x = -6, -10$

$\quad Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$

Hence vertices are $(-6, 2), (-10, 2)$

Directrices: $X = \pm \frac{a}{e}$

$\qquad X = x + 8 \Rightarrow x + 8 = \pm \frac{a}{e}$

$\Rightarrow x + 8 = \pm \frac{2}{\frac{\sqrt{3}}{2}}$

$\Rightarrow x = -8 \pm \frac{4}{\sqrt{3}}$

(vi) $25x^2 + 4y^2 - 250x - 16y + 541 = 0$

Solution:

$25x^2 + 4y^2 - 250x - 16y + 541 = 0$

$25(x^2 - 10x) + 4(y^2 - 4y) = -541$

$25(x^2 - 10x + 25 - 25) + 4(y^2 - 4y + 4 - 4) = -541$

$25[(x - 5)^2 - 25] + 4[(y - 2) - 4] = -541$

$25(x - 5)^2 - 625 + 4(y - 2)^2 - 16 = -541$

$25(x - 5)^2 + 4(y - 2)^2 = -451 + 625 + 16$

$\Rightarrow 25(x - 5)^2 + 4(y - 2)^2 = 100$

$or \quad \frac{(x - 5)^2}{4} + \frac{(y - 2)^2}{25} = 1 \quad (\div by \ 100)$

$\Rightarrow$ compare with $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

$\Rightarrow X = x - 5, \quad Y = y - 2 \qquad a^2 = 25$

$\Rightarrow a = 5 \because c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 4 = 21$

$\Rightarrow c = \sqrt{21}$

$\Rightarrow b^2 = 4 \Rightarrow b = 2$

Centre: $(X = 0, Y = 0)$

$\quad \because X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$

$\quad Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

$\Rightarrow$ so centre $(5, 2)$

Foci= Major axis is along $y - axis$

Foci $(X = 0, Y = \pm c)$

$\quad \because X = x - 5 \Rightarrow x - 5 = x = 5$

$\quad Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$

Hence $foci \left(5, 2 \pm \sqrt{21}\right)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{21}}{5}$

Vertices: $(X = \pm 0, Y = a)$

$\quad \because X = x + 5 \Rightarrow x + 5$

$\qquad x = -5$

$\quad Y = y - 2 \Rightarrow y - 2 = a \Rightarrow y - 2 = \pm 5$

$\qquad \Rightarrow y = 2 \pm 5 \Rightarrow y = -3, 7$

Hence vertices are $(5, 7), (7, -3)$

Directrices: $Y = \pm \frac{a}{e}$

$\qquad Y = y - 2 \Rightarrow y - 2 = \pm \frac{a}{e}$

$\Rightarrow y - 2 = \pm \frac{5}{\frac{\sqrt{21}}{5}} \qquad \left(\begin{matrix} \because a = 5 \\ e = \frac{\sqrt{21}}{5} \end{matrix}\right)$

$\Rightarrow$ $y = 2 \pm 25/\sqrt{21}$

Question No.3 let "a" be a positive number and $0 < c <$

$a.$ let $F(c, o)$ and $F'(-c, 0)$ be two $given$ $points.$ $prove$ $that$ the $locus$ of $points$ $P(x, y)$ $such$ $that$ $|PF| + |PF'| =$ $2a$ is an $Ellipse.$

$Solution:$

$Given$ $that$

$$|PF'| + |PF|$$
$$= 2a \quad F'(-c, o) \ F(c, o), P(x, y)$$

$\Rightarrow$ $\sqrt{(x + c)^2 + (y - o)^2} =$
$\sqrt{(x - c)^2 + (y - o)^2}$
$$= 2a$$

$\Rightarrow$ $\sqrt{(x + c)^2 + (y)^2} = 2a - \sqrt{(x - c)^2 + (y)^2}$

Squaring both sides by

$\Rightarrow$ $x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x - c)^2 + (y)^2}$

$\Rightarrow$ $2cx + 2cx - 4a^2 = -4a\sqrt{x^2 + c^2 - 2cx + y^2}$

$\Rightarrow$ $4(cx - a^2) = -4a\sqrt{x^2 + y^2 + c^2 - 2cx}$

$\Rightarrow$ $cx - a^2 = -a\sqrt{x^2 + y^2 + c^2 - 2cx}$

$\Rightarrow$ $a^2 - cx = a\sqrt{x^2 + y^2 + c^2 - 2cx}$

$Again$ $squaring$

$$a^2 + c^2x^2 - 2a^2cx = a^2(x^2 + y^2 + c^2 - 2cx)$$
$$a^4 + c^2x^2 - 2a^2cx = a^2x^2 + a^2y^2 + a62 \ c^2 - 2a^2cx$$

$\Rightarrow$ $c^2a^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$
$$\div by \ a^2(a^2 - c^2)$$

$\Rightarrow$ $\frac{x^2}{a^2(a^2-c^2)} + \frac{y^2}{a^2-c^2} = 1$

$\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\because a^2 - b^2 = c^2)$
$$\Rightarrow (a^2 - c^2 = b^2)$$

Which is an ellipse. Hence proved.

Question No.4 let a be a $positive$ $number$ and $0 < c < a.$ let $F(0,0)$ and $F'(1,1)$ be two $given$ $points$ $P(x, y)$ is an $ellipse,$ $such$ $that$ $|PF| + |PF'| = 2$ $find$ $equation$ of $ellipse.$

Solution:
$$0 < c < a$$
$$P(x, y), F(0,0), F'(1,1)$$

Given that
$$|PF| + |PF'| = 2$$

$\Rightarrow$ $\sqrt{(x - 0)^2 + (y - o)^2} =$
$\sqrt{(x - 1)^2 + (y - 1)^2}$

$\Rightarrow$ $\sqrt{x^2 + y^2} + \sqrt{x^2 + 1 - 2x + y^2 + 1 - 2y} = 2$

$\Rightarrow$ $\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 - 2x - 2y + 2} = 2$

$\Rightarrow$ $\sqrt{x^2 + y^2 - 2x - 2y + 2} = 2 - \sqrt{x^2 + y^2}$

$squring$ $both$ $sides$

$$x^2 + y^2 - 2x - 2y + 2$$
$$= 4 + x^2 + y^2 - 4\sqrt{x^2 + y^2}$$
$$-2x - 2y + 2 = 4 - 4\sqrt{x^2 + y^2}$$
$$\Rightarrow 4\sqrt{x^2 + y^2} = 2 + 2x + 2y$$
$$4(x^2 + y^2) = x^2 + y^2 + 1 + 2xy + 2y + 2x = 0$$

$\Rightarrow$ $4x^2 + 4y^2 - x^2 - y^2 - 2xy - 2x - 2y - 1 = 0$

or $3x^2 + 3y^2 - 2x - 2y - 2xy - 1 = 0$

Which is ellipse.

Question No.5 prove that letusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Proof:
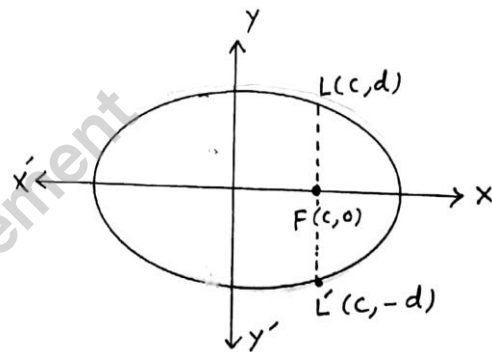
Let

$L(c, d)$ and $L'(c, d)$ be $points$ of $Letusrectum$ LL' of

Given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (i)$

$\because$ $Length$ of $letusrectum = |LL'| = 2d$

$\because$ $L(c, d)$ $lies$ on (i) so

$$\frac{c^2}{a^2} + \frac{d^2}{b^2} = 1$$

$\Rightarrow$ $c^2b^2 + a^2d^2 = a^2b^2$

$\Rightarrow$ $a^2d^2 = a^2b^2 - b^2c^2$

$\Rightarrow$ $a^2d^2 = b^2(a^2 - c^2)$

$\Rightarrow$ $a^2d^2 = b^2(b^2)$

$\because$ $c^2 = a^2 - b^2$

$\Rightarrow$ $b^2 = a^2 - c^2$

or $d^2 = \frac{b^4}{a^2} \Rightarrow d = \sqrt{\frac{b^4}{a^2}}$

$$\Rightarrow d = \frac{b^2}{a} \quad so$$

$Length$ of $letusrectum = 2d = \frac{2b^2}{a}$

Hence proved.

Question No.6 the major axis of an ellipse in standard form lies along the x-axis and has length of the minor axis. Write an equation of ellipse.

Solution:

Given $2a = 4\sqrt{2}$

$\Rightarrow a = 2\sqrt{2}$

$\Rightarrow a^2 = 8$

$\qquad$ *also given that* $2c = 2b \Rightarrow c = b$

$\qquad$ *Now using* $c^2 = a^2 - b^2$

$\qquad\qquad b^2 = a^2 - b^2$

$\qquad\qquad \Rightarrow 2b^2 = a^2$

$\qquad\qquad \Rightarrow 2b^2 = 8 \Rightarrow b^2 = 4$

$\qquad$ *Thus eq. of ellipse is* $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$

Question No.7 an asteroid has elliptic orbit with the sun at one focus, its distance from the sun ranges 17milions to 183 million miles.

Write an equation of the orbit of the asteroid.

Solution:

$\qquad\qquad a - c = 17 \to (i)$

$\qquad\qquad a + c = 183 \to (ii)$

By $(i) + (ii)$

$\qquad\qquad 2a = 200 \Rightarrow a = 100$

$\qquad (ii) \Rightarrow 100 + c = 183 \Rightarrow c = 83$

Using $c^2 = a^2 - b^2$

$\qquad\qquad b^2 = a^2 - c^2$

$\qquad\qquad b^2 = (100)^2 - (83)^3$

$\qquad\qquad b^2 = 10000 - 6889$

$\qquad\qquad b^2 = 3111$

$\qquad \because$ *the eqution of elliptic orbits is*

$\qquad\qquad \dfrac{x^2}{10,000} + \dfrac{y^2}{3111} = 1$

Question No.8 An arch in the shape of a semi –ellipse is 90m wide at the base and 30m high at the Centre. At what distance from the Centre is the arch $20\sqrt{2}m\ high$?

Solution:

Here $2a = 90 \Rightarrow a = 45$ *and* $b = 30$

$\because$ *Eq. of ellipse is*

$\qquad\qquad \dfrac{x^2}{(45)^2} + \dfrac{y^2}{(30)^2} = 1 \to (i)$

At the high

$20\sqrt{2}m$ *let* x_1 *be the distance from the Centre then the point* $(x_1, 20\sqrt{2})$ *lies on* Ellipse (i)

$$\frac{x_1^2}{(45)^2} + \frac{(20\sqrt{2})^2}{30} = 1$$

$\Rightarrow \dfrac{x_1^2}{2025} + \dfrac{800}{900} = 1$

$\Rightarrow \dfrac{x_1^2}{2025} = 1 - \dfrac{800}{900}$

$\Rightarrow \dfrac{x_1^2}{2025} = 1 - \dfrac{8}{9}$

$\Rightarrow \dfrac{x_1^2}{2025} = \dfrac{1}{9} \Rightarrow x_1^2 = \dfrac{2025}{9}$

$\Rightarrow x_1^2 = 225 \Rightarrow x_1 = \pm 15$

$\Rightarrow x_1 = 15m$ *(neglect* $-ve$ *value of* x_1)

$\qquad \because Req\ distance\ from\ centre = 15m$

Question No9. The moon orbits the earth in an elliptic path with the earth at one focus. The major and minor axes of the orbit are $7,68,806\ km\ and$

$7,6,7,746 km\ respectively,.Find\ the\ greatest$ **And least distance.**

Solution:

Let the earth be at F

Given that $2a = 768,806$

$\qquad\qquad \Rightarrow a = 384403\ km$

$\qquad\qquad 2b = 767,746$

$\Rightarrow b = 383873 km$

Using $c^2 = a^2 - b^2$

$\Rightarrow c^2 = (a - b)(a + b)$

$\Rightarrow c^2 = (530)(768276)$

$\Rightarrow c^2 = 407186280$

$\Rightarrow c = 20178.86$

Now Required greatest distance

$\qquad = a + c = 404582\ km (Approx)$

$\qquad and\ least\ distance = a - c$

$\qquad\qquad = 364224 km (Approx)$

Hyperbola

"A set of all points in a plane such that distance of each point from a fixed point bars a constant ratio (greater than one) to a distance from a fixed line."
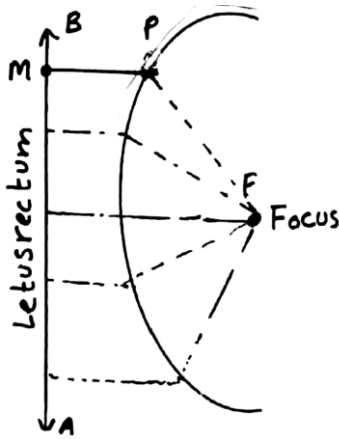
Note:

- Fixed point is called focus.
- Fixed line is called directrix.
- Constant ratio is called eccentricity, dented by e.
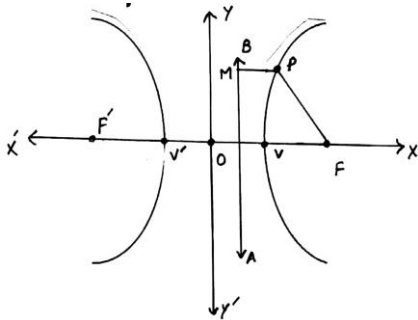- In fig. $\dfrac{|FP|}{|PM|} = e$
- $|FP| = e|PM|$ where $|FP| > |PM|$

$\Rightarrow \dfrac{|FP|}{|PM|} > 1$

$\qquad\qquad so\ for\ hyperbola\ e > 1$

Definition:

1) The midpoint of line segment joining the foci is called Centre of hyperbola. In fig. F and F' are foci and O is Centre.

2) The line passing through the foci of hyperbola is called focal axis or Transverse axis.

3) The line passing through Centre of hyperbola and perpendicular to transverse axis is called conjugate axis. In fig. y-axis is conjugate axis.

4) The points where the hyperbola meets its transverse axis are called vertices of hyperbola.in fig.
   V and V' are vertices of hyperbola.

5) Length of $leturectum$ is $\frac{2b^2}{a}$

6) If the point $asec\theta, bTan\theta)$ lies on hyperbola
   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $\theta \in R$ then $x = asec\theta$
   $y = atan\theta$ are parametric equations of hyperbola

   Standard Equation of hyperbola:
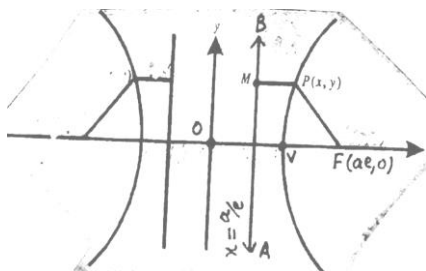
Let
$F(ae, o)$ be focus and AB be directed with equation

be directrix with equation $x = \frac{a}{e} \Rightarrow ex - a = 0$ of req. hyperbola.

Let $|PM|$ be $\perp$
ar distance of any point $P(x, y)$ on
Hyperbola to the $directrix\ ex - a = 0$
By $def.$ of hyperbola

$$\frac{|FP|}{|PM|} = e \Rightarrow |FP| = e|PM|$$

$\Rightarrow \sqrt{(x - ae)^2 + (y - 0)^2} = e\frac{|ex - a|}{\sqrt{(e)^2 + (o)^2}}$

$\Rightarrow \sqrt{x^2 + a^2e^2 - 2aex + y^2} = \frac{e|ex - a|}{e}$

$\Rightarrow$ or $x^2 + a^2e^2 - 2aex + y^2 = |ex - a|^2$

$\Rightarrow x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 + a^2 - 2aex$

$\Rightarrow a^2e^2 - a^2 + y^2 = e^2x^2 - x^2$

$\Rightarrow a^2(e^2 - 1) = (e^2 - 1)x^2 - y^2$

$\Rightarrow (e^2 - 1)x^2 - \frac{y^2}{a^2(e^2 - 1)} = 1 \div by\ a^2(e^2 - 1)$

Or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ let $a^2(e^2 - 1) = b^2$

Note:

$$\because b^2 = a^2(e^2 - 1)$$
$$\Rightarrow b^2 = a^2e^2 - a^2 = (ae)^2 - a^2$$
$$\Rightarrow b^2 = c^2 - a^2 \quad \because ae = c \quad or \quad c^2 = a^2 + b^2$$

Summary of Standard Hyperbolas

Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x = \pm\frac{a}{e}$ or $x = \pm\frac{c}{e^2}$	$y = \pm\frac{a}{e}$ or $y = \pm\frac{c}{e^2}$
Transverse axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Eccentricity	$e = \frac{c}{a} > 1$	$e = \frac{c}{a} > 1$
Centre	$(0, 0)$	$(0, 0)$
Graph		

Exercise 6.6

Question No.1 find an equation of hyperbola with the given data, and sketch its graph

(i) **Centre** $(0,0)$ **, focus** $(6,0)$ **Vertex** $(4,0)$

Solution:

Centre

$(0,0), Focus\ (6,0)\ so\ other\ focus\ (-6,0)$

Foci $(\pm c, 0) = (\pm 6,0) \Rightarrow c = 6$

$\Rightarrow c^2 = 36$

$Vertex = (4,0), other\ vertex (-4,0)$

$Vertices = (\pm a, 0) = (\pm 4,0) \Rightarrow a = 4$

$\Rightarrow a = 16$

$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$

$$\Rightarrow b^2 = 36 - 16 \Rightarrow b^2 = 20 \Rightarrow b = \sqrt{20}$$

$clearly\ from\ foci\ and\ vertces\ x$
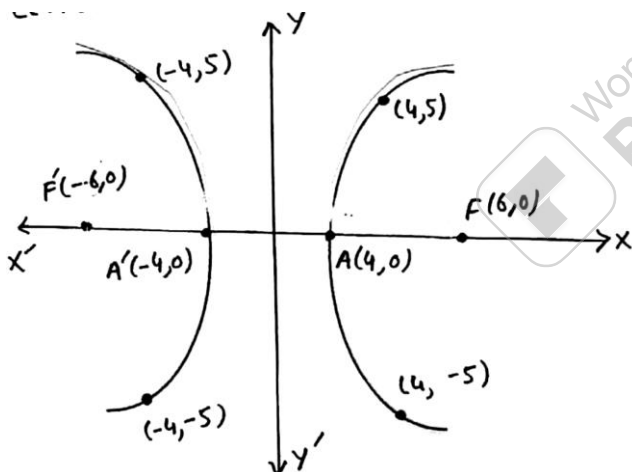$\qquad - axis\ is\ transvrerse\ axis$

So, eq.is

$$\frac{x^2}{a2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{16} - \frac{y^2}{20} = 1$$

Sketch:

$A(4,0), A'(-4,0), F(6,0), F'(-6,0)$

Length of l$letusrectum = \frac{2b^2}{a} = \frac{2(20)}{4} = 10$

$$Centre = (0,0)$$

(ii) **Foci** $(\pm 5, 0)$, **Vertex** $(3, 0)$

Solution:

$Foci\ (\pm c, 0) = (\pm 5,0) i.e\ F(5,0), F'(-5,0)$

$\Rightarrow c = 5, c^2 = 25$

Centre $= midpoint\ of\ foci$

$$= \left(\frac{5-5}{2}, \frac{0+0}{2}\right) = (0,0)$$

$vertex = (3,0), other\ vertex = (-3,0)$

$vertex\ (\pm a, o) = (\pm 3,0) \Rightarrow a = 3$

$$a^2 = 9 \quad \because c^2 = a^2 + b^2$$

$\Rightarrow b^2 = c^2 - a^2 = 25 - 9 = 16$

Clearly from foci and vertices $x - axis$ is

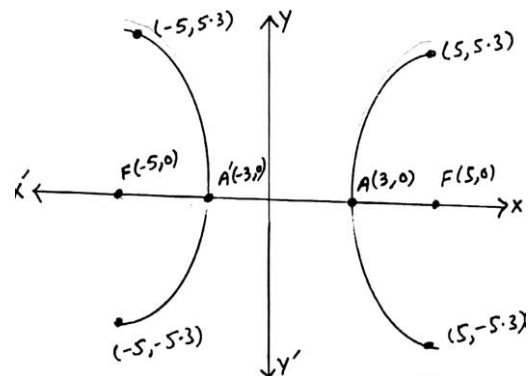Transverse axis. So eq. is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$$

Sketch:

Vertices $F\ (5,0), F'(-5,0)$

Length of $letusrectum: \frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$

$$Centre = (0,0)$$

(iii)

　Foci $(2$
　$\pm 5\sqrt{2}, -7)$, **length of transverse axis** 10

Solution:

$$Foci\ (2 \pm 5\sqrt{2}, -7) \Rightarrow F(2 + 5\sqrt{2}, -7)$$
$$F'(2 - 5\sqrt{2}, -7)$$

$Centre = midpoint\ of\ foci$

$(h, k) = \dfrac{2 + 5\sqrt{2}, +2 - 5\sqrt{2}}{2}, \dfrac{-7-7}{2} = (2, -7)$

Length of transverse axis $F(2 + 5\sqrt{2}, -7)$

And $Centre\ (2, -7) is$

$$c = \sqrt{\left(2 + 5\sqrt{2} - 2\right)^2 + (-7 + 7)^2} = 5\sqrt{2}$$

$\Rightarrow c^2 = \left(5\sqrt{2}\right)^2 = 25(2) \Rightarrow c^2 = 50$

$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$
$$= 50 - 25$$
$$\Rightarrow b^2 = 25$$

$foci\ have\ same\ ordinate\ so\ eq. of\ hyperbola\ is$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$\Rightarrow \frac{(x-2)^2}{25} - \frac{(y+7)^2}{25} = 1$

$$compare\ with\ \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$X = x - 2, Y = y + 7, \qquad a^2 = 25, b^2 = 25$

$\Rightarrow a = 5, b = 5$

Sketch:

$$vertics\ (X = \pm a, Y = 0)$$

$X = x - 2 \Rightarrow x - 2 = \pm 5 \Rightarrow x = 2 \pm 5$

$$x = 7, -3$$

$Y = y + 7 \Rightarrow y + 7 = 0 \Rightarrow y = -7$

$so\ vertices\ (7, -7), (-3, -7)$
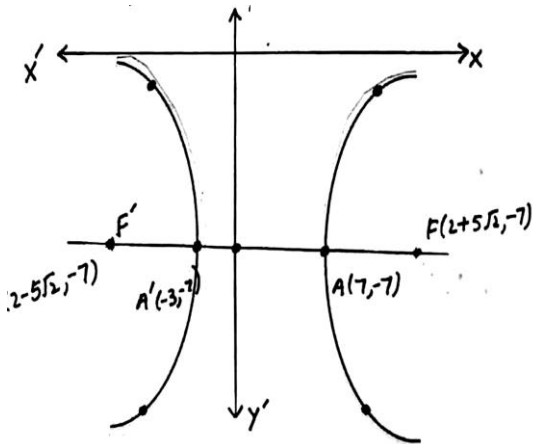
$foci\ (X = \pm c, Y = 0)$

$$X = x - 2 \Rightarrow x - 2 = \pm x$$
$$x = 2 \pm 5\sqrt{2}$$
$$y = 0 \Rightarrow y + 7 = 0y = -7$$
$$so\ foci\ (2 + 5\sqrt{2}, -7), F'(2 - 5\sqrt{2}, -7)$$
$$length\ of\ letusretum : \frac{2b^2}{a} = \frac{2(25)}{5} = 10$$

(iv) Foci $(0, \pm 6), e = 2$

Solution:

$$foci = (0, \pm c) = (0, \pm 6)$$

$$\Rightarrow\ c = 6 \Rightarrow c^2 = 36$$

$$Centre = midpoint\ of\ foci$$
$$= \left(\frac{0 - 0}{2}, \frac{6 - 6}{2}\right) = (0,0)$$
$$\because e = 2 \Rightarrow e = \frac{c}{a}\ or\ a = \frac{c}{e}$$
$$\Rightarrow a = \frac{c}{2} = 3\ \Rightarrow a^2 = 9$$
$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$
$$\Rightarrow b^2 = 36 - 9 = 27$$
$$\because e\ foci\ have\ same\ abscissa.$$

So $y -$

$axis\ is\ trasverse\ axis. so\ eq. of\ hyperbola$
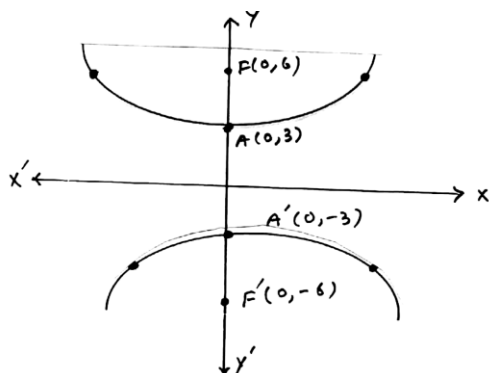
Is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $\Rightarrow \frac{y^2}{9} - \frac{x^2}{27} = 1$

Sketch:

Vertices: $A(0,3), A'(0, -3)$

Foci: $F(0,0),\ \ F'(0, -6)$

Length of $letusrectum = \frac{2b^2}{a} = \frac{2(37)}{3}$
$$= 18\ and\ centre\ (0,0)$$

(v) $Foci\ (0, \pm 9), directed\ y = \pm 4$

Solution:

$$foci = (0, \pm c) = (0, \pm 9)$$
$$\Rightarrow\ c = 9 \Rightarrow c^2 = 81$$

$$Centre = midpoint\ of\ foci$$
$$= \left(\frac{0 + 0}{2}, \frac{9 - 9}{2}\right) = (0,0)$$
$$Directrices: y = \pm 4$$
$$\because\ \ y = \pm \frac{a}{3}$$
$$\Rightarrow \frac{a}{e} = 4 \Rightarrow a = 4e \rightarrow (i) also$$

Also $c = ea$

$$ae = 9\ (\because c = 9)$$
$$\Rightarrow\ e = \frac{9}{a}\ put\ in\ (i)$$
$$\Rightarrow\ a = 4\left(\frac{9}{a}\right) \Rightarrow a^2 = 36 \Rightarrow a = 6$$
$$\Rightarrow\ c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$
$$\Rightarrow\ b^2 = 81 - 36 = 45$$
$$\because foci\ have\ same\ abscissa\ so\ y$$
$$- axis\ is\ transverse\ axis.$$

Thus eq. of req. hyperbola is
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 1$$

Sketch:

Vertices $A(0,6), A'(0, -6)$ $Fcoi: F(0,9)$

Length of $letusrectum : \frac{2b^2}{a} = \frac{2(45)}{6}$
$$= 15\ \ centre\ (0,0)$$
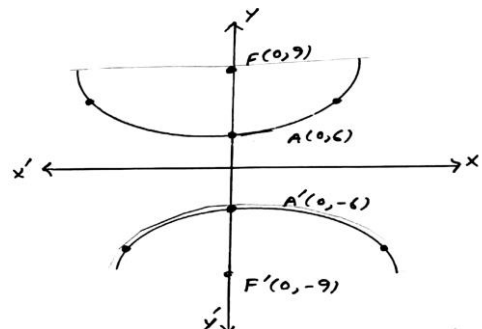
(vi) $Centre\ (2, 2), horizontal\ transverse\ axis\ of$
Length 6 and eccentricity $e = 2$

Solution:

Centre $= (h, k) = (2,3)$

$$length\ of\ transerse\ axis = 2a = 6$$
$$\Rightarrow\ e = 2\ \because c = ea \Rightarrow c = (3)(2) = 6$$
$$or\ c^2 = 36\ \because c^2 = a^2 + b^2$$
$$\Rightarrow b^2 = c^2 - a^2 = 36 - 9 = 27$$
$$\because$$
$$transverse\ axis\ is\ horizonatal\ axis\ i.e$$
$$x - axis\ so\ eq. of\ required\ hyperbola\ is$$
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$
$$\Rightarrow \frac{(x - 2)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

sketch:

vertices: $(X = \pm a, Y = 0)$

$X = x - 2 \Rightarrow x = 2 = \pm a$

$\Rightarrow x - 2 = \pm 3 \Rightarrow x = 2 \pm 3$

or $x = 5, -1$

$Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

so vertices $A(5,2), A'(-1,2)$

Foci: $(X = \pm c, Y = 0)$

$X = x - 2 \Rightarrow x - 2 = \pm 6$

$\Rightarrow x = 2 \pm 6 \Rightarrow x = 8, -4$

$Y = 0 \Rightarrow y - 2 \Rightarrow y = 0$

so foci are $F(8,2), F'(-4,2)$

Centre: $(2,2)$

length of letusrectum $= \dfrac{2b^2}{a} = \dfrac{2(27)}{3}$

$= 18$

(vii) **Vertices** $(2, \pm 3), (0, 5)$ **lies on the curve.**

solution:

vertices $= (2, \pm 3) \Rightarrow A(2,3), A'(2,-3)$

Centre: *midpoint of vertices*

$(h, k) = \left(\dfrac{2 + 2}{2}, \dfrac{3 - 3}{2}\right) = (2,0)$

$\because$ *Distance b\w vertices* $= 2a$

$\Rightarrow 2a = \sqrt{(2 - 2)^2 + (-3 - 3)^2} = \sqrt{36} = 6$

$\Rightarrow a = 3 \Rightarrow a^2 = 9$

$\because$ *vertices have same abscssa so transverse axis is along* $y - axis$. *so eq. of hyperbola is*

$\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$

$\Rightarrow$ (i) *passes throuh* $(0,5)$ *so*

(i) $\dfrac{(5)^2}{9} - \dfrac{(0 - 2)^2}{b^2} = 1$

$\Rightarrow \dfrac{25}{9} - \dfrac{4}{b^2} = 1 \Rightarrow \dfrac{4}{b^2} = \dfrac{25}{9} - 1$

$\Rightarrow \dfrac{4}{b^2} = \dfrac{16}{9} \Rightarrow 36 = 16b^2$

$\Rightarrow b^2 = \dfrac{36}{16} = \dfrac{9}{4}$

$\because c^2 = a^2 + b^2 \Rightarrow c^2 = 9 + \dfrac{9}{4}$

$\Rightarrow c^2 = \dfrac{45}{4} \Rightarrow c = \sqrt{\dfrac{45}{4}} = \dfrac{3\sqrt{2}}{2}$

so $(i) \Rightarrow \dfrac{y^2}{9} - \dfrac{(x - 2)^2}{\frac{9}{4}} = 1$

Sketch:

vertices: $(X = 0, Y = \pm a)$

$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

$\Rightarrow Y = y \Rightarrow y = \pm c \Rightarrow y = \pm \dfrac{3\sqrt{5}}{2}$

so foci $F\left(2, \dfrac{3\sqrt{5}}{2}\right), F'\left(2, -\dfrac{3\sqrt{5}}{2}\right)$

lehgth *of letusrectum*: $\dfrac{2b^2}{a} = \dfrac{2\left(\frac{9}{4}\right)}{2} = \dfrac{6}{4}$

$= \dfrac{3}{2}$
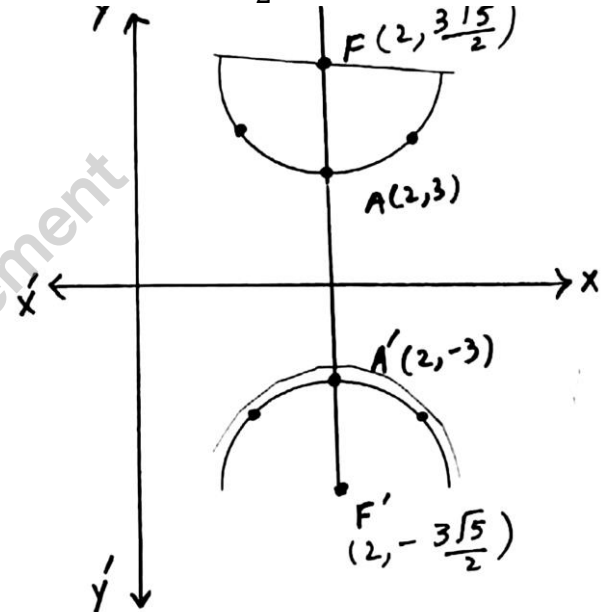
$(viii)$ **Foci** $(5, -2), (5, 4)$ **and one vertex** $(5, 3)$

Foci: $F'(5, -2), F(5, 4)$

Centre $= (h, k) = $ *midpoit of foci*

$= \left(\dfrac{5 + 5}{2}, \dfrac{4 - 2}{2}\right) = (5, 1)$

$\because$ *one vertex* $= (5, 3)$

$\because$ *Distance b\w foci* $= 2c$

$2c = \sqrt{(5 - 5)^2 + (4 + 2)^2} = 6$

$c = 3 \Rightarrow c^2 = 9$

$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$

$b^2 = 9 - 4 = 5$

as foci have same abscissa so transverse axis is along $y - axis$. *so eq is*

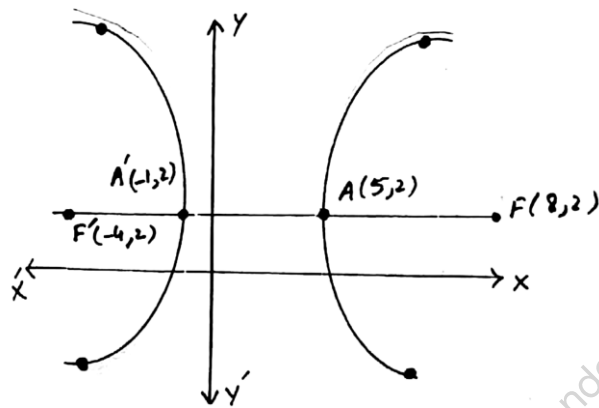$\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$

$\dfrac{(y - 1)^2}{4} - \dfrac{(x - 5)^2}{5} = 1$

Sketch:

$$Vertices: (X = 0, Y = \pm a)$$
$$X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$$
$$Y = y - 1 \Rightarrow y - 1 = \pm a \Rightarrow y - 1 = \pm 2$$
$$y = 1 \pm 2 \Rightarrow y - 1 = \pm a \Rightarrow y - 1 = \pm 2$$
$$y = 1 \pm 2 \Rightarrow y = 3, -1$$

So vertices $A(5,3), A'(5,-1)$

Foci$(X = 0, Y = \pm c)$
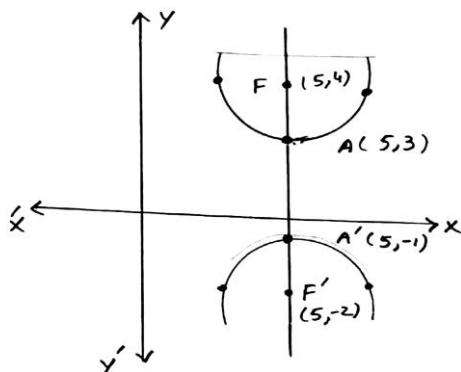$$X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$$
$$Y = y - 1 \Rightarrow y - 1 = \pm c \Rightarrow y - 1 = \pm 3$$
$$y = 1 \pm 3, \quad y = 4, -2$$
$$so\ foci\ (5,4), F'(5,-2)$$

Centre $(5,1)$

Length of $latusrectum : \dfrac{2b^2}{a}$

$$= \frac{2(5)}{2} = 5$$

Question No.2 find Centre, foci eccentricity, vertices and equation of $directrices$ of

(i) $\quad x^2 - y^2 = 9$

Solution:

$$x^2 - y^2 = 9 \Rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1$$

Here $\quad a^2 = 9 \Rightarrow a = 3, b^2 = 9 \Rightarrow b = 3$
$$\because c^2 = a^2 + b^2 = 9 + 9 = 18$$
$$\Rightarrow \quad c = \sqrt{18} = 3\sqrt{2}$$
$$Centre: (0,0)$$
$$Eccentricity: e = \frac{c}{a}$$
$$\Rightarrow \quad e = \frac{3\sqrt{2}}{3} = \sqrt{2}$$
$$Foci: (\pm c, 0) = (\pm 3\sqrt{2}, 0)$$
$$Directrices: x = \pm \frac{a}{e} = \pm \frac{3}{\sqrt{2}}$$
$$Vertices: (\pm a, 0) = (\pm 3, 0)$$

(ii) $\quad \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Here $a^2 = 4 \Rightarrow a = 2, b^2 = 9 \Rightarrow b = 3$
$$\because c^2 = a^2 + b^2 = 4 + 9 = 13$$
$$\Rightarrow \quad c = \sqrt{13}$$

Centre: $(0,0)$

Eccentricty:
$$e = \frac{c}{a} \Rightarrow c = \frac{\sqrt{13}}{2}$$

Foci: $(\pm c, 0) = (\pm\sqrt{13}, 0)$
$$Directrices: x = \pm \frac{a}{e}$$
$$\Rightarrow \quad x = \pm \frac{2}{\frac{\sqrt{13}}{2}} = \pm \frac{4}{\sqrt{13}}$$
$$Vertices: (\pm a, 0) = (\pm 2, 0)$$

(iii)
$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$solution:$
$$\frac{y^2}{16} - \frac{x^2}{9} = 1 \quad here\ a^2 = 16 \Rightarrow a = 4$$
$$b^2 = 9 \Rightarrow b = 3$$
$$\because c^2 = a^2 + b^2$$
$$c^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Centre: $(\mathbf{0, 0})$

$$eccentricity: e = \frac{c}{a} = \frac{5}{4}$$
$$foci: (o, \pm c) = (0, \pm 5)$$
$$Directricity: y = \pm \frac{a}{e}$$
$$\Rightarrow y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$$
$$Vertices: (0, \pm a) = (0, \pm 4)$$

(iv) $\quad \dfrac{y^2}{4} - x^2 = 1$

Solution:

$$\frac{y^2}{4} - x^2 = 1 \quad \Rightarrow \frac{y^2}{4} - \frac{x^2}{1} = 1$$
$$a^2 = 4 \Rightarrow a = 2, b^2 = 1 \Rightarrow b = 1$$
$$\because \quad c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 1 = 5$$
$$c = \sqrt{5}$$

Centre: $(0,0)$

$$Eccentricity: e = \frac{c}{e} = \frac{\sqrt{5}}{2}$$

Foci $= (0, \pm c) = (0, \pm\sqrt{5})$
$$Directrices: y = \pm \frac{a}{e}$$
$$\Rightarrow \quad Diectricity: y = \pm \frac{a}{e}$$
$$\Rightarrow \quad y = \pm \frac{2}{\frac{\sqrt{5}}{2}} \Rightarrow y = \pm \frac{4}{\sqrt{5}}$$
$$Vertices: (0, \pm a) = (0, \pm 2)$$

(v) $\quad \frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$

Solution:

$$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x-1, \qquad Y = y-1, \qquad a^2 = 2,$

$\qquad b^2 = 9 \ , \quad a = \sqrt{2}, \quad b = 3$

$\therefore \ c^2 = a^2 + b^2 = 2 + 9 = 11 \Rightarrow c = \sqrt{11}$

Centre:

$$(X = 0, \qquad Y = 0)$$

$X = x-1 \Rightarrow x-1 = 0 \Rightarrow x = 1$

$Y = y-1 \Rightarrow y-1 = 0 \Rightarrow y = 1$

So Centre $(1,1)$

Eccentricity:

$$e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{11}}{\sqrt{2}} = \sqrt{\frac{11}{2}}$$

Foci: $(X = \pm c, Y = 0)$

$\therefore X = x-1 \Rightarrow x-1 = \pm c \Rightarrow x = 1 \pm \sqrt{11}$

$\qquad Y = y-1 \Rightarrow y-1 = 0 \Rightarrow y = 1$

So $foci(1 \pm \sqrt{11} , 1)$

Directrices: $X = \pm \dfrac{a}{e}$

$$\Rightarrow x-1 = \pm \frac{\sqrt{2}}{\frac{\sqrt{11}}{\sqrt{2}}}$$

$$x - 1 \pm \frac{2}{\sqrt{11}}$$

Vertices:

$$(X = \pm a, \qquad Y = 0)$$

$\Rightarrow X = x-1 \Rightarrow x-1 = \pm a, \Rightarrow x = 1 \pm \sqrt{2}$

$\qquad Y = y-1 \Rightarrow y-1 = 0 \Rightarrow y = 1$

$\qquad$ so $Vertices(1 \pm \sqrt{2}, 1)$

(vi) $\dfrac{(y+2)^2}{9} - \dfrac{(x-2)^2}{16} = 1$

Solution:

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$X = x-2, Y = y+2, a^2 = 9 \Rightarrow a = 3$

$\qquad \therefore c^2 = 9 + 16 = 25$

$\qquad\qquad \Rightarrow c = 5$

$\qquad b^2 = 16 \Rightarrow b = 4$

Centre:

$$(X = 0, Y = 0)$$

$X = x-2 \Rightarrow x-2 = 0 \Rightarrow x = 2$

$Y = y+2 \Rightarrow y+2 = 0 \Rightarrow y = -2$

So $Centre$ $(2,-2)$

$\qquad$ *Eccentricity*: $\because \ e = \frac{c}{a} \Rightarrow e = \frac{5}{3}$

foci: $(X = 0, Y = \pm c)$

$\qquad X = x-2 \Rightarrow x-2 = 0 \Rightarrow x = 2$

$\qquad Y = y+2 \Rightarrow y+2 = \pm c \Rightarrow y = -2 \pm 5$

$\qquad\qquad \Rightarrow y = -7, 3$

$\qquad$ *Foci are* $(2,3), \qquad (2,-7)$

Directrices:

$$Y = \pm \frac{a}{e}$$

$\Rightarrow \ y + 2 = \pm \frac{3}{\frac{5}{3}} \Rightarrow y = -2 \pm \frac{9}{5}$

Vertices: $(X = 0, Y = \pm a)$

$\qquad \Rightarrow X = x-2 \Rightarrow x-2 = 0 \Rightarrow x = 2$

$\qquad \Rightarrow Y = y+2 \Rightarrow y+2 = \pm a$

$\Rightarrow y = -2 \pm 3 \Rightarrow y = -5, 1$

So vertices $(2,-5), (2,1)$

(vii) $9x^2 - 12x - y^2 - 2y + 2 = 0$

Solution:

$$9x^2 - 12x - y62 - 2y + 2 = 0$$

$$9\left(x^2 - \frac{12}{9}x\right) - (y^2 + 2y) = -2$$

$$9\left(x^2 - \frac{12}{9}x\right) - (y^2 + 2y) = -2$$

$$9\left\{x^2 - 2(x)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right\}$$
$$- \{y^2 + 2y + 1 - 1\} = -2$$

$$9\left\{\left(x - \frac{2}{3}\right)^2 - \frac{4}{9}\right\} - (y+1)^2 + 1 = -2$$

$$9\left(x - \frac{2}{3}\right)^2 - (y+1)^2 = -2 + 4 - 1$$

$$\Rightarrow 9\left(x - \frac{2}{3}\right)^2 - (y+1)^2 = 1$$

$$or \ \ \frac{\left(x - \frac{2}{3}\right)^2}{\frac{1}{9}} - \frac{(y+1)^2}{1} = 1 \rightarrow (i)$$

$$compare\ with\ \ \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$X = x - \frac{2}{3}, \qquad Y = y + 1, \ a^2 = \frac{1}{9} \Rightarrow a$

$$= \frac{1}{3}$$

$$b^2 = 1 \Rightarrow b = 1$$

$$\therefore c^2 = a^2 + b^2 = \frac{1}{9} + 1 = \frac{10}{9}$$

$$\Rightarrow c = \frac{\sqrt{10}}{3}$$

Centre:

$$X = 0, Y = 0$$

$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = 0 \Rightarrow x = \frac{2}{3}$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$

So $Centre \left(\frac{2}{3}, -1\right)$

Eccentricity:

$$e = \frac{c}{a} \Rightarrow e = \frac{\frac{\sqrt{10}}{3}}{\frac{1}{3}} = \sqrt{10}$$

Foci:

$$(X = \pm c, Y = 0)$$
$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = \pm c \Rightarrow x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$$
$$\Rightarrow x = \frac{2 \pm \sqrt{10}}{3}$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$

So $foci \left(\frac{2 \pm \sqrt{10}}{3}, -1\right)$

Directrices:

$$X = \pm \frac{a}{e}$$
$$\Rightarrow x - \frac{2}{3} = \pm \frac{\frac{1}{3}}{\sqrt{10}} \Rightarrow x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$$

Vertices:

$$(X = \pm a, Y = 0)$$
$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = \pm a \Rightarrow x = \frac{2}{3} \pm \frac{1}{3}$$
$$\Rightarrow x = \frac{2}{3} + \frac{1}{3} \Rightarrow x = \frac{3}{3} = 1 \Rightarrow x = 1$$
$$also\ x = \frac{2}{3} - \frac{1}{3} \Rightarrow x = \frac{1}{3}$$

$hence\ vertices\ are\ (1, -1) \left(\frac{1}{3}, -1\right)$

(viii) $4y^2 + 12y - x^2 + 4x + 1 = 0$

Solution:

$$4y^2 + 12y - x^2 + 4x + 1 = 0$$
$$4(y^2 + 3y) - (x^2 - 4x) = -1$$
$$4\left\{y^2 + 2(y)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\}$$
$$- \{x^2 - 4x + 4 - 4\} = -1$$
$$4\left(y + \frac{3}{2}\right)^2 - 9 - (x - 2)^2 + 4 = -1$$
$$4\left(y + \frac{3}{2}\right)^2 - 9 - (x - 2)^2 = -1 - 4 + 9$$
$$4\left(y + \frac{3}{2}\right)^2 - (x - 2)^2 = 4$$

$$or\ \left(y + \frac{3}{2}\right)^2 - \frac{(x - 2)^2}{4} = \frac{4}{4} = 1$$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$$X = x - 2, \qquad Y = y + \frac{3}{2},$$
$$a^2 = 1 \Rightarrow a = 1$$
$$b^2 = 4 \Rightarrow b = 2$$
$$\because\ c^2 = a^2 + b^2 = 1 + 4 \Rightarrow c^2 = 5$$
$$or\ c = \sqrt{5}$$

Centre:

$$(X = 0, Y = 0)$$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = 0 \Rightarrow y = -\frac{3}{2}$$

So Centre $\left(2, -\frac{3}{2}\right)$

Eccentricity:

$$e = \frac{c}{a}$$
$$\Rightarrow e = \frac{\sqrt{5}}{1} = \sqrt{5}$$
$$\Rightarrow$$

$Foci$: $(X = 0, Y = \pm c)$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm c \Rightarrow y = -\frac{3}{2} \pm \sqrt{5}$$

So foci $\left(2, -\frac{3}{2} \pm \sqrt{5}\right)$

Directriecs:

$$Y = \pm \frac{a}{c}$$
$$\Rightarrow\ Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm \frac{a}{c}$$
$$\Rightarrow\ y = -\frac{3}{2} \pm \frac{1}{\sqrt{5}}$$

Vertices:

$(X = 0, Y = \pm a)$
$X = x - 2 \Rightarrow x - 2 = 0, x = 2$
$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm a$
$$\Rightarrow\ y = -\frac{3}{2} \pm 1 \Rightarrow y = -\frac{1}{2}, -\frac{5}{2}$$

$so\ vertices\ \left(2, \frac{1}{2}\right), \left(2, -\frac{5}{2}\right)$

(ix)

$$x^2 - y^2 + 8x - 2y - 10 = 0$$

Solution:

$$x^2 + 8x - y^2 - 2y - 10 = 0$$
$$\Rightarrow x^2 + 2(x)(4) + (4)^2 - (4)^2 - (y^2 + 2y + 1 - 1) = 10$$
$$\Rightarrow (x + 4)^2 - 16 - (y + 1)^2 + 1 = 10$$
$$\Rightarrow (x + 4)^2 - (y + 1)^2 = 10 - 1 + 16$$
$$\Rightarrow (x + 4)^2 - (y + 1)^2 = 25$$
$$\Rightarrow \frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1$$
$compare\ with$
$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$
$$X = x + 4, Y = y + 1, a^2 = 25 \Rightarrow a = 5$$

$$b^2 = 25 \Rightarrow c = 5$$
$$\therefore c^2 = a^2 + b^2 = 25 + 25 = 50$$
$$c = \sqrt{50} = 5\sqrt{2}$$

Centre:
$$(X = 0, Y = 0)$$
$$X = x + 4 \Rightarrow x + 4 = 0 \Rightarrow x = -4$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$
$$so\ centre\ (-4, -1)$$
$$Eccentricity: e = \frac{c}{a}$$
$$\Rightarrow\ e = \frac{5\sqrt{2}}{5} = \sqrt{2}$$

Foci:
$$(X = \pm c, Y = 0)$$
$$X = x + 4 \Rightarrow x + 4 = \pm c$$
$$\Rightarrow\ x = -4 \pm 5\sqrt{2}$$
$$\Rightarrow\ Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$
$$So\ foci\ (-4 \pm 5\sqrt{2}, -1)$$

Directrices:
$$X = \pm \frac{a}{c}$$
$$\Rightarrow\ x + 4 = \pm \frac{5}{\sqrt{2}} \Rightarrow x = -4 \pm \frac{5}{\sqrt{2}}$$

Vertices:
$$(X = \pm a\ , Y = 0)$$
$$X = x + 4 \Rightarrow x + 4 = \pm a$$
$$\Rightarrow\ x = -4 \pm 5 \Rightarrow x = -9, 1$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$
$$So\ vertices\ (1, -1)(-9, -1)$$

$(x)\ 9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution:
$$9x^2 - 36x - y^2 - 6y = -18$$
$$9(x^2 - 4x + 4 - 4) - (y^2 + 6y + 9 - 9) = -18$$
$$9(x - 2)^2 - 36 - (y + 3)^2 + 9 = -18$$
$$9(x - 2)^2 - (y + 3)^2 = -18 - 9 + 36$$
$$\Rightarrow\ 9(x - 2)^2 - (y + 3)^2 = 9$$
$$\Rightarrow\ (x - 2)^2 - \frac{(y+3)^2}{9} = 1$$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$
$$X = x - 2, Y = y + 3, a^2 = 1, b^2 = 9$$
$$\Rightarrow\ a = 1, b = 3,$$
$$\therefore c^2 = a^2 + b^2$$
$$25 + 25 = 50$$
$$c = \sqrt{50} = 5\sqrt{2}$$

Centre:
$$(X = 0, Y = 0)$$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + 3 \Rightarrow y = -3$$
$$So\ Centre(2, -3)$$
Foci:

$$(x = \pm c, Y = 0)$$
$$\boldsymbol{X = x - 2 \Rightarrow x - 2 = \pm c \Rightarrow x - 2 = \pm\sqrt{10}}$$
$$\boldsymbol{x = 2 \pm \sqrt{10}}$$
$$Y = y + 3 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$
Hence foci are $\left(2 \pm \sqrt{10}, -3\right)$

Directrices:
$$X = \pm \frac{a}{e}$$
$$\Rightarrow\ x - 2 = \pm \frac{1}{\sqrt{10}} \Rightarrow x = 2 \pm \frac{1}{\sqrt{10}}$$

Vertices:
$$\boldsymbol{X = x - 2 \Rightarrow x - 2 = \pm 1}$$
$$\Rightarrow\ X = x - 2 \Rightarrow x - 2 = \pm 1$$
$$\Rightarrow\ x = 2 \pm 1 \Rightarrow x = 3, 1$$
$$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$

Question No.3

Let $0 < c < a$ and $F(c, 0)$
　$F'(-c, 0)$ be two fixed points. show that
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$
$(F\ and\ F'\ are\ foci\ of\ hyperbola)$
By given condition.
$$|PF| - |PF'| = \pm 2a$$
$$\sqrt{x - (-c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = \pm 2a$$
$$\sqrt{(x - c)^2 + (y)^2} - \sqrt{(x - c)^2 + (y)^2} = \pm 2a$$
$$\sqrt{x^2 + c^2 - 2cx} - \sqrt{x^2 + c^2 - 2cx + y^2} = \pm 2a$$
$$\sqrt{x^2 + c^2 - 2cx} = \pm 2a + \sqrt{x^2 + c^2 - 2cx + y^2}$$
Squaring both sides
$$x^2 + c^2 + 2cx + y^2$$
$$= 4a^2 + x^2 + c^2 - 2cx + y^2$$
$$\pm 2(2a)\sqrt{x^2 + c^2 - 2cx + x^2 + y^2}$$
$$4cx = 4a^2 \pm 4a\sqrt{x^2 - 2cx + x^2 + y^2}$$
$$\Rightarrow\ 4cx = 4a^2 \pm 4a\sqrt{x^2 - 2cx + x^2 + y^2}$$
$$\Rightarrow\ cx = a^2 \pm a\sqrt{x^2 - 2cx + x^2 + y^2}$$
$$\Rightarrow\ cx - a^2 = \pm\sqrt{x^2 - 2cx + x^2 + y^2}$$
$$squaring\ both\ sides$$
$$c^2x^2 + a^4 - 2cxa^2$$
$$= a^2(x^2 - 2xc + x^2 + y^2)$$
$$c^2x^2 + a^4 - 2cxa^2$$
$$= a^2x^2 - 2exa^2 + a^2x^2$$
$$+ a^2y^2$$
$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$
$$\Rightarrow (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$
$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1 \div by\ a^2(c^2 - a^2)$$
$$req.\ eq.\ of\ hyperbola.$$

Question No. 4
Let $0 < c < a$ and $F(5, 5)\ F'(-5, -5)$ be foci.

also the set of points $P(x, y)$ *such that* $|PF| - |PF'| = \pm 2a$ *is the hyperbola with* **Vertices** $A(3\sqrt{2}, 3\sqrt{2}) A'(-3\sqrt{2}, -3\sqrt{2})$. *find* **Equation of hyperbola.**

Solution:

$$\because 2a = |AA'|$$

$$= \sqrt{\left(3\sqrt{2} + 3\sqrt{2}\right)^2 + \left(3\sqrt{2} + 3\sqrt{2}\right)^2}$$

$$\Rightarrow 2a = \sqrt{\left(6\sqrt{2}\right)^2 + \left(6\sqrt{2}\right)^2} = \sqrt{72 + 72}$$

$$= \sqrt{144}$$

$$\Rightarrow 2a = 12 \Rightarrow a = 6$$

$$using |PF| - |PF'| = \pm 2a$$

$$|PF| = \pm 2a + |PF^{\wedge'}|$$

$$\sqrt{(x+5)^2 + (y+5)^2} \pm \mathbf{12}$$
$$+ \sqrt{(x-5)^2 + (y-5)^2}$$

squaring both sides by

$$x^2 + 25x + 10x + y^2 + 25 + 2$$
$$= 144 + x^2 + 25 - 10x + y^2 + 25 - 10y$$
$$\pm 24 \sqrt{(x-5)^2 + (y-5)^2}$$

$$\Rightarrow 20x + 20y = 144 \pm 24\sqrt{(x-5)^2 + (y-5)^2}$$

$$5x + 5y - 36 = \pm 6\sqrt{(x-5)^2 + (y-5)^2}$$

squaring both sides by

$$36(x^2 + 25 - 10x + y^2 + 25 - 10y) =$$
$$25x^2 + 25y^2 + 1296 + 50xy - 360x - 360y$$

$$36x^2 + 36y^2 - 360x - 360y + 1800$$
$$= 25x^2 + 25y^2 + 1296 + 50xy - 360x$$
$$- 360y$$

$$11x^2 + 11y^2 - 50xy + 504 = 0$$

Req. equation of hyperbola.

Question No.5

For any point on the hyperbola the difference of its distance from the points $(2, 2)$ *and* $(110, 2)$ **is 6. Find an equation of hyperbola.**

Solution:

Let $P(x, y)$ be any point on hyperbola so

$$\sqrt{(x-2)^2 + (y-2)^2} - \sqrt{(x-10)^2 + (y-10)^2}$$
$$= 6$$

$$\sqrt{(x-2)^2 + (y-2)^2} = \sqrt{(x-10)^2 + (y-10)^2}$$
$$+ 6$$

Squaring both sides

$$x^2 + 4 - 4x + y^2 + 4 - 4y$$
$$= 36 + x^2 + 100 - 20x$$

$$y^2 + 4 - 4y + 12\sqrt{(x-10)^2 + (y-2)^2}$$

$$\Rightarrow -4x + 20x + 8 - 36 - 104 =$$
$$12\sqrt{(x-10)^2 + (y-2)^2}$$

$$\Rightarrow 16x - 132 =$$
$$12\sqrt{x^2 + 100 - 20x + y^2 - 4y + 4}$$

$$\Rightarrow 4x - 33 =$$
$$3\sqrt{x^2 + y^2 - 20x + y^2 - 4y + 104}$$

Again squaring

$$16x^2 + 1089 - 264x$$
$$= 9(x^2 + y^2 - 20x - 4y + 104)$$

$$\Rightarrow 16x^2 + 1089 - 264x = 9x^2 + 9y^2 - 180x - 36y + 936$$

$$\Rightarrow 7x^2 - 9y^2 - 84x + 36y + 153 = 0$$

Which is the required equation.

Q6. *two listing posts hear the sound of an enemy* **Gun. The difference in time is one second. If the listening posts are 1400ft apart, write an equation of the hyperbola passing through the position of the enemy gun. (Sounds)** *travels at* $1080 \, ft \backslash sec$.

Solution:

Let two

listening F_1 *and* F_2 *hear the sound of enemy* Gun after t and $t - 1$ *second respectively here* *listening posts are* $1400m$ *apart. i.e* $2c$

$$= 1400 \Rightarrow c = 700$$

If p *is the position of enemy gun.*

So sound travels at 1080 ft/sec so we have

$$|pF_1| - |pF_2| = 2a$$

$$\Rightarrow 1080t - 1080(t - 1) = 2a$$

$$\Rightarrow 1080t - 1080t + 1080 = 2a$$

$$or \, 2a = 1080$$

$$a = 540$$

Using $c^2 = a^2 + b^2$

$$b^2 = c^2 - a^2$$

$$b^2 = (700)^2 - (540)^2$$

$$b^2 = 490000 - 291600$$

$$b^2 = 198400$$

The equation of hyperbola is

$$\frac{x^2}{(540)^2} - \frac{y^2}{198400} = 1$$

$$or \, \frac{x^2}{291600} - \frac{y^2}{198400} = 1$$

Note:

Ex 6.7, 6.8 and 6.9 are not include in paper pattern of all BISE's of Punjab.

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