MATHEMATICS 12 INTERMEDIATE

PART II

Bilal Article

Chapter 3.

INTEGRATION

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Contents

Integration:

The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti derivative or integration.

Differential of variable:

Let f be a differentiable function defined as

$$y = f(x) \ \Rightarrow y + \delta y = f(x + \delta x)$$
$$\Rightarrow \delta x = f(x + \delta x) - y \Rightarrow \delta y$$
$$= f(x + \delta x) - f(x)$$
$$Now \ \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$
$$\Rightarrow \frac{dy}{dx} = f'(x)$$

$\because$ *before the limit is reached.* $\frac{\delta y}{\delta x}$ *differs from* $f'(x)$*bysmall real number* $\in . i.e \frac{\delta y}{\delta x} = f'(x) + \in$

$\Rightarrow f'(x)$*is called differntial of dependent varibale y we denoted differential of y as dy.*

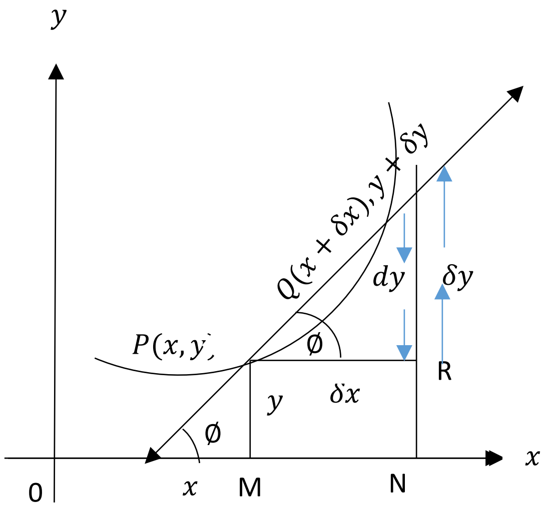
so $dy = f'(x)\delta x \Rightarrow dx = \delta y = f'(x)dx$

Note: 1.The differential of x is denoted by dx and defined as $dx = \delta x$

$$\boldsymbol{i.e \ for \ y = x \Rightarrow dy = \frac{d}{dx}(x)\delta x}$$
$$\boldsymbol{\Rightarrow dy = 1.\delta x \Rightarrow dx = \delta x \quad \because y = x}$$

2. $\boldsymbol{f'(x) is \ used \ differential \ coefficeient.}$

Distinguishing between dy and δ



Let us draw the graph of curve of a function $y = f(x)$*Let* $P(x, y)$ *and* $Q(x + \delta x, y + \delta y)$*be two neighbouring points on the curve at point* $P(x, y)$ *s.that it makes an angle* $\emptyset$ *with x − axis. Also*

Draw $\perp$ *PM and QN on x − axix also draw* $\perp$ PR *on QN on x − axis.in fig.* $|PR| = dx$

$$|QR| = |QT| + |TR|$$
$$\Rightarrow \delta y = |QT| + |TR| \to (i)$$
$$In \ \triangle TPR, \quad tan\emptyset dx = \frac{|TR|}{|PR|} = \frac{|TR|}{dx}$$
$$\Rightarrow |TR| = tan\emptyset dx$$

So$(i) \Rightarrow \delta y = \ tan\emptyset dx + |QT|$

$$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) dx + |QT| \qquad \because \frac{dy}{dx} = tan\emptyset$$
$$\delta y = dy + |QT| \quad \because |QT| \ is \ very \ small$$

So *by neglecting* $|QT|$

$$\Rightarrow \delta y \approx dy$$

Example:

Find$\boldsymbol{\delta y \ and \ dy}$ of the function defined as

$$\boldsymbol{f(x) = x^2 \ when \ x = 2 \ and \ dx = 0.01}$$

Solution:

$$Let \ y = f(x) \qquad dy = ?$$

⇨ $y = x^2$

⇨ $\frac{dy}{dx} = 2x \Rightarrow dy = 2dx$

Take $x = 2 \ and \ dx = 0.01$

$$dy = 2(2)(0.01) = 0.04$$

Now we find $\delta y, y + \delta y = (x + \delta x)^2$

⇨ $\delta y = (x + \delta x)^2 - y \quad , y = (x)^2 = (2)^2 = 4$

$$= (2 + 0.01)^2 - 4 \qquad \because dx = \delta x = 0.01$$

$\delta y = 4.041 - 4 = 0.0401$

Example:

Use differentials find $\boldsymbol{\frac{dy}{dx} \ when \frac{y}{x} - lnx = lnc}$

Solution:

$$\boldsymbol{\frac{y}{x} - lnx = lnc}$$

⇨ $\boldsymbol{d\left(\frac{y}{x} - lnx\right) = d(lnc)}$

⇨ $\boldsymbol{d\left(\frac{y}{x}\right) - d(lnx) = 0}$

⇨ $\boldsymbol{\frac{xdy - ydx}{x^2} - \frac{1}{x}dx = 0}$

⇨ $\boldsymbol{\frac{xdy - ydx}{x^2} = \frac{1}{x}dx}$

⇨ $\boldsymbol{xdy - ydx = xdx}$

⇨ $\boldsymbol{xdy = xdx + ydx}$

⇨ $\boldsymbol{dy = \frac{x + y}{x}dx}$

⇨ $\boldsymbol{\frac{dy}{dx} = \frac{x + y}{x}}$

Exercise 3.1

$Q.1$: *Find δy and dy in the following cases:*

$$\boldsymbol{i)\ y = x^2 - 1}$$

when x changes from 3 to 3.02

SOLUTION:

$y = x^2 - 1$ *As x changes from 3 to 3.02, so*

$y = x^2 - 1$

$d(y) = d(x^2 - 1)$

$\mathrm{d}y = 2x\,dx - 0 = 2x\,dx$

Put value of x and dx

$\mathrm{d}y = 2(3)(0.02)$

$\mathrm{d}y = 0.12$

Now

$y + \delta y = (x + \delta x)^2 - 1$

$\delta y = (x + \delta x)^2 - 1 - y$

Put value of y

$\delta y = (x + \delta x)^2 - 1 - (x^2 - 1)$

$\delta y = (x + \delta x)^2 - 1 - x^2 + 1$

$\delta y = (x + \delta x)^2 - x^2$

$x = 3$

$\delta x = dx = 3.02 - 3 = 0.02$

Put value of x and δx

$\delta y = (3 + 0.02)^2 - (3)^2$

$\delta y = 0.1204$

$\boldsymbol{ii)\ y = x^2 + 2x}$

when x changes from 2 to 1.8

SOLUTION:

$y = x^2 + 2x$

Now

$y = x^2 + 2x$

$d(y) = d(x^2 + 2x)$

$\mathrm{d}y = 2x\,dx + 2dx$

Put value of x and dx

$\mathrm{d}y = 2(2)(-0.2) + 2(-0.2)$

$\mathrm{d}y = -1.2$

Now

$y + \delta y = (x + \delta x)^2 + 2(x + \delta x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - y$

Put value of y

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - (x^2 + 2x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - x^2 - 2x$

$\delta y = (x + \delta x)^2 + 2\delta x - x^2$

$x = 2\,,$

$\delta x = dx = 1.8 - 2 = -0.2$

Put value of x and δx

$\delta y = (2 - 0.2)^2 + 2(-0.2) - (2)^2$

$\delta y = -1.16$

$\boldsymbol{iii)\ y = \sqrt{x}}$

when x changes from 4 to 4.01

SOLUTION:

$y = \sqrt{x}$

Now

$y = \sqrt{x}$

$d(y) = d(\sqrt{x})$

$\mathrm{d}y = \frac{1}{2\sqrt{x}}\,dx$

Put value of x and dx

$\mathrm{d}y = \frac{1}{2\sqrt{4}}\,(0.41)$

$\mathrm{d}y = 0.1025$

Now.

$y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

Put value of y

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

$x = 4\,,$

$\delta x = dx = 4.41 - 4 = 0.41$

Put value of x and δx

$\delta y = \sqrt{4 + 0.41} - \sqrt{4}$

$\delta y = 0.1$

$Q.2$: *Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.*

$$\boldsymbol{i)\ xy + x = 4}$$

Taking differentials on both sides

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$

$x\,dy + y\,dx + dx = 0$

$x\,dy + (y + 1)dx = 0$

$x\,dy = -(y\ + 1)dx$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y+1}{x}$ *and*

$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x}{y+1}$

$\boldsymbol{ii)\ x^2 + 2y^2 = 16}$

Taking differentials on both sides

$d(x^2 + 2y^2) = d(16)$

$d(x^2) + 2d(y^2) = 0$

$2x\,dx + 2.2y^{2-1}.dy = 0$

$2x\,dx + 4y\,dy = 0$

$4y\,dy = -2x\,dx$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{4y} = -\frac{x}{2y}$ *and*

$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{2y}{x}$

$\boldsymbol{iii)\ x^4 + y^2 = xy^2}$

Taking differentials on both sides

$d(x^4 + y^2) = d(xy^2)$

$d(x^4) + d(y^2) = (x)'(y^2) + (y^2)'x$

$4x^3\,dx + 2y\,dy = dx.y^2 + (2y\,dy)x$

$4x^3\,dx + 2y\,dy = y^2\,dx + 2xy\,dy$

$2y\,dy - 2xy\,dy = y^2\,dx - 4x^3\,dx$

$(2y\ - 2xy)\,dy = (y^2\ - 4x^3\ )dx$

$\frac{\mathrm{D}y}{\mathrm{d}x} = \frac{y^2 - 4x^3}{2y - 2xy}$ *taking reciprocal*

$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2y - 2xy}{y^2 - 4x^3}$

$\boldsymbol{iv)\ xy - \ln x = c}$

Taking differentials on both sides

$d(xy - \ln x) = d(c)$

$d(xy) - d(\ln x) = 0$

$x\,dy + y\,dx - \frac{1}{x}dx = 0$

$x\,dy = -y\,dx + \frac{1}{x}dx$

$x\,dy = -\left(y - \frac{1}{x}\right)dx$

$x\,dy = -\left(\frac{xy-1}{x}\right)dx$

$\frac{dy}{dx} = \frac{1-xy}{x^2}$ and

$\frac{dx}{dy} = \frac{x^2}{1-xy}$

Q.3: Use differentials to approximate the values of:

i) $\sqrt[4]{17}$

SOLUTION:

Let $y = \sqrt[4]{x} = x^{\frac{1}{4}}$

We take $x = 16$,

$\delta x = dx = 17 - 16 = 1$

$y = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

Now $y = x^{\frac{1}{4}}$

$d(y) = d\left(x^{\frac{1}{4}}\right)$

$dy = \frac{1}{4}x^{\frac{1}{4}-1}\,dx$

$dy = \frac{1}{4}x^{-\frac{3}{4}}\,dx$

Put $x = 16$, $dx = 1$

$dy = \frac{1}{4}(16)^{-\frac{3}{4}}\,(1) = \frac{1}{4}(2^4)^{-\frac{3}{4}}$

$dy = \frac{1}{4}(2)^{-3} = \frac{1}{4}.\frac{1}{8} = \frac{1}{32}$

$dy = 0.03125$

Thus $\sqrt[4]{17} \approx y + dy$

$= 2 + 0.03125$

$= 2.03125$

ii) $(31)^{\frac{1}{5}}$

SOLUTION:

Let $y = x^{\frac{1}{5}}$

We take $x = 32$,

$\delta x = dx = 31 - 32 = -1$

$y = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$

Now $y = x^{\frac{1}{5}}$

$d(y) = d\left(x^{\frac{1}{5}}\right)$

$dy = \frac{1}{5}x^{\frac{1}{5}-1}\,dx$

$dy = \frac{1}{5}x^{-\frac{4}{5}}\,dx$

Put $x = 32$, $dx = -1$

$dy = \frac{1}{5}(32)^{-\frac{4}{5}}\,(-1) = -\frac{1}{5}(2^5)^{-\frac{4}{5}}$

$dy = \frac{1}{5}(2)^{-4} = \frac{1}{5}.\frac{1}{16} = \frac{1}{80}$

$dy = -0.0125$

Thus $(31)^{\frac{1}{5}} \approx y + dy$

$= 2 - 0.0125$

$= 1.9875$

iii) **cos 29°**

SOLUTION:

Let $y = \cos x$

We take $x = 30°$,

$\delta x = dx = 29° - 30° = -1° = -0.01745$

$y = \cos 30° = 0.866$

Now $y = \cos x$

$d(y) = d(\cos x)$

$dy = -\sin x\;dx$

$dy = -\sin 30°\,(-0.01745)$

$dy = -(0.5)\,(-0.01745)$

$dy = 0.0087$

Thus $\cos 29° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

iv) **sin 61°**

SOLUTION:

Let $y = \sin x$

We take $x = 60°$,

$\delta x = dx = 61° - 60° = 1° = 0.01745$

$y = \sin 60° = 0.866$

Now $y = \sin x$

$d(y) = d(\sin x)$

$dy = \cos x\;dx$

$dy = \cos 60°\,(0.01745)$

$dy = (0.5)\,(0.01745)$

$dy = 0.0087$

Thus $\sin 61° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

Q.4: Find the approximate increase in the volume of a cube if the lenght of each edge changes from 5 to 5.02.

SOLUTION:

Lenght of each edge of cube $= x$ *unit*

Volume of a cube $= L.W.H$

$V = x.x.x$

$V = x^3$

$d(V) = (x^3)$

$dV = 3x^2\,dx$

when x *changes from* 5 *to* 5.02, *so*

$x = 5$, $dx = 5.02 - 5 = 0.02$

$dV = 3(5)^2\,(0.02) = 1.5$ *cubic units*

Q.5: Find the approximate increase in the area of a circular disc if its diameter is increased from 44 cm to 44.4 cm.

SOLUTION:

Let radius of circular disc $= x$ *cm*

Area of a disc $= \pi r^2$

$A = \pi x^2$

$d(A) = d(\pi x^2)$

$dA = \pi.2x\,dx$

As diameter changes from 44 *to* 44.4,

so radius changes from 22 *to* 22.4, *so*

$x = 22$, $dx = 22.2 - 22 = 0.2$

$dA = \pi(2)(22)(0.2)$

$dA = 27.646\ \ cm^2$

Integration as anti-derivative (inverse of derivative)

$\textit{Integration}$: $\quad v.v.v.\textit{important defination}(***)$

$\textit{The inverse process of differentiation is called anti} - \textit{differentiation or integration.}$

Consider $F(x)$ $\textit{is antidervative of a function if}$

$$F'(x) = f(x)\ \ then \int f(x)dx = \int F'^{(x)}dx$$

$$= \int \frac{d}{dx}F(x)dx$$

$$\int f(x)dx = F(x) + c$$

$\because \frac{d}{dx}$ and $\int dx$ $\textit{are inverse operations of each other.}$

*The symbol

$\int \ldots dx$ $\textit{indicates that integrand is two integrated w.r.t "x"}$

*The anti- derivative of a function is also called integrated is called integrand of the integral.

*The function which is to be integrated is called integrand of the integral.

Some standard formulae for Anti-derivatives

$$\int 1dx = x + c\ ,\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1)$$

$$\int sindx = -cosx + c\ ,\int cosxdx = sinx + c$$

$$\int \sec^2 x\, dx = tanx + c,\quad \int cosec^2 dx = -cotx + c$$

$$\int secxtanxdx = secx + c, \int cosecxcotxdx$$

$$= -cosecx$$

$$\int e^x dx = e^x + c\quad ,\int a^x dx = \frac{1}{lna}.a^x + c$$

$$\int \frac{1}{x}dx = ln|x| + c, x \neq 0\ \ ,\int tanxdx$$

$$= lnsec|x| + c = -ln|cosx| + c$$

$$\int cotxdx = ln|sinx| + c$$

$$\int secxdx = \ln|secx + tanx| + c$$

$$\int cosecxdx = \ln|cosecx - cots| + c$$

Here c is constant of integration. These formulae can be verified by showing that the derivatives of right hand side of each w.r.t “x” is equal to the corresponding integral

Examples:

1. $\int x^5 dx \qquad \because \int x^n dx = \frac{x^{(n+1)}}{\boldsymbol{n+1}} + c$

$$= \frac{\boldsymbol{x^{5+1}}}{\boldsymbol{5+1}} + \boldsymbol{c} = \frac{\boldsymbol{x^6}}{\boldsymbol{6}} + \boldsymbol{c}$$

2. $\int \frac{1}{\sqrt{x^3}} dx$

$$= \int \frac{1}{x^{\left(\frac{2}{3}\right)}} dx = \int x^{-\frac{3}{2}} dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -\frac{2}{\sqrt{x}} + c$$

3. $\int \frac{1}{(2x+3)^4} dx =$

$$\int (2x+3)^{-4} dx = \frac{1}{2}.\frac{(2x+3)^{-4+1}}{-4+1} + c$$

$$= -\frac{1}{6(2x+3)^3} + c$$

4. $\int cos2xdx \qquad \because \int cosaxdx = \frac{sinax}{a} + c$

$$= \frac{sin2x}{2} + c$$

5. $\int sin3xdx \qquad \because \int sinaxdx = -\frac{cosax}{a} + c$

$$= -\frac{cos3x}{3} + c$$

6. $\int cosec^2 xdx$

$= -cotx + c$

7. $\int sec5xtan5xdx$

$$= \frac{sec5x}{5} + c \qquad \because \int secaxtanaxdx$$

$$= \frac{secax}{a} + c$$

$$= \frac{se5x}{5} + c$$

8. $\int e^{ax+b} dx \qquad \because \int e^{ax} dx = \frac{e^{ax}}{a} + c$

$$\frac{e^{ax+b}}{a} + c$$

9. $\int 3^{\lambda x} dx = \qquad \because \int e^{ax} dx = \frac{e^{ax}}{a} + c$

$$= \frac{3^{\lambda x}}{\lambda ln3}$$

10. $\int \frac{1}{ax+b} dx$

$$\int (ax+b)^{-1} dx = \frac{1}{a}\ln(ax+b) + c$$

1. $\int af(x)dx = a\int f(x)dx$
2. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

Prove that

$$\int [\boldsymbol{f(x)}]^{\boldsymbol{n}} \boldsymbol{f'(x)dx} = \frac{[\boldsymbol{f(x)}]^{\boldsymbol{n+1}}}{\boldsymbol{n+1}} + \boldsymbol{c}, (\boldsymbol{n} \neq -\boldsymbol{1}$$

Proof:

We know that $\frac{d}{dx}\left(f^{n+1}(x)\right)$

$= (n+1)f^n(x).\frac{d}{dx}f(x)$

⇨ $\frac{d}{dx}(f^n(n+1) = (n+1)f^n(x).f'(x)dx$

Taking integration

$$\int \frac{d}{dx}f^{(n+1)}(x)dx = (n+1)\int f^n(x).f'(x)dx$$

⇨ $f^{n+1}(x) = (n+1)\int f^n(x)f'(x)dx$

⇨ $\int f^n(x)\,f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c \quad by\ def.$

Hence proved.

Prove that $\boldsymbol{\int \frac{f\prime(x)}{f(x)}dx = lnf(x) + c}$

Proof:

We know that

$$\frac{d}{dx}[lnf(x)] = \frac{1}{f(x)}.f'(x)$$

Taking integration both sides

$$\int \frac{d}{dx}[lnf(x)]dx = \int \frac{1}{f(x)}.f'(x)dx$$

⇨ $\boldsymbol{lnf(x) = \int \frac{f'(x)}{f(x)}\,dx}$

⇨ $\boldsymbol{\int \frac{f'(x)}{f(x)}\,dx = lnf(x) +}$

$\boldsymbol{c\ \ by\ defination}$

$$\boldsymbol{\left(\int f(x)dx = F(x) + c\right)}$$

Hence proved.

Exercise 3.2

$\boldsymbol{Q.1: Evaluate\ the\ following\ indefinate\ integrals}$:

$\boldsymbol{i)\ \ \int(3x^2 - 2x + 1)\ dx}$

$\boldsymbol{SOLUTION}$:

$= \int 3x^2\,dx - \int 2x\,dx + \int 1\,dx$

$= 3\int x^2\,dx - 2\int x\,dx + \int 1\,dx$

$= 3.\frac{x^{2+1}}{2+1} - 2.\frac{x^{1+1}}{1+1} + x + c$

$= 3.\frac{x^3}{3} - 2.\frac{x^2}{2} + x + c$

$= x^3 - x^2 + x + c$

$\boldsymbol{ii)\ \ \int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\ dx}$

$\boldsymbol{SOLUTION}$:

$= \int \sqrt{x}\,dx + \int \frac{1}{\sqrt{x}}dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x + c$

$\boldsymbol{iii)\ \ \int x(\sqrt{x} + 1)\ dx}$

$\boldsymbol{SOLUTION}$:

$= \int x(\sqrt{x} + 1)\ dx$

$= \int x\sqrt{x}\,dx + \int x\,dx$

$= \int x^{1+\frac{1}{2}}\,dx + \int x\,dx$

$= \int x^{\frac{3}{2}}\,dx + \int x\,dx$

$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$

$= \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$

$\boldsymbol{iv)\ \ \int(2x + 3)^{\frac{1}{2}}\ dx}$

$\boldsymbol{SOLUTION}$:

$= \ \int(2x + 3)^{\frac{1}{2}}\ dx$

$\times$ *and* $\div$ *by 2 to make derivative*

$= \frac{1}{2}\int(2x + 3)^{\frac{1}{2}}.2\,dx$

$= \frac{1}{2}\frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{1}{2}\frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{2}.\frac{2}{3}\ (2x + 3)^{\frac{3}{2}} + c$

$= \frac{1}{3}\ (2x + 3)^{\frac{3}{2}} + c$

$\boldsymbol{v)\ \ \int(\sqrt{x} + 1)^2\ dx}$

$\boldsymbol{SOLUTION}$:

$= \int(\sqrt{x} + 1)^2\ dx$

$= \int((\sqrt{x})^2 + 2\sqrt{x}.1 + (1)^2)\ dx$

$= \int[x + 2\sqrt{x} + 1]$

$= \int x\,dx + 2\int x^{\frac{1}{2}}\,dx + \int 1\,dx$

$= \frac{x^{1+1}}{1+1} + 2.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \ x + c$

$= \frac{x^2}{2} + 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \ x + c$

$= \frac{1}{2}x^2 + 2.\frac{2}{3}x^{\frac{3}{2}} + x + c$

$= \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + x + c$

$\boldsymbol{vi)\ \ \int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2\ dx}$

$\boldsymbol{SOLUTION}$:

$= \ \ \int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2\ dx$

$= \ \int\left[\left(\sqrt{x}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}.\frac{1}{\sqrt{x}}\right]\ dx$

$= \ \int\left[x + \frac{1}{x} - 2\right]\ dx$

$= \int x\,dx + \int \frac{1}{x}dx - 2\int 1\,dx$

$= \frac{x^{1+1}}{1+1} + \ln x - 2\,x + c$

$= \frac{1}{2}x^2 + \ln x - 2\,x + c$

$\boldsymbol{NOTE: FOR\ \ Q.(vi)}$

AGAR FUNCTION OVER M HO AUR FUNCTION KI POWER 1 HO TU AP US PAR POWER RULE NI LAGA SAKTAY.

FOR EXAMPLE:

$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \infty$

IS M ANSWER ∞ A GIA SO NOT SOLVED?
THEN AGAR FUNCTION KA DERIVATIVE UPER MAJOOD H T US K ln K SATH LIKH DE.

$\boldsymbol{vii)}\ \int \frac{3x+2}{\sqrt{x}}\,dx$

SOLUTION:

$$\int \frac{3x+2}{\sqrt{x}}\,dx$$
$$= \int \left[\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx$$
$$= \int \left[\frac{3\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx \quad \because x = \sqrt{x}.\sqrt{x}$$
$$= \int \left[3\sqrt{x} + \frac{2}{\sqrt{x}}\right] dx$$
$$= \int \left[3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right] dx$$
$$= 3\int x^{\frac{1}{2}}\ dx + 2\int x^{-\frac{1}{2}}\ dx$$
$$= 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$
$$= 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 3\frac{2}{3}\ x^{\frac{3}{2}} + 2\,.2x^{\frac{1}{2}} + c$$
$$= 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c$$

$$\boldsymbol{viii)}\ \int \frac{\sqrt{y}(y+1)}{y}\,dx$$

SOLUTION:

$$\int \frac{\sqrt{y}(y+1)}{y}\,dx$$
$$= \int \frac{\sqrt{y}(y+1)}{\sqrt{y}\sqrt{y}}\,dx$$
$$= \int \frac{y+1}{\sqrt{y}}\,dx$$
$$= \int \left[\frac{y}{\sqrt{y}}dx + \frac{1}{\sqrt{y}}\ dx\right]$$
$$= \int \left[\sqrt{y}dx + \frac{1}{\sqrt{y}}\ dx\right]$$
$$= \int \left[y^{\frac{1}{2}}\ dx + y^{-\frac{1}{2}}\ dx\right]$$
$$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$
$$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + x + c$$

$$\boldsymbol{ix)}\ \int \frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta$$

SOLUTION:

$$\int \frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta$$
$$= \int \frac{(\sqrt{\theta})^2+(1)^2-2\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$$
$$= \int \frac{\theta+1-2\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$$
$$= \int \left[\frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}}\right] d\theta$$
$$= \int \left[\sqrt{\theta} + \frac{1}{\sqrt{\theta}} - 2\right] d\theta$$
$$= \int \theta^{\frac{1}{2}}\ d\theta + \int \theta^{-\frac{1}{2}}\ d\theta - 2\int 1\ d\theta$$
$$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\theta + c$$
$$= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + c$$
$$= \frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c$$

$$\boldsymbol{x)}\ \int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\,dx$$

SOLUTION:

$$\int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\,dx$$
$$= \int \frac{(\sqrt{x})^2+(1)^2-2\sqrt{x}}{\sqrt{x}}\,dx$$
$$= \int \frac{x+1-2\sqrt{x}}{\sqrt{x}}\,dx$$
$$= \int \left[\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}}\right] dx$$
$$= \int \left[\sqrt{x} + \frac{1}{\sqrt{x}} - 2\right] dx$$
$$= \int x^{\frac{1}{2}}\ dx + \int x^{-\frac{1}{2}}\ dx - 2\int 1\ dx$$
$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2x + c$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + c$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2x + c$$

$\boldsymbol{xi)}\ \int \frac{e^{2x}+e^x}{e^x}\,dx$

SOLUTION:

$$\int \frac{e^{2x}+e^x}{e^x}\,dx$$
$$= \int \left[\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right] dx$$
$$= \int [e^x + 1]\ dx$$
$$= \int e^x\ dx + \int 1\ dx$$
$$= \frac{e^x}{1} + x + c$$
$$= e^x + x + c$$

NOTE: DERIVATION M

EXPONENTIAL FUNCTION KA JAB DERIVATIVE LATY H T FUNCTION AS IT AUR POWER KA DERIVATIVE MULTIPLY KARTY H. LAKIN INTEGRATION M DIVIDE KARE GAI.

Q. 2: *Evaluate*:

i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

SOLUTION:

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$
$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\ \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\ dx$$
$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}\ dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\ dx$$
$$= \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$
$$= \frac{1}{a-b}\left\{\int (x+a)^{\frac{1}{2}}\ dx + \int (x+b)^{\frac{1}{2}}\ dx\right\}$$

$\boldsymbol{using\ \int [f(x)]^n . f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c}$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x+b)^{\frac{3}{2}}\right\} + c$$

$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} + (x+b)^{\frac{3}{2}}\right\} + c$$

ii) $\int \frac{1-x^2}{1+x^2}\ dx$

***SOLUTION*:**

$$\int \frac{1-x^2}{1+x^2}\ dx$$

$$= \int \frac{2-1-x^2}{1+x^2}\ dx$$

$$= \int \frac{2-(1+x^2)}{1+x^2}\ dx$$

$$= \int \frac{2}{1+x^2}\ dx - \int \frac{1+x^2}{1+x^2}\ dx$$

$$= 2\int \frac{1}{1+x^2}\ dx - \int 1\ dx$$

$$= 2\tan^{-1} x - x + c$$

iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

***SOLUTION*:**

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\ \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\ dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2}\ dx = \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\ dx$$

$$= \frac{1}{a}\int\left(\sqrt{x+a} - \sqrt{x+}\ \right)dx$$

$$= \frac{1}{a}\left\{\int (x+a)^{\frac{1}{2}}\ dx + \int (x)^{\frac{1}{2}}\ dx\right\}$$

using $\int [f(x)]^n . f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$$

$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$

$$= \frac{1}{a}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x)^{\frac{3}{2}}\right\} + c$$

$$= \frac{2}{3a}\left\{(x+a)^{\frac{3}{2}} + (x)^{\frac{3}{2}}\right\} + c$$

iv) $\int (a-2x)^{\frac{3}{2}}\ dx$

***SOLUTION*:**

$$\int (a-2x)^{\frac{3}{2}}\ dx$$

$\times$ *and* $\div$ *by* 2

$$= \frac{1}{-2}\int (a-2x)^{\frac{3}{2}}.(-2)\ dx$$

$$= -\frac{1}{2}\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$= -\frac{1}{2}\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= -\frac{1}{2}.\frac{2}{5}(a-2x)^{\frac{5}{2}} + c$$

$$= -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$$

FUNCTION AS IT AUR POWER KE DERIVATIVE S DIVIDE KARNA H.

$\int e^x\, dx = \frac{e^x}{1} + c = e^x + c$

v) $\int \frac{(1+e^x)^3}{e^x}\ dx$

***SOLUTION*:**

$$\int \frac{(1+e^x)^3}{e^x}\ dx$$

$$\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$= \int \frac{1^3+(e^x)^3+3(1)(e^x)(1+e^x)}{e^x}\ dx$$

$$= \int \frac{1+e^{3x}+3e^x(1+e^x)}{e^x}\ dx$$

$$= \int \left[\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x(1+e^x)}{e^x}\right] dx$$

$$= \int [e^{-x} + e^{2x} + 3 + 3e^x]\ dx$$

$$= \int e^{-x}\ dx + \int e^{2x}\ dx + 3\int 1\ dx + 3\int e^x\ dx$$

$$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3\frac{e^x}{1} + c$$

$$= -e^{-x} + \frac{1}{2}e^{2x} + 3x + 3e^x + c$$

vi) $\int \sin(a+b)x\ dx$

***SOLUTION*:**

$$\int \sin(a+b)x\ dx$$

$$= \frac{-\cos(a+b)x}{a+b} + c$$

$$= -\frac{1}{a+b}\cos(a+b)x + c$$

DERIVATION M FUNCTION KA DERIVATIVE LENA HOTA H AUR SATH ANGLE KE DERIVATIVE KO MULTIPLY KARTY H. BUT INTEGRATION M ANGLE KE DERIVATIVE K DIVIDE KARE GAI.

vii) $\int \sqrt{1-\cos 2x}\ dx$

***SOLUTION*:**

$$\int \sqrt{1-\cos 2x}\ dx$$

As $\sin^2 x = \frac{1-\cos 2x}{2}$

So $1 - \cos 2x = 2\sin^2 x$

$$= \int \sqrt{2\sin^2 x}\ dx$$

$$= \int \sqrt{2}\sqrt{\sin^2 x}\, dx$$

$$= \sqrt{2}\int \sin x\ dx$$

$$= \sqrt{2}(-\cos x) + c$$

$$= -\sqrt{2}\cos x + c$$

viii) $\int \ln x\ \frac{1}{x}\ dx$

SOLUTION:

$$\int \ln x\ .\ \frac{1}{x}\ dx$$

As $f(x) = \ln x$

And $f'(x) = \frac{1}{x}$, ***so***

using $\int [f(x)]^n = \frac{[f(x)]^{n+1}}{n+1}$

$$= \frac{(\ln x)^{1+1}}{1+1} + c$$

$$= \frac{(\ln x)^2}{2} + c$$

ix) $\int \sin^2 x\ dx$

***SOLUTION*:**

$$\int \sin^2 x\ dx$$

As $\sin^2 x = \frac{1-\cos 2x}{2}$

$$= \int \frac{1-\cos 2x}{2}\ dx$$

$= \frac{1}{2}\int(1-\cos 2x)dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$$\boldsymbol{x)}\ \int \frac{\mathbf{1}}{\mathbf{1+\cos x}}\,\boldsymbol{dx}$$

SOLUTION:

$\int \frac{1}{1+\cos x}\ dx$

$\boldsymbol{As\ \cos^2\frac{x}{2} = \frac{1+\cos x}{2}}$

$\boldsymbol{So\ 1+\cos x = 2\cos^2\ \frac{x}{2}}$

$= \int \frac{1}{2\ \cos^2\frac{x}{2}}\ dx$

$= \frac{1}{2}\int \sec^2\frac{x}{2}\ dx$

$= \frac{1}{2}\ \frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = \tan\frac{x}{2} + c$

$\sin^2 x\ ,\cos^2 x\ ,\tan^2 x\ ,\cot^2 x$ *in functions k derivative exist ni karty jab b ye function a jay t ap ye formula use kare.*

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\tan^2 x = \sec^2 x - 1$

$\cot^2 x = \csc^2 x - 1$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x\ ,\cos^2 x\ ,\tan^2 x\ ,\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$

$(\operatorname{cosec} x)' = -\csc x \cot x$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x\ ,\cos^2 x\ ,\tan^2 x\ ,\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

$\boldsymbol{xi)\ \int \frac{ax+b}{ax^2+2bx+c}\ dx}$

SOLUTION:

$\int \frac{ax+b}{ax^2+2bx+c}\ dx$

× & ÷ *by 2 to make derivative uper*

$= \frac{1}{2}\int \frac{2(ax+b)}{ax^2+2bx+c}\ dx$

$= \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\ dx$

Using $\int \frac{f'(x)}{[f(x)]} = \boldsymbol{ln}[f(x)]$

$= \frac{1}{2}\ \boldsymbol{ln}(ax^2+2bx+c) + \boldsymbol{c}$

$\boldsymbol{xii)\ \int \cos 3x \sin 2x\ \ dx}$

SOLUTION:

$\int \cos 3x \sin 2x\ \ dx$

× & ÷ *by 2 to make formula*

$= \frac{1}{2}\int 2\cos 3x \sin 2x\ \ dx$

As $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= \frac{1}{2}\int[\sin(3x+2x) - \sin(3x-2x)]\ \ dx$

$= \frac{1}{2}\int[\sin(5x) - \sin(x)]\ \ dx$

$= \frac{1}{2}\{\int \sin 5x\ dx - \int \sin x\ dx\}$

$= \frac{1}{2}\left\{\frac{-\cos 5x}{5} - \frac{-\cos x}{1}\right\} + c$

$= -\frac{1}{2}\left\{\frac{\cos 5x}{5} - \cos x\right\} + c$

$\boldsymbol{xiii)\ \int \frac{\cos 2x-1}{1+\cos 2x}\ \ dx}$

SOLUTION:

$= \int \frac{\mathrm{Cos}\, 2x-1}{1+\cos 2x}\ \ dx$

$= -\int \frac{1-\cos 2x}{1+\cos 2x}\ \ dx$

$\because \boldsymbol{\sin^2 x = \frac{1-\cos 2x}{2} \Rightarrow 2\sin^2 x = 1 - \cos 2x}$

$\because \boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow 2\cos^2 x = 1 + \cos 2x}$

$= -\int \frac{2\sin^2 x}{2\cos^2 x}\ \ dx = -\int \tan^2 x\ \ dx$

$= -\int(\sec^2 x - 1)\ \ dx \quad \because 1+\tan^2 x = \sec^2 x$

$= -\int \sec^2 x\ dx + \int 1\, dx$

$= -\tan x + x + c$

$\boldsymbol{xiv)\ \int \tan^2 x\ \ dx}$

SOLUTION:

$\int \mathrm{Tan}^2 x\ \ dx$

$= \int(\sec^2 x - 1)\ \ dx \quad \because 1+\tan^2\theta = \sec^2\theta$

$= \int \sec^2 x\ dx - \int 1\, dx = \tan x - x + c$

Integration by method of substitution

Sometimes it is possible to convert an integral into standard form by a suitable change of a variable. This is called substitution method.

i.e Evaluate $\int f(x)dx$ *by method of subsitution*

$Let\ x = \emptyset(t) \Rightarrow dx = \emptyset''(t)dt$

So $\int f(x)dx = \int f(\emptyset(t))\emptyset'(t)dt$

Some useful substitutions:

1. $\boldsymbol{\sqrt{a^2-x^2}}$ $\quad \boldsymbol{put x = asin\theta}$ $\quad (\because \boldsymbol{1 - sin^2\theta = cos^2\theta})$
2. $\sqrt{x^2-a^2}$ $\quad put\ x = asec\theta$ $\quad (\because \sec^2\theta - 1 = \tan^2\theta$
3. $\sqrt{a^2+x^2}$ $\quad put\ x = atan\theta$ $\quad (\because \sec^2\theta = 1 + \tan^2\theta$
4. $\sqrt{x+a}(0r)\sqrt{x-a}$ $\quad put\ \sqrt{x+a} = t$ $\quad or(\sqrt{x-a}) = t$
5. $\sqrt{2ax-x^2}$ $\quad put\ x - a = asin\theta$
6. $\sqrt{2ax+x^2}$ $\quad put\ x + a = asec\theta$

Exercise 3.2

Evaluate the following integrals:

$\boldsymbol{Q.1:}\ \int \frac{-2x}{\sqrt{4-x^2}}\ dx$

SOLUTION:

$\int \frac{-2x}{\sqrt{4-x^2}}\ dx$

$= \int (4-x^2)^{-\frac{1}{2}}\ (-2x)\ \ dx$

$Here\ f(x) = 4 - x^2$

$f'(x) = -2x$

$= \frac{(4-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$= \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \ 2\sqrt{4-x^2} + c \quad \because t = 4 - x^2$

$\boldsymbol{Q.2:}\ \int \frac{dx}{x^2+4x+13}$

SOLUTION:

By completing square

$= \int \frac{dx}{x^2+4x+4-4+13}$

$= \int \frac{dx}{(x+2)^2+9}$

$= \int \frac{1}{(x+2)^2+(3)^2}\ dx$

$\because \int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

$\boldsymbol{Q.3:}\ \int \frac{x^2}{4+x^2}\ dx$

SOLUTION:

$(+)\ and\ (-)\ 4$

$= \int \frac{4+x^2-4}{4+x^2}\ dx$

$= \int \left(\frac{4+x^2}{4+x^2} - \frac{4}{4+x^2}\right)\ dx$

$= \int 1\ dx - \int \frac{4}{4+x^2}\ dx$

$= \int 1\ dx - 4\int \frac{1}{2^2+x^2}\ dx$

$= x - 4.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$

$\boldsymbol{Q.4:}\ \int \frac{1}{x\ln x}\ dx$

SOLUTION:

$\int \frac{1}{\ln x} . \frac{1}{x}\ dx$

$As\ f(x) = ln\ x$

$And\ f'(x) = \frac{1}{x}\ ,\ so$

$using\ \int \frac{f'(x)}{[f(x)]} = ln[f(x)]$

$= ln[\ln x] + c$

$\boldsymbol{Q.5:}\ \int \frac{e^x}{e^x+3}\ dx$

SOLUTION:

$\int \frac{e^x}{e^x+3}\ dx$

$Here\ f(x) = e^x$

$And\ f'(x) = e^x\ ,\ so$

$using\ \int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln(e^x + 3) + c$

$\boldsymbol{Q.6:}\ \int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

SOLUTION:

$\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

$\int\ (x^2 + 2bx + c)^{-\frac{1}{2}}.(x+b)\ \ dx$

$Here\ f(x) = x^2 + 2bx + c$

$Here\ f'(x) = 2x + 2b = 2(x+b)$

$\times\ and\ \div\ by\ 2$

$= \frac{1}{2}\int\ (x^2 + 2bx + c)^{-\frac{1}{2}}.2(x+b)\ \ dx$

$= \frac{1}{2}\ \frac{(x^2+2bx+c)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\ \frac{(x^2+2bx+c)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \ \sqrt{x^2 + 2bx + c} + c$

$\boldsymbol{Q.7:}\ \int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

SOLUTION:

$\int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

$= \int (\tan x)^{-\frac{1}{2}}\ \ \sec^2 x$

$Here\ f(x) = \tan x$

$Here\ f'(x) = \sec^2 x$

$= \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{(\tan x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{\tan x} + c$

$\boldsymbol{Q.8:(a)\ \ Show\ that}$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \boldsymbol{ln}\left(x + \sqrt{x^2-a^2}\right) + c$

SOLUTION:

$L.H.S = \int \frac{dx}{\sqrt{x^2-a^2}}$

$Put\ x = a\sec\theta \implies dx = a\sec\theta\ \tan\theta\ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta\ d\theta}{\sqrt{(a\sec\theta)^2-a^2}} = \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2sec^2\ \theta-a^2}}\ \ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2(sec^2\ \theta-1)}}\ \ d\theta = \int \frac{\sec\theta\ \tan\theta}{\sqrt{tan^2\ \theta}}\ \ d\theta$

$= \int \frac{\sec\theta\ \tan\theta}{\tan\theta}\ \ d\theta = \int \sec\theta\ \ d\theta$

$= ln|\sec\theta + \ \tan\theta| + c_1$

$Then\ back\ substitution:$

$x = a\sec\theta \quad \implies \quad \frac{x}{a} = \sec\theta$

$And\ \ 1 + tan^2\ = sec^2\ \theta$

$tan^2\ \theta = sec^2\ \theta - 1$

$\tan\theta = \sqrt{sec^2\ \theta - 1}$

$\tan\theta\ = \sqrt{\left(\frac{x}{a}\right)^2 - 1}$

$\tan\theta\ = \sqrt{\frac{x^2-a^2}{a^2}}$

$\tan\theta\ = \frac{\sqrt{x^2-a^2}}{a}$

$Now\ put\ values$

$= ln\left|\frac{x}{a} + \ \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$

$= ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$

Using $\ln\frac{A}{B} = lnA - lnB$

$= ln\left| x+\sqrt{x^2-a^2}\right| - lna + c_1$

Where $c = -lna + c_1$

$= ln\left| x+\sqrt{x^2-a^2}\right| + c$

Q.9: $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

SOLUTION:

$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

Put $x = \tan\theta$

$\Longrightarrow \quad d(x) = \sec^2\theta \, d\theta$

$= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}} \, d\theta$

$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}} \, d\theta$

$= \int \frac{\sec^2\theta}{\sec^3\theta} \, d\theta$

$= \int \frac{1}{\sec\theta} \, d\theta$

$= \int \cos\theta$

$= \frac{\sin\theta}{1} + c$

$= \frac{\sin\theta}{\cos\theta} . \cos\theta + c$

$= \tan\theta . \cos\theta + c$

$= \frac{\tan\theta}{\sec\theta} + c$

$= \frac{\tan\theta}{\sqrt{\sec^2\theta}} + c$

$= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + c$

Put $\tan\theta = x$

$= \frac{x}{\sqrt{1+x^2}} + c$

Q.10: $\int \frac{1}{(1+x^2)Tan^{-1}x} \, dx$

SOLUTION:

$\int \frac{1}{(1+x^2)Tan^{-1}x} \, dx$

$\int \frac{1}{Tan^{-1}x} \cdot \frac{1}{(1+x^2)} \, dx$

Here $f(x) = Tan^{-1}x$

Here $f'(x) = \frac{1}{(1+x^2)}$

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln|Tan^{-1}x| + c$

Q.11: $\int \sqrt{\frac{1+x}{1-x}} \, dx$

SOLUTION:

By rationalizing

$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}} \, dx$

$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}} \, dx$

$= \int \sqrt{\frac{(1+x)^2}{1-x^2}} \, dx$

$= \int \frac{1+x}{\sqrt{1-x^2}} \, dx$

$= \int \frac{1}{\sqrt{1-x^2}} \, dx + \int \frac{x}{\sqrt{1-x^2}} \, dx$

$= \sin^{-1}x + \frac{1}{-2}\int (1-x^2)^{-\frac{1}{2}}(-2x) \, dx$

$= \sin^{-1}x - \frac{1}{2} . \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \sin^{-1}x - \frac{1}{2} . \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \sin^{-1}x - \sqrt{1-x^2} + c$

Q.12: $\int \frac{\sin\theta}{1+\cos^2\theta} \, d\theta$

SOLUTION:

$\int \frac{\sin\theta}{1+\cos^2\theta} \, d\theta = \int \frac{1}{1+\cos^2\theta} \sin\theta \, d\theta$

Put $\cos\theta = t \Longrightarrow -\sin\theta \, d\theta = dt$

$\int \frac{1}{1+t^2} .-dt = -\tan^{-1}t + c$

Put $t = \cos\theta$

$= -\tan^{-1}(\cos\theta) + c$

Q.13: $\int \frac{ax}{\sqrt{a^2-x^4}} \, dx$

SOLUTION:

$\int \frac{ax}{\sqrt{a^2-x^4}} \, dx = a\int \frac{x}{\sqrt{a^2-(x^2)^2}} \, dx$

Put $x^2 = t \Longrightarrow 2x \, dx = dt \Longrightarrow x \, dx = \frac{1}{2} \, dt$

$= \frac{a}{2}\int \frac{1}{\sqrt{a^2-t^2}} \, dt = \frac{a}{2} \sin^{-1}\frac{t}{a} + c$

using $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\frac{x}{a}$

$= \frac{a}{2} \sin^{-1}\frac{x^2}{a} + c \qquad \because x^2 = t$

Q.14: $\int \frac{dx}{\sqrt{7-6x-x^2}}$

SOLUTION:

$\int \frac{dx}{\sqrt{7-6x-x^2}}$

By completing square

$= \int \frac{dx}{\sqrt{7-x^2-6x-9+9}}$

$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$

$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$

Using $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$

$= \sin^{-1}\frac{x+3}{a} + c$

Q.15: $\int \frac{\cos x}{\sin x ln \sin x} \, dx$

SOLUTION:

$\int \frac{1}{ln \sin x} \frac{\cos x}{\sin x} \, dx$

Here $f(x) = ln \sin x$

And $f'(x) = \frac{\cos x}{\sin x}$, *so*

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln[ln \sin x] + c$

Q.16: $\int \cos x \frac{ln \sin x}{\sin x} \, dx$

SOLUTION:

$\int ln \sin x . \frac{\cos x}{\sin x} \, dx$

Here $f(x) = ln \sin x$

And $f'(x) = \frac{\cos x}{\sin x}$, *so*

$= \frac{[ln \sin x]^{1+1}}{1+1} + c$

$= \frac{1}{2}[ln \sin x]^2 + c$

$Q.17$: $\int \frac{x\,dx}{4+2x+x^2}$

***SOLUTION*:**

$$\int \frac{x\,dx}{4+2x+x^2}$$

$$= \frac{1}{2}\int \frac{2x}{4+2x+x^2}\quad dx$$

$$= \frac{1}{2}\int \frac{2x+2-2}{4+2x+x^2}\quad dx$$

$$= \frac{1}{2}\left\{\int \frac{2x+2}{4+2x+x^2}dx - \int \frac{2}{4+2x+x^2}\quad dx\right\}$$

$$= \frac{1}{2}\left\{\ln(4+2x+x^2) - \int \frac{2}{x^2+2x+1^2+4-1^2}\quad dx\right\}$$

$$= \frac{1}{2}\ln(4+2x+x^2) - \frac{1}{2}\int \frac{2}{(x+1)^2+(\sqrt{3})^2}\quad dx$$

$$\boldsymbol{using\ \int \frac{1}{a^2+x^2}\quad dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$= \frac{1}{2}\ln(x^2+2x+4) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{(x+1)}{\sqrt{3}}\quad + c$$

$Q.18$: $\int \frac{x}{x^{4+2x^2+5}}\ dx$

***SOLUTION*:**

$$= \int \frac{x}{(x^2)^2+2x^2+5}\ dx$$

$Put\quad x^2 = t$

$2x\ dx = dt$

$x\ dx = \frac{1}{2}\ dt$

$$= \int \frac{\frac{1}{2}}{t^2+2t+5}\ dt$$

$$= \frac{1}{2}\int \frac{1}{t^2+2t+1+5-1}\ dt$$

$$= \frac{1}{2}\int \frac{1}{(t+1)^2+2^2}\ dt$$

$$= \frac{1}{2}\cdot\frac{1}{2}\ \tan^{-1}\frac{t+1}{2} + c$$

$Put\quad x^2 = t$

$$= \frac{1}{4}\ \tan^{-1}\frac{x^2+1}{2} + c$$

$Q.19$: $\int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}}-1\right)\right]\ dx$

***SOLUTION*:**

$$\int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}}-1\right)\right]\ dx$$

$Put\ \sqrt{x}-\frac{x}{2} = t$

$$\Longrightarrow\quad d\left(\sqrt{x}-\frac{x}{2}\right) = d(t)$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{2} = dt$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right) = dt$$

$$\left(\frac{1}{\sqrt{x}}-1\right) = 2dt$$

$$= \int[\cos t \times 2\ dt]$$

$$= 2\int[\cos t\ dt]$$

$$= 2\ \frac{\sin t}{1} + c$$

$Put\ value\ of\ t$

$$= 2\ \sin\left(\sqrt{x}-\frac{x}{2}\right) + c$$

$Q.20$: $\int \frac{x+2}{\sqrt{x+3}}\quad dx$

[$Q.19$: *solve on page* 9]

***SOLUTION*:**

$\int \frac{x+2}{\sqrt{x+3}}\quad dx = \int \frac{x+2+1-1}{\sqrt{x+3}}\quad dx = \int \frac{x+3}{\sqrt{x+3}}\quad dx - \int \frac{1}{\sqrt{x+3}}\quad dx = \int \sqrt{x+3}\quad dx - \int \frac{1}{\sqrt{x+3}}\quad dx = \int (x+3)^{\frac{1}{2}}.1\ dx - \int (x+3)^{-\frac{1}{2}}\ .1\ dx$

Now integrate

$$\frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}(x+3)^{\frac{3}{2}} + 2\sqrt{x+3} + c$$

$Q.21$: $\int \frac{\sqrt{2}}{\sin x+\cos x}\ dx$

***SOLUTION*:**

$$\int \frac{1}{\frac{1}{\sqrt{2}}(\cos x+\sin x)}\ dx$$

$$= \int \frac{1}{\cos x\ .\frac{1}{\sqrt{2}}+\sin\mathrm{x}\ .\frac{1}{\sqrt{2}}}\ dx$$

$$= \int \frac{1}{\cos x\ \cos 45°+\sin x\ \ \sin 45°}\ dx$$

$using\ \cos(\alpha-\beta) = \cos\alpha\ \sin\beta + \sin\alpha\ \ \sin\beta$

$$= \int \frac{1}{\cos(x-45°)}\ dx$$

$$= \int \sec(x-45°)\ dx$$

$using\ \int\ \sec x\ \ dx = ln|\sec x + \tan x| + c$

$ln|\sec(x-45°) + \tan(x-45°)| + c$

$Q.22$: $\int \frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$

***SOLUTION*:**

$$= \int \frac{1}{\sin x\ .\ \frac{1}{2}+\cos x\ .\ \frac{\sqrt{3}}{2}}\ dx$$

$$= \int \frac{1}{\sin x\ .\ \ \cos 60°+\cos x\ .\ \ \sin 60°}\ dx$$

$using\quad \sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \ \sin\beta$

$$= \int \frac{1}{\sin(x+60°)}\ dx$$

$$= \int\ \mathrm{cosec}(x+60°)\ \ dx$$

$using\ \int\ \mathrm{cosec}\, x\ \ dx = ln|\mathrm{cosec}\, x - \cot x| + c$

$= ln|\mathrm{cosec}(x-60°) + \cot(x-60°)| + c$

Integration by parts.

We know that for two functions $f\ and\ g$

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$⇨\quad f(x)g'(x) = \frac{d}{dx}\big((f(x)g(x)\big) = f'(x)g(x)$$

Taking integrations w.r.t $x we\ get$

$$\int f(x)g'(x)dx = \int\left[\frac{d}{dx}f(x)g(x) - f'(x)g(x)\right]dx$$

$$= \int (\frac{d}{dx}\big(f(x)g(x)\big) - \int f'(x)g(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Or $\int f(x)g'(x)dx = f(x)\int g'(x)\,dx - \int (g'(x)dx)f'(x)dx$

In other words.

$$\int (1st\ function\,)(2nd\ function)dx$$

$$= (1st\ funct).\int (2nd\ funct.)\,dx$$

$$-\int (integrated\ funct.)\frac{d}{dx}(istfunction)dx$$

This is called “integrations by parts”

Some basic rules for Integration by parts.

*some the function as 2nd function whose integration is known or possible.

*if integration of both given functions are known but one of the given function is polynomial functions then whose polynomial function as first function.

*if integration of both given function are known but no one is polynomial function. Then we may choose any function as 1st.

*if we are given only one function whose integration is unknown or cannot be easily find.

$$i.e, \sin^{-1} x, \cos^{-1} x, \sqrt{a^2-x^2}, \frac{1}{\sqrt{x^2-a^2}} e.t.c$$

Then we take 1 as 2nd function.

"Review above Rules"

	1st function	2nd function
$\int x^n cosdx$	x^n	$cosx$
$\int x^n sinxdx$	x^n	$sinx$
$\int x^n \sin^{-1} xdx$	$\sin^{-1} x$	x^n
$\int x^n \tan^{-1} xdx$	$\tan^{-1} x$	x^n
$\int e^x sinxdx$	e^x or $sinx$	$sinx$ e^x
$\int lnxx^n dx$	x^n	knx
$\int \tan^{-1} xdx$	$\tan^{-1} x$	1
$\int \sqrt{a^2+x^2}dx$	$\sqrt{a^2+x^2}$	1

You may remember the word "ILATE"
I=inverse function
L=logarithmic function
A=algebraic function
T=trigonometric functions
E=exponential functions.

Remember useful formulas

1. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^1\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x+\sqrt{x^2-a^2}\right| + c$
3. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left|x+\sqrt{x^2+a^2}+c\right|$

Prove that $\int e^x\left(f(x)+f'(x)\right)dx + e^x f(x) + c$

Prove: $\int e^x f(x)dx = f(x)e^x - \int e^x f'(x)dx$

⇨ $\int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x)$

⇨ $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x) + c$

Hence proved.

Exercise 3.4

i) $\int x \sin x\ dx$

SOLUTION:

$\int x\ \sin x\ dx$

$Here\ U = x\ ,\ V = \sin x$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x.\int \sin x\,dx - \int[(x)'.\int \sin x\ dx]\,dx$

$= x.(-\cos x) - \int[1.(-\cos x)]\ dx$

$= -x\ \cos x - \int[-\cos x]\ dx$

$= -x\ \cos x + \int[\cos x]\ dx$

$= -x\ \cos x + \sin x + c$

$= \sin x - x\ \cos x + c$

ii) $\int \ln x\ dx$

SOLUTION:

$\int \ln x.\ 1\,dx$

$Here\ U = \ln x\ ,\ V = 1$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx$

$= \ln x.x - \int\left[\frac{1}{x}.x\right]\ dx$

$= \ln x.x - \int 1\,dx$

$= x\ \ln x - x + c$

iii) $\int x \ln x\ dx$

SOLUTION:

$\int x\ \ln x\ dx$

$Here\ U = \ln x\ ,\ V = x$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x\,dx - \int[(\ln x)'.\int x\,dx]\,dx$

$= \ln x.\frac{x^2}{2} - \int\left[\frac{1}{x}.\frac{x^2}{2}\right]\ dx$

$= \ln x.\frac{x^2}{2} - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}.\ \frac{x^2}{2} + c$

$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$

iv) $\int x^2 \ln x\ dx$

SOLUTION:

$\int x^2\ \ln x\ dx$

$Here\ U = \ln x\ ,\ V = x^2$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x^2\,dx - \int[(\ln x)'.\int x^2\,dx]\,dx$

$= \ln x.\frac{x^3}{3} - \int\left[\frac{1}{x}.\frac{x^3}{3}\right]\ dx$

$= \ln x.\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}.\ \frac{x^3}{3} + c$

$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + c$

v) $\int x^3 \ln x\ dx$

SOLUTION:

$\int x^2\ \ln x\ dx$

$Here\ U = \ln x\ ,\ V = x^3$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x^3\,dx - \int[(\ln x)'.\int x^3\,dx]\,dx$

$= \ln x.\frac{x^4}{4} - \int\left[\frac{1}{x}.\frac{x^4}{4}\right]\ dx$

$= \ln x.\frac{x^4}{4} - \frac{1}{4}\int x^3\,dx$

$$= \frac{x^4}{4} \ln x \ - \frac{1}{4} . \ \frac{x^4}{4} + c$$

$$= \frac{x^4}{4}\left(\ln x \ - \frac{1}{4} \right) + c$$

vi) $\boldsymbol{\int x^4 \ \ln x \ dx}$

***SOLUTION*:**

$\int x^4 \ \ln x \ dx$

Here $U = \ln x \ \ , V = x^4$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \ln x \ . \int x^4\, dx - \int [(\ln x)' . \int x^4\, dx]\, dx$

$$= \ln x \ . \frac{x^5}{5} - \int \left[\frac{1}{x} . \frac{x^5}{5} \right] dx$$

$$= \ln x \ . \frac{x^5}{5} - \frac{1}{5} \int x^4\, dx$$

$$= \frac{x^5}{5} \ln x \ - \frac{1}{5} . \ \frac{x^5}{5} + c$$

$$= \frac{x^5}{5}\left(\ln x \ - \frac{1}{5} \right) + c$$

vii) $\boldsymbol{\int \tan^{-1} x \ dx}$

***SOLUTION*:**

$\int 1 . \tan^{-1} x \ dx$

Here $U = \tan^{-1} x \ \ , V = 1$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \tan^{-1} x \ . \int 1\, dx - \int \left[(\tan^{-1} x)' . \int 1\, dx \right] dx$

$$= \tan^{-1} x \ . x \ - \int \left[\frac{1}{1+x^2} . x \right] dx$$

$$= x \tan^{-1} x \ - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx$$

$$= x \tan^{-1} x \ - \frac{1}{2} \ln|1 + x^2| +$$

viii) $\boldsymbol{\int x^2 \ \sin x \ dx}$

***SOLUTION*:**

$\int x^2 \ \sin x \ dx$n

Here $U = x^2 \ \ , V = \sin x$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= x^2 . \int \sin x\, dx - \int \left[(x^2)' . \int \sin x \ dx \right] dx$

$= x^2 . (-\cos x) - \int [2x . (-\cos x)] \ dx$

$= -x^2 \ \cos x + 2 \int [x . \cos x] \ dx$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= -x^2 \ \cos x + 2\{x . \int \cos x\, dx -$
$\int [(x)' . \int \cos x \ dx]\, dx\}$

$= -x^2 \ \cos x + 2\{x . \sin x - \int [1 . \sin x]\}$

$= -x^2 \ \cos x + 2x . \sin x - 2 \int \sin x \ dx$

$= -x^2 \ \cos x + 2x . \sin x - 2(-\cos x) + c$

$= -x^2 \ \cos x + 2x \ \sin x + 2 \cos x + c$

ix) $\boldsymbol{\int x^2 \ \tan^{-1} x \ dx}$

***SOLUTION*:**

$\int x^2 . \tan^{-1} x \ dx$

Here $U = \tan^{-1} x \ \ , V = x^2$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \tan^{-1} x \ . \int x^2\, dx - \int \left[(\tan^{-1} x)' . \int x^2\, dx \right] dx$

$$= \tan^{-1} x . \frac{x^3}{3} - \int \left[\frac{1}{1+x^2} . \frac{x^3}{3} \right] dx$$

$$= \frac{x^3}{3} \tan^{-1} x \ - \frac{1}{3} \int \frac{x^3}{1+x^2}\, dx \quad ,$$

$$= \frac{x^3}{3} \tan^{-1} x \ - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x \ - \frac{1}{3} \int x\, dx + \frac{1}{3.2} \int \frac{2x}{1+x^2}\, dx$$

$$= \frac{x^3}{3} \tan^{-1} x \ - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \ \ln|1 + x^2| + c$$

x

$1 + x^2 \sqrt{x^3}$

$\pm x^3 \pm 1$

-1

$$\because \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

x) $\boldsymbol{\int x \ \tan^{-1} x \ dx}$

***SOLUTION*:**

$\int x . \tan^{-1} x \ dx$

Here $U = \tan^{-1} x \ \ , V = x$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \tan^{-1} x \ . \int x\, dx - \int \left[(\tan^{-1} x)' . \int x\, dx \right] dx$

$$= \tan^{-1} x . \frac{x^2}{2} \ - \int \left[\frac{1}{1+x^2} . \frac{x^2}{2} \right] dx$$

$$= \frac{x^2}{2} \tan^{-1} x \ - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \tan^{-1} x \ - \frac{1}{2} \int \left(\frac{1+x^2-1}{1+x^2} \right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x \ - \frac{1}{2} \int \frac{1+x^2}{1+x^2}\, dx - \frac{1}{2} \int \frac{1}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \tan^{-1} x \ - \frac{1}{2} \int 1\, dx - \frac{1}{2} \ \tan^{-1} x$$

$$= \frac{x^2}{2} \tan^{-1} x \ - \frac{1}{2} \ x - \frac{1}{2} \ \tan^{-1} x + c$$

$$= \left(\frac{1}{2} \ \tan^{-1} x \right) (x^2 + 1) - \frac{1}{2} x + c$$

1

$1 + x^2 \sqrt{x^2}$

$\pm x^2 \pm 1$

-1

xi) $\boldsymbol{\int x^3 \ \tan^{-1} x \ dx}$

***SOLUTION*:**

$\int x^3 . \tan^{-1} x \ dx$

Here $U = \tan^{-1} x \ \ , V = x^3$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \tan^{-1} x \ . \int x^3\, dx - \int \left[(\tan^{-1} x)' . \int x^3\, dx \right] dx$

$$= \tan^{-1} x . \frac{x^4}{4} \ - \int \left[\frac{1}{1+x^2} . \frac{x^4}{4} \right] dx$$

$$= \frac{x^4}{4} \tan^{-1} x \ - \frac{1}{4} \int \frac{x^4}{1+x^2}\, dx$$

$$= \frac{x^4}{4} \tan^{-1} x \ - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^4}{4} \tan^{-1} x \ - \frac{1}{4} \int x^2\, dx + \frac{1}{4} \int 1 - \frac{1}{4} \int \frac{1}{1+x^2}\, dx$$

$$= \frac{x^4}{4} \tan^{-1} x \ - \frac{1}{4} \ \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \ \tan^{-1} x + c$$

$$= \frac{1}{4} \left[x^4 \ \tan^{-1} x \ - \ \frac{x^3}{3} + x - \ \tan^{-1} x \right] + c$$

$$= \frac{1}{4} \left[(x^4 - 1) \ \tan^{-1} x \ - \ \frac{x^3}{3} + x \right] + c$$

$$= \frac{1}{4} \left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x \right] + c$$

$x^2 - 1$

$1 + x^2 \sqrt{x^4}$

$\pm x^4 \pm \ {}^2$

$-x^2$

$\mp x^2 \mp 1$

1

xii) $\boldsymbol{\int x^3 \ \cos x \ dx}$

***SOLUTION*:**

$\int x^3 \ \cos x \ dx$n

Here $U = x^3 \ \ , V = \cos x$

Using $\int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= x^3 . \int \cos x\, dx - \int \left[(x^3)' . \int \cos x \ dx \right] dx$

$= x^3 . (\sin x) - \int [3x^2 . (\sin x)] \ dx$

$= x^3 \ \sin x - 3 \int [x^2 \ \sin x] \ dx$

Using $\int U.V = U \ \int V\, dx - \int [U' \int V\, dx]\, dx$

$= x^3 \ \sin x - 3\{x^2 . \int \sin x\, dx -$
$\int \left[(x^2)' . \int \sin x \ dx \right] dx\}$

$= x^3 \ \sin x - 3\{x^2 . (- \ \cos x) - \int [2x \, (- \ \cos x) \,]\}$

$= x^3 \ \sin x + 3x^2 \ \cos x - 6 \int \ x \cos x \ dx$

$= x^3 \ \sin x + 3x^2 \ \cos x - 6\{x \int \cos x \ dx -$
$\int [(x)' \int \cos x\, dx]\, dx\}$

$= x^3 \ \sin x + 3x^2 \ \cos x - 6\{x \ \sin x - \int [1. \sin x]\, dx\}$

$$= x^3 \ \sin x + 3x^2 \ \cos x - 6\{x \ \sin x - \int \sin x \ dx\}$$

$$= x^3 \ \sin x + 3x^2 \ \cos x - 6\{x \ \sin x - (-\cos x)\} + c$$

$$= x^3 \ \sin x + 3x^2 \ \cos x - 6x \ \sin x - 6 \cos x + c$$

$$= (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c$$

$xiii) \displaystyle\int \sin^{-1} x \ dx$

SOLUTION:

$$\int 1 . \sin^{-1} x \ dx$$

Here $U = \sin^{-1} x \quad , V = 1$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$= \sin^{-1} x . \int 1 \, dx - \int \left[(\sin^{-1} x)' . \int 1 \, dx\right] dx$$

$$= \sin^{-1} x . x - \int \left[\frac{1}{\sqrt{1-x^2}} . x\right] dx \qquad skip$$

$$= x \sin^{-1} x - \frac{1}{-2} \int (1 - x^2)^{-\frac{1}{2}} (-2x) \, dx$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + c$$

$xiv) \displaystyle\int x \ \sin^{-1} x \ dx$

SOLUTION:

$$\int x . \sin^{-1} x \ dx$$

Here $U = \sin^{-1} x \quad , V = x$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$= \sin^{-1} x . \int x \, dx - \int \left[(\sin^{-1} x)' . \int x \, dx\right] dx$$

$$= \sin^{-1} x . \frac{x^2}{2} - \int \left[\frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2}\right] dx$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \ dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} \ dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} \ dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} \ dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \ dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1 - x^2} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \ dx + c$$

Using $\sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left\{\frac{1}{2} \sin^{-1} x + \frac{x}{2} \sqrt{1 - x^2}\right\} - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1 - x^2} - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \left(\frac{1}{4} - \frac{1}{2}\right) \sin^{-1} x + \frac{x}{4} \sqrt{1 - x^2} + c$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1 - x^2} + c$$

$xv) \int e^x \sin x \cos x \ dx$

SOLUTION:

Let $I = \int e^x \sin x \cos x \ dx$

Multiply and divide by 2

$$I = \frac{1}{2} \int e^x 2 \sin x \cos x \ dx$$

$$I = \frac{1}{2} \int e^x \ \sin 2x \ dx$$

Here $U = \sin 2x \quad , V = e^x$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$I = \sin 2x \int e^x \ dx - \int [(\sin 2x)' . \int e^x \, dx] \, dx$$

$$I = \sin 2x \ e^x - \int [\cos 2x . 2 . e^x] \, dx$$

$$I = \sin 2x \ e^x - 2 \int \cos 2x \ e^x \, dx$$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$I = \sin 2x \ e^x - 2\left\{\cos 2x \int e^x \ dx - \int \left[(\cos 2x)' . \int e^x \, dx\right] dx \right\}$$

$$I = \sin 2x \, e^x - 2\{\cos 2x \ e^x - \int (-\sin 2x) \ e^x \, \mathrm{dx} \}$$

$$I = e^x \sin 2x - 2 \cos 2x \ e^x - 2 \int 2 \ \sin x \ \cos x \ e^x \ \mathrm{dx}$$

$$I = e^x \sin 2x - 2 \cos 2x \ e^x - 4 \int \sin x \ \cos x \ e^x \ \mathrm{dx}$$

Put $I = \int e^x \sin x \cos x \ dx$

$$I = e^x \sin 2x - 2 \cos 2x \ e^x - 4 \, I$$

$$5I = e^x (\sin 2x - 2 \cos 2x)$$

$$I = \frac{e^x}{5} (\sin 2x - 2 \cos 2x)$$

$xvi) \displaystyle\int x \ \sin x \cos x \ dx$

SOLUTION:

$$\int x \ \sin x \cos x \ dx = \frac{1}{2} \int x . 2 \sin x \cos x \ dx = \frac{1}{2} \int x \ \sin 2x \ dx$$

Here $U = x \quad , V = \sin 2x$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$= \frac{1}{2} [x . \int \sin 2x \ dx - \int [(x)' . \int \sin 2x \ dx] \, dx]$$

$$= \frac{1}{2} x . \left(-\frac{\cos 2x}{2}\right) - \frac{1}{2} \int \left[1 \left(-\frac{\cos 2x}{2}\right)\right] dx = -\frac{1}{4} x \ \cos 2x + \frac{1}{4} \int \cos 2x \ dx$$

$$= -\frac{1}{4} x \ \cos 2x + \frac{1}{4} \frac{\sin 2x}{2} + c = \frac{1}{4} \left[-x \ \cos 2x + \frac{\sin 2x}{2}\right] + c = \frac{1}{4} \left[-x \ \cos 2x + \frac{2 \sin x \cos x}{2}\right] + c$$

$$= \frac{1}{4} [-x \ \cos 2x + \sin x \cos x] + c = \frac{1}{4} [\sin x \cos x - x \ \cos 2x] + c$$

$xvii) \displaystyle\int x \ \cos^2 x \ dx$

SOLUTION:

$$\int x \ \cos^2 x \ dx = \int x . \frac{1+\cos 2x}{2} \ dx \ \boldsymbol{As} \ \boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$= \frac{1}{2} \int x . (1 + \cos 2x) \ dx = \frac{1}{2} \int (x + x \cos 2x) \ dx$$

$$= \frac{1}{2} \int x \ dx + \frac{1}{2} \int x \cos 2x \ dx = \frac{1}{2} \frac{x^2}{2} + \frac{1}{2} \int x \cos 2x \ dx$$

Here $U = x \quad , V = \cos 2x$

Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$

$$= \frac{x^2}{4} + \frac{1}{2} [x . \int \cos 2x \ dx - \int [x' . \int \cos 2x \, dx] \, dx]$$

$$= \frac{x^2}{4} + \frac{1}{2} \left[x . \frac{\sin 2x}{2} - \int \left[1 . \frac{\sin 2x}{2}\right] dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{2} x . \frac{\sin 2x}{2} - \frac{1}{4} \int \sin 2x \, dx$$

$$= \frac{x^2}{4} + \frac{x \ \sin 2x}{4} - \frac{1}{4} \frac{-\cos 2x}{2}$$

$$= \frac{1}{4} \left(x^2 + \mathrm{x} \sin 2x + \frac{1}{2} \cos 2x\right) + c$$

$xviii) \displaystyle\int x \ \sin^2 x \ dx$

SOLUTION:

$\int x\ \sin^2 x\ dx = \int x\,.\frac{1-\cos 2x}{2}\ dx$ *As* $\sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2}\int x\,.(1-\cos 2x)\ dx = \frac{1}{2}\int\ (x - x\cos 2x)\ dx$

$= \frac{1}{2}\int x\ dx - \frac{1}{2}\int\ x\cos 2x\ dx = \frac{1}{2}\ \frac{x^2}{2} - \frac{1}{2}\int\ x\cos 2x\ dx$

Here $U = x\ \ ,\ V = \cos 2x$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \frac{x^2}{4} - \frac{1}{2}\ [x\,.\int \cos 2x\ dx - \int[x'.\int \cos 2x\,dx]\,dx]$

$= \frac{x^2}{4} - \frac{1}{2}\left[x\,.\frac{\sin 2x}{2} - \int\left[1.\frac{\sin 2x}{2}\right] dx\right]$

$= \frac{x^2}{4} - \frac{1}{2}x\,.\frac{\sin 2x}{2} + \frac{1}{4}\int \sin 2x\,dx$

$= \frac{x^2}{4} - \frac{\text{x}\sin 2x}{4} + \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$

$= \frac{1}{4}\left(x^2 - \text{x}\sin 2x - \frac{1}{2}\cos 2x\right) + c$

xix) $\int(\ln x)^2\ dx$

SOLUTION:

$\int(\ln x)^2\,.1\,dx$

Here $U = (\ln x)^2\ \ ,\ V = 1$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= (\ln x)^2\,.\int 1\,dx - \int[((\ln x)^2)'.\int 1\,dx]\,dx$

$= (\ln x)^2\,.x - \int\left[2(\ln x)\frac{1}{x}.x\right]\ dx$

$= x(\ln x)^2\ - 2\int \ln x\ dx$

$= x(\ln x)^2\ - 2[\int (\ln x)\,.1\ dx]$

Here $U = \ln x\ \ ,\ V = 1$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x(\ln x)^2\ - 2[\ln x\ .\int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx]$

$= x(\ln x)^2\ - 2\left[\ln x\ .x\ - \int\left[\frac{1}{x}.x\right]\ dx\right]$

$= x(\ln x)^2\ - 2[\ln x\ .x\ - \int 1\,dx]$

$= x(\ln x)^2\ - 2[x\ \ln x\ \ - x] + c$

$= x(\ln x)^2\ - 2x\ \ln x\ + 2x + c$

$= x\ln x\ (\ln x - 2\,) + 2x + c$

xx) $\int \ln(\tan x)\ \sec^2 x\ dx$

SOLUTION:

$\int \ln(\tan x)\ \sec^2 x\ dx$

Here $U = \ln(\tan x)\ \ \ ,\ V = \sec^2 x$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln(\tan x)\ \ .\int \sec^2 x\ dx - \int[[\ln(\tan x)\,]'.\int \sec^2 x\ dx]\,dx$

$= \ln(\tan x)\,.\ \tan x - \int\left[\frac{\sec^2 x}{\tan x}.\tan x\right]\ dx$

$= \tan x\,.\ln(\tan x)\ \ - \int \sec^2 x\ dx$

$= \tan x\,.\ln(\tan x)\ \ - \tan x + c$

xxi) $\int \frac{x\,.\ \sin^{-1} x}{\sqrt{1-x^2}}\ dx$

SOLUTION:

$\int \frac{x\,.\ \sin^{-1} x}{\sqrt{1-x^2}}\ dx = \frac{1}{-2}\int \sin^{-1} x\left[(1-x^2)^{-\frac{1}{2}}(-2x)\right]$

Here $U = \sin^{-1} x\ \ \ ,\ V = (1-x^2)^{-\frac{1}{2}}(-2x)$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= -\frac{1}{2}\left\{\sin^{-1} x\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx - \int\left[(\sin^{-1} x)'.\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx\right]\,dx\right\}$

$= -\frac{1}{2}\left\{\sin^{-1} x\,\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int\left[\frac{1}{\sqrt{1-x^2}}\,.\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]\ dx\right\} = -\frac{1}{2}\left\{\sin^{-1} x\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int\left[\frac{1}{\sqrt{1-x^2}}\,.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]\ dx\right\}$

$= -\frac{1}{2}\{2\ \sin^{-1} x\sqrt{1-x^2} - \int[2]\,dx\} = -\frac{1}{2}\{2\ \sin^{-1} x\sqrt{1-x^2} - 2\int 1\,dx\} = -\ \sin^{-1} x\sqrt{1-x^2} + x + c$

$= x - \sqrt{1-x^2}\ \sin^{-1} x + c$

Q. 2: *Evaluate the following integrals*:

i) $\int \tan^4 x\ dx$

SOLUTION:

$\int \tan^4 x\ dx$

$= \int \tan^2 x\ .\tan^2 x\ dx$

$= \int \tan^2 x\ (\sec^2 x - 1)\,dx$

$= \int \tan^2 x\ \sec^2 x\,dx - \int(\sec^2 x - 1)\ \ dx$

$= \frac{\tan^3 x}{3} - \int\ \sec^2 x\,dx + \int 1\ \ dx$

$= \frac{\tan^3 x}{3} - \tan x + x + c$

ii) $\int \sec^4 x\ dx$

SOLUTION:

$\int \sec^4 x\ dx$

$= \int \sec^2 x\ .\sec^2 x\ dx$

$= \int \sec^2 x\ (1 + \tan^2 x)\,dx$

$= \int \sec^2 x\ \ dx + \int\ \sec^2 x\ \tan^2 x\ \ dx$

$= \tan x + \frac{\tan^3 x}{3} + c$

$= \tan x + \frac{1}{3}\tan^3 x + c$

iv) $\int \tan^3 x \sec x\ dx$

SOLUTION:

$\int \tan^3 x\sec x\ dx$

$\int \tan^2 x\ \tan x\sec x\ dx$

$= \int(\sec^2 x - 1)\ \tan x\sec x\ dx$

$= \int \sec^2 x\sec x\tan x\ dx - \int \sec x\tan x\ \ dx$

$= \frac{1}{3}\sec^3 x - \sec x + c$

v) $\int \mathrm{x}^3\,e^{5x}\,dx$

SOLUTION:

$\int \mathrm{x}^3\,e^{5\mathrm{x}}\,dx$

Here $U = \mathrm{x}^3\ ,\ V = e^{5x}$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \mathrm{x}^3\ \int e^{5x}\,dx - \int[(\mathrm{x}^3)'.\int e^{5x}\,dx]\,dx$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \int 3x^2\ \frac{e^{5x}}{5}\ \ dx$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{5}\{\int x^2\ e^{5x}\ \ dx\}$

Again integrating by parts

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{5}\{\mathrm{x}^2\ \int e^{5x}\,dx - \int[(\mathrm{x}^2)'.\int e^{5x}\,dx]\,dx\}$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{5}\left\{\mathrm{x}^2\ \frac{e^{5x}}{5} - \int 2x\ \frac{e^{5x}}{5}\ dx\right\}$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{25}\mathrm{x}^2\ e^{5x} + \frac{6}{25}\int x\,e^{5x}\ dx$

Again integrating by parts

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{25}\mathrm{x}^2\ e^{5x} + \frac{6}{25}\{x\ \int e^{5x}\ dx -$
$\int[(\mathrm{x})'.\int e^{5x}\ dx]\ dx\}$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{25}\mathrm{x}^2\ e^{5x} + \frac{6}{25}\left\{x\ \frac{e^{5x}}{5} - \int 1\,.\frac{e^{5x}}{5}\ dx\right\}$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{25}\mathrm{x}^2\ e^{5x} + \frac{6}{125}x\ e^{5x} - \frac{6}{125}\int e^{5x}\ dx$

$= \mathrm{x}^3\ \frac{e^{5x}}{5} - \frac{3}{25}\mathrm{x}^2\ e^{5x} + \frac{6}{125}x\ e^{5x} - \frac{6}{125}\frac{e^{5x}}{5} + c$

$= \frac{e^{5x}}{5}\left(\mathrm{x}^3\ - \frac{3}{5}\mathrm{x}^2\ + \frac{6}{25}x - \frac{6}{125}\right) + c$

vi) $\int e^{-x}\ \sin 2x\,dx$

SOLUTION:

Let $I = \int\ \sin 2x\,e^{-x}\ dx$

Here $U = \sin 2x\ \ ,\ V = e^{-x}$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$I = \sin 2x\ \int e^{-x}\ dx - \int[(\sin 2x)'\int e^{-x}\ dx]\ dx$

$I = \sin 2x\ \frac{e^{-x}}{-1} - \int\left[(\cos 2x\,.\,2)\frac{e^{-x}}{-1}\right]\ dx$

$I = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\ \ dx$

Again integrating by parts

$I = -e^{-x}\sin 2x + 2\{\cos 2x\ \int e^{-x}\ dx -$
$\int[(\cos 2x)'\int e^{-x}\ dx]\ dx\}$

$I = -e^{-x}\sin 2x + 2\left\{\cos 2x\ \frac{e^{-x}}{-1} -\right.$
$\left.\int\left[(-\sin 2x.\,2)\frac{e^{-x}}{-1}\right]\ dx\right\}$

$I = -e^{-x}\sin 2x - 2\cos 2x\ \ e^{-x} -$
$4\int\ e^{-x}\sin 2x\ \ dx$

$I = -e^{-x}\sin 2x - 2\cos 2x\ \ e^{-x} - 4I + c_1$

$5I = -e^{-x}\sin 2x - 2\cos 2x\ \ e^{-x} + c_1$

$I = -\frac{2}{5}\cos 2x\ \ e^{-x} - \frac{1}{5}e^{-x}\sin 2x + \frac{c_1}{5}$

$I = -\frac{2}{5}e^{-x}\ (\cos 2x + \frac{1}{2}e^{-x}\sin 2x) + c$ *where* $c =$
$\frac{c_1}{5}$

vii) $\int e^{2x}\ \cos 3x\,dx$

SOLUTION:

Let $I = \ \int e^{2x}\ \cos 3x\,dx$

Here $U = \cos 3x\ \ ,\ V = e^{2x}$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$I = \cos 3x\ \int e^{2x}\ dx - \int[(\cos 3x)'\int e^{2x}\ dx]\ dx$

$I = \cos 3x\ \frac{e^{2x}}{2} - \int\left[(-\sin 3x\,.\,3)\frac{e^{2x}}{2}\right]\ dx$

$I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{2}\int[\sin 3x\,e^{2x}]\ dx$

Again integrating by parts

$I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{2}\{\sin 3x\ \int e^{2x}\ dx -$
$\int[(\sin 3x)'\int e^{2x}\ dx]\ dx\}$

$I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{2}\left\{\sin 3x\,\frac{e^{2x}}{2} -\right.$
$\left.\int\left[\cos 3x\,.\,3\,.\frac{e^{2x}}{2}\right]\ dx\right\}$

$I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\,e^{2x} - \frac{9}{4}\int\cos 3x\ \ e^{2x}\ dx$

$I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\,e^{2x} - \frac{9}{4}I + c_1$

$I + \frac{9}{4}I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\,e^{2x} + c_1$

$\frac{13}{4}I = \cos 3x\ \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\,e^{2x} + c_1$

$I = \ \frac{4}{13}\frac{e^{2x}}{2}(\cos 3x\ \ + \frac{3}{2}\sin 3x) + \frac{4}{13}c_1$

$I = \ \frac{2}{13}e^{2x}\left(\cos 3x\ \ + \frac{3}{2}\sin 3x\right) +$
c *where* $\frac{4}{13}c_1 = c$

$I = \ \frac{3}{13}e^{2x}(\sin 3x + \frac{2}{3}\cos 3x\ ) + c$

viii) $\int \mathbf{cosec^3}\,\boldsymbol{x}\,\boldsymbol{dx}$

SOLUTION:

Let $I = \ \int \mathrm{cosec}^2\,x\ \mathrm{cosec}\,x\ dx$

Here $U = \mathrm{cosec}\,x\ \ ,\ V = \mathrm{cosec}^2\,x$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= \mathrm{cosec}\,x\ \int \mathrm{cosec}^2\,x\ \ dx -$
$\int[(\mathrm{cosec}\,x)'.\int \mathrm{cosec}^2\,x\,dx]\,dx$

$I = \mathrm{cosec}\,x\ \ (-\cot x) - \int[(-\mathrm{cosec}\,x\cot x)(-\cot x)]\ dx$

$I = -\mathrm{cosec}\,x\ \ \cot x - \int \mathrm{cosec}\,x\cot^2 x\ \ dx$

$I = -\mathrm{cosec}\,x\ \ \cot x - \int \mathrm{cosec}\,x\,(\mathrm{cosec}^2\,x - 1)\ dx$

$I = -\mathrm{cosec}\,x\ \ \cot x - \ \int \mathrm{cosec}^3\,x\,dx + \ \int \mathrm{cosec}\,x\,dx$

$I = -\mathrm{cosec}\,x\ \ \cot x - \mathrm{I} + \ \int \mathrm{cosec}\,x\,dx$

$2I = -\mathrm{cosec}\,x\ \ \cot x + \ln|\mathrm{cosec}\,x - \cot x| + c_1$

$I = -\frac{1}{2}[\ \cot x\ \mathrm{cosec}\,x - \ln|\mathrm{cosec}\,x - \cot x|] + \frac{1}{2}c_1$

$I = -\frac{1}{2}[\ \cot x\ \mathrm{cosec}\,x - \ln|\mathrm{cosec}\,x - \cot x|] + c$

TIT BIT:

Jab pure quadratic equation h aur us ka derivative b majood na h t substituition s solve karty h aur substitution m trigonometry functions hi let karty lakin j c s start hu w let nai karny nai t book answer ni aye ga ut jin pure quadratic equation walay questions ki power $\frac{1}{2}$ h t un k ap by parts integration k method s b kar saktay h.

Q.3: ***Show that*** $\int \boldsymbol{e^{ax}}\,\mathbf{sin}\,\boldsymbol{bx}\,\boldsymbol{dx}$

$$= \frac{\mathbf{1}}{\sqrt{\boldsymbol{a^2 + b^2}}}\boldsymbol{e^{ax}}\,\mathbf{sin}(\boldsymbol{bx} - \mathbf{tan}^{-1}\left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)$$
$$+\ \boldsymbol{c}$$

SOLUTION:

Let $I = \ \int e^{ax}\sin bx\,dx$

Here $U = \sin bx\ \ ,\ V = e^{ax}$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$I = \sin bx\int e^{ax}\ dx - \int[(\sin bx)'.\int e^{ax}\ dx]\ dx$

$I = \sin bx\,\frac{e^{ax}}{a} - \int\left[\cos bx\ .\ b.\frac{e^{ax}}{a}\right]\ dx$

$I = \sin bx\,\frac{e^{ax}}{a} - \frac{b}{a}\int\cos bx\ \ e^{ax}\ dx$

Again integrating by parts

$I = \sin bx\,\frac{e^{ax}}{a} - \frac{b}{a}\{\cos bx\int e^{ax}\ dx -$
$\int[(\cos bx)'.\int e^{ax}\ dx]\ dx\}$

$I = \sin bx\,\frac{e^{ax}}{a} - \frac{b}{a}\left\{\cos bx\,\frac{e^{ax}}{a} - \int\left[-\sin bx\ .\,b.\frac{e^{ax}}{a}\right]\ dx\right\}$

$I = \sin bx\,\frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\,e^{ax} - \frac{b^2}{a^2}\int\sin bx\ \ e^{ax}\ dx$

$I = \sin bx\,\frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\,e^{ax} - \frac{b^2}{a^2}I + c_1$

$$I+\frac{b^2}{a^2}I=\sin bx\,\frac{e^{ax}}{a}-\frac{b}{a^2}\cos bx\,e^{ax}+c_1$$

$$\left(\frac{a^2+b^2}{a^2}\right)I=e^{ax}\left(\sin bx\,\frac{1}{a}-\frac{b}{a^2}\cos bx\right)+c_1$$

$$I=\frac{a^2}{a^2+b^2}e^{ax}\left(\sin bx\,\frac{1}{a}-\frac{b}{a^2}\cos bx\right)+\frac{a^2}{a^2+b^2}c_1$$

$$I=\frac{1}{a^2+b^2}e^{ax}(a\ \sin bx-b\cos bx)+$$

c (A) where $\frac{a^2}{a^2+b^2}c_1=c$

Let $a=r\cos\theta$ (1) , $b=r\sin\theta$ (2)

Squaring and adding (1) and (2)

dividing (1) and (2)

$$a^2+b^2=r^2\cos^2\theta+r^2\sin^2\theta \qquad \frac{r\sin\theta}{r\cos\theta}=\frac{b}{a}$$

$$a^2+b^2=r^2(\cos^2\theta+\sin^2\theta) \qquad \tan\theta=\frac{b}{a}$$

$$a^2+b^2=r^2 \quad\Rightarrow\quad r=\sqrt{a^2+b^2} \qquad \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

Now Put values in (A)

$$I=\frac{1}{a^2+b^2}e^{ax}(r\cos\theta\ \ \sin bx-r\sin\theta\cos bx)+c$$

$$I=\frac{r}{a^2+b^2}e^{ax}(\cos\theta\ \ \sin bx-\sin\theta\cos bx)+$$

c (Take r common)

$$I=\frac{\sqrt{a^2+b^2}}{a^2+b^2}e^{ax}(\sin bx\cos\theta-\cos bx\sin\theta)+$$

c (Put value r)

$$I=\frac{1}{\sqrt{a^2+b^2}}e^{ax}(\sin(bx-\theta))c$$

Using $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$I=\frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx-\tan^{-1}\left(\frac{b}{a}\right)+c\right.$$

Put $\theta=\tan^{-1}\left(\frac{b}{a}\right)$ have proved.

Q.4: *Evaluate the following indefinate integrals*:

i) $\int\sqrt{a^2-x^2}\,dx$

SOLUTION:

Let $I=\int\sqrt{a^2-x^2}\,.1\,dx$

Here $U=\sqrt{a^2-x^2}$, $V=1$

Using $\int U.V=U.\int V\,dx-\int[U'.\int V\,dx]\,dx$

$$I=\sqrt{a^2-x^2}\int 1\,dx-\int\left[\left(\sqrt{a^2-x^2}\right)'.\int 1\,dx\right]dx$$

$$I=\sqrt{a^2-x^2}\,.x-\int\left[\frac{-2x}{2\sqrt{a^2-x^2}}.x\right]dx$$

$$I=x\sqrt{a^2-x^2}-\int\frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$I=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$I=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx+\int\frac{a^2}{\sqrt{a^2-x^2}}\,dx$$

$$I=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$$

Using $\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a}+c$

$$I=x\sqrt{a^2-x^2}-I+a^2\sin^{-1}\frac{x}{a}+c_1$$

$$I+I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}+c_1$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}+c_1$$

$$I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{c_1}{2}$$

$I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$ where $\frac{c_1}{2}=$

c

ii) $\int\sqrt{x^2-a^2}\,dx$

SOLUTION:

Let $I=\int\sqrt{x^2-a^2}\,.1\,dx$

Here $U=\sqrt{x^2-a^2}$, $V=1$

Using $\int U.V=U.\int V\,dx-\int[U'.\int V\,dx]\,dx$

$$I=\sqrt{x^2-a^2}\int 1\,dx-\int\left[\left(\sqrt{x^2-a^2}\right)'.\int 1\,dx\right]dx$$

$$I=\sqrt{x^2-a^2}\,.x-\int\left[\frac{2x}{2\sqrt{x^2-a^2}}.x\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx-\int\frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$I=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,dx-a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx$$

Using $\int\frac{1}{\sqrt{x^2-a^2}}\,dx=ln\left|x+\sqrt{x^2-a^2}\right|+c$

$$I=x\sqrt{x^2-a^2}-I-a^2ln\left|x+\sqrt{x^2-a^2}\right|+c_1$$

$$I+I=x\sqrt{x^2-a^2}-a^2ln\left|x+\sqrt{x^2-a^2}\right|+c_1$$

$$2I=x\sqrt{x^2-a^2}-a^2ln\left|x+\sqrt{x^2-a^2}\right|+c_1$$

$$I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}ln\left|x+\sqrt{x^2-a^2}\right|+\frac{c_1}{2}$$

$$I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}ln\left|x+\sqrt{x^2-a^2}\right|+c$$

where $\frac{c_1}{2}=c$

iii) $\int\sqrt{4-5x^2}\,dx$

SOLUTION:

Let $I=\int\sqrt{4-5x^2}\,.1\,dx$

Here $U=\sqrt{4-5x^2}$, $V=1$

Using $\int U.V=U.\int V\,dx-\int[U'.\int V\,dx]\,dx$

$$I=\sqrt{4-5x^2}\int 1\,dx-\int\left[\left(\sqrt{4-5x^2}\right)'.\int 1\,dx\right]dx$$

$$I=\sqrt{4-5x^2}\,.x-\int\left[\frac{-10x}{2\sqrt{4-5x^2}}.x\right]dx$$

$$I=x\sqrt{4-5x^2}-\int\frac{-5x^2}{\sqrt{4-5x^2}}\,dx$$

$$I=x\sqrt{4-5x^2}-\int\frac{4-5x^2-4}{\sqrt{4-5x^2}}\,dx$$

$$I=x\sqrt{4-5x^2}-\int\frac{4-5x^2}{\sqrt{4-5x^2}}\,dx+\int\frac{4}{\sqrt{4-5x^2}}\,dx$$

$$I=x\sqrt{4-5x^2}-\int\sqrt{4-5x^2}\,dx+4\int\frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}\,dx$$

$$I=x\sqrt{4-5x^2}-\int\sqrt{4-5x^2}\,dx+$$

$$\frac{4}{\sqrt{5}}\int\frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2-x^2}}\,dx$$

Using $\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a}+c$

$$I=x\sqrt{4-5x^2}-I+\frac{4}{\sqrt{5}}\sin^{-1}\left(\frac{x}{\frac{2}{\sqrt{5}}}\right)+c_1$$

$$2I=x\sqrt{4-5x^2}+\frac{4}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)+c_1$$

$$I=\frac{x}{2}\sqrt{4-5x^2}+\frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)+\frac{c_1}{2}$$

$I=\frac{x}{2}\sqrt{4-5x^2}+\frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)+c$ where $\frac{c_1}{2}=$

c

iv) $\int\sqrt{3-4x^2}\,dx$

SOLUTION:

Let $I=\int\sqrt{3-4x^2}\,.1\,dx$

Here $U=\sqrt{3-4x^2}$, $V=1$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{3-4x^2}\ \int 1\,dx - \int\left[\left(\sqrt{3-4x^2}\,\right)'.\int 1\,dx\right]\,dx$

$I = \sqrt{3-4x^2}\,.x - \int\left[\frac{-8x}{2\sqrt{3-4x^2}}.x\right]\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{-4x^2}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{3-4x^2-3}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{3-4x^2}{\sqrt{3-4x^2}}\,dx + \int\frac{3}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\sqrt{3-4x^2}\,dx + 3\int\frac{1}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}\,dx$

$I = x\sqrt{3-4x^2} - \int\sqrt{3-4x^2}\,dx + \frac{3}{2}\int\frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-x^2}}\,dx$

Using $\int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{3-4x^2} - I + \frac{3}{2}\sin^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) + c_1$

$2I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c_1$

$I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$ where $\frac{c_1}{2} =$

v) $\int\sqrt{x^2+4}\,dx$

SOLUTION:

Let $I = \int\sqrt{x^2+4}\,.1\,dx$

Here $U = \sqrt{x^2+4}$, $V = 1$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{x^2+4}\ \int 1\,dx - \int\left[\left(\sqrt{x^2+4}\,\right)'.\int 1\,dx\right]\,dx$

$I = \sqrt{x^2+4}\,.x - \int\left[\frac{2x}{2\sqrt{x^2+4}}.x\right]\,dx$

$I = x\sqrt{x^2+4} - \int\frac{x^2}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4}{\sqrt{x^2+4}}\,dx + \int\frac{4}{\sqrt{\sqrt{x^2+4}}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\ \sqrt{x^2+4}\,dx + 4\int\frac{1}{\sqrt{x^2+4}}\,dx$

Using $\int\frac{1}{\sqrt{x^2+a^2}}\,dx = ln\left|x+\sqrt{x^2+a^2}\right| + c$

$I = x\sqrt{x^2+4} - I + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I + I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$2I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}ln\left|x+\sqrt{x^2+2^2}\right| + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{x^2+4} + 2\ ln\left|x+\sqrt{x^2+2^2}\right| + c$

where $\frac{c_1}{2} = c$

vi) $\int x^2 e^{ax}\,dx$

SOLUTION:

$\int x^2 e^{ax}\,dx$

Here $U = x^2$, $V = e^{ax}$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x^2\int e^{ax}\,dx - \int[(x^2)'.\int e^{ax}\,dx]\,dx$

$= x^2\,\frac{e^{ax}}{a} - \int 2x\,\frac{e^{ax}}{a}\,dx$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a}\{\int x\ e^{ax}\,dx\}$

Again integrating by parts

$$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a}\left\{x\,\frac{e^{ax}}{a} - \int 1.\,\frac{e^{ax}}{a}\,dx\right\}$$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\int e^{ax}\,dx$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\frac{e^{ax}}{a} + c$

Take common $\frac{e^{ax}}{a}$

$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c$

Q. 5: ***Evaluate the following integrals***:

i) $\int e^x\left(\frac{1}{x} + \ln x\right)dx$

SOLUTION:

$= \int e^{1\,.\,x}\left(1\,.\ln x + \frac{1}{x}\right)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\ \ln x + c$

$= e^x\ \ln x + c$

ii) $\int e^x(\cos x + \sin x)dx$

SOLUTION:

$= \int e^{1\,.\,x}(1.\sin x + \cos x)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\sin x + c$

$= e^x\sin x + c$

iii) $\int e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx$

SOLUTION:

$\int e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{ax}\sec^{-1}x + c$

iv) $\int e^{3x}\left(\frac{3\sin x - \cos x}{sin^2\,x}\right)dx$

SOLUTION:

$= \int e^{3x}\left(\frac{3\sin x}{sin^2\,x} - \frac{\cos x}{sin^2\,x}\right)dx$

$= \int e^{3x}\left(\frac{3}{\sin x} - \frac{\cos x}{\sin x.\sin x}\right)dx$

$= \int e^{3x}(3\,\mathrm{cosec}\,x - \cot x\,\mathrm{cosec}\,x)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{3x}\,\mathrm{cosec}\,x + c$

v) $\int e^{2x}(-\sin x + 2\cos x)dx$

SOLUTION:

$= \int e^{2x}(2\cos x - \sin x)dx$

$= \int e^{2x}(2\cos x + (-\sin x))dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{2x}\cos x + c$

vi) $\int\frac{x\,e^x}{(1+x)^2}dx$

SOLUTION:

$= \int e^x\left[\frac{1+x-1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right]dx$

$= e^x.\frac{1}{1+x} + c$

vii) $\int e^{-x}(\cos x - \sin x)dx$

***SOLUTION*:**

$= \int e^{-x}(-\sin x + \cos x)dx$

$= \int e^{-1\,.\,x}(-1.\sin x + \cos x)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{-1.x}\sin x + c$

$= e^{-x}\sin x + c$

viii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

***SOLUTION*:**

$= \int e^{m\tan^{-1}x}\frac{1}{1+x^2}dx$

Put $y = \tan^{-1}x$

$dy = \frac{1}{1+x^2}\,dx$

$= \int e^{my}dy = \frac{e^{my}}{m} + c = \frac{e^{m\tan^{-1}x}}{m} + c$

Put $y = \tan^{-1}x$

ix) $\int \frac{2x}{1-\sin x}dx$

***SOLUTION*:**

$\int \frac{2x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\,dx = \int \frac{2x(1+\sin x)}{1-\sin^2 x}dx =$
$\int \frac{2x(1-\sin x)}{\cos^2 x}dx = \int 2x\left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x\cos x}\right)dx$

$= \int 2x(\sec^2 x + \tan x\sec x)dx = \int 2x\ \sec^2 x\ dx -$
$\int 2x\ \tan x\sec x\ dx$

Here $U = 2x\ ,\ V = \sec^2 x$ *and* $U = 2x\ ,$
$V = \tan x\sec x$

Using $\int U.V = U\,.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= [2x\ \tan x - \int 2(1)\tan x\ dx] + [2x\ \sec x -$
$\int 2(1)\sec x\ dx]$

$= 2x\,.\tan x - 2\ln|\sec x| + 2x\,.\sec x - 2\,ln|\sec x +$
$\tan x| + c$

x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$

***SOLUTION*:**

$= \int e^x\left[\frac{2-1+x}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{(2+x)-1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{2+x} - \frac{1}{(2+x)^2}\right]dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x.\frac{1}{2+x} + c$

xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x dx$

***SOLUTION*:**

$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2}\mathrm{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$

$= \int e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\mathrm{cosec}^2\frac{x}{2}\right)dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x\left(-\cot\frac{x}{2}\right) + c = -e^x\cot\frac{x}{2} + c$

Integration involving Partial Fraction

$If\ P(x), Q(x) are\ two\ polynomial\ function\ and\ Q(x) \neq 0$

In rational fraction

$\frac{P(x)}{Q(x)}$ *can be factorized into linear and*
Quadratic (irreducible) factors then the rational function is written as as a sum of simpler rational functions, each of which can be integrated by methods already known.
Here we discuss examples of the three cases of partial fraction and then apply integrated.

Case1.

when $Q(x)$*contain non − repeated linear factors. e. g;*

$$\frac{P(x)}{(x-a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Or $\frac{-x+6}{(x-2)(x-3)(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-4}$ *e. tc*

Case2.

when $Q(x)$*contain non repeated and repeates linear factors.*

$$\frac{P(x)}{(x-a)(x+b)^2} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$$

$$\frac{2x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\ e.t.c$$

Case3.

When $Q(x)$*contain non repeated irreducible quadratic factors.*

$$\frac{P(x)}{(x+b)(x^2+c)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+c}$$

$$\frac{1}{(x-1)(x^2+1+2x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+2x}$$

Exercise 3.5

Evaluate the following integrals.

Q1. $\int \frac{3x+1}{x^2-x-6}dx$

Solution: $\int \frac{3x+1}{x^2-x-6}dx$

Now

$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$

$$⇨\ 3x + 1 = A(x+2) + B(x-3) \rightarrow (i)$$

$$Put\ x - 3 = 0 \Rightarrow x = 3\ in(i)$$

$$3(3) + 1 = A(3+2) + B(0) \Rightarrow 5A = 10 \Rightarrow A = 2$$

$$Putx + 2 = 0 \Rightarrow x = -2\ in(i)$$

$$3(-2) + 1 = A(0) + B(-2-3) \Rightarrow -5B = -6 + 1$$

$\Rightarrow -5B = -5 \Rightarrow B = 1$

$$so\ \frac{3x+1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$$

$$\Rightarrow \int \frac{3x+1}{x^2-x-6}dx = 2\int \frac{1}{x-3} + \int \frac{1}{x+2}dx$$

$$= 2\ln|x-3| + \ln|x+2| + c$$

Q2. $\int \frac{5x+8}{(x+3)(2x-1)} dx$

Solution: $\int \frac{5x+8}{(x+3)(2x-1)} dx$

Now.

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

⇨ $5x + 8 = A(2x-1) + B(x+3) \to (i)$

$$Put\ 2x - 1 = 0 \Rightarrow x = \frac{1}{2}\ in(i)$$

⇨ $5\left(\frac{1}{2}\right) + 8 = A(0) + B\left(\frac{1}{2} + 3\right)$

⇨ $\frac{5+16}{2} = B\left(\frac{1+6}{2}\right) \Rightarrow 7B = 21 \Rightarrow B = 3$

Put $x + 3 = 0 \Rightarrow x = -3\ in\ (i)$

⇨ $5(-3) + 8 = A(2(-3) - 1) + B(0)$

⇨ $-15 + 8 = -7A \Rightarrow -7 = -7A \Rightarrow A = 1$

So $\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$

$$\int \frac{5x+8}{(x+3)(2x-1)} dx$$

$$= \int \frac{1}{x+3} dx + 3 \int \frac{1}{2x-1} dx$$

$$= \ln|x+3| + \frac{3}{2} \int \frac{2}{2x-1} dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)} dx$$

$$= \ln|x+3| + \frac{3}{2} \ln|2x-1| + c$$

Q3. $\int \boldsymbol{\frac{x^2+3x-34}{x^2+2x-15} dx}$

Solution: $\int \frac{x^2+3x-34}{x^2+2x-15} dx$

So $\int \left(1 + \frac{x-19}{x^2+2x-15}\right) dx$

$= \int 1dx + \int \frac{x-19}{x^2+2x-15}$

$$\begin{array}{r} 1 \\ x^2+2x-15\overline{\big)\, x^2+3x-34} \\ \underline{\pm x^2 \pm 2x \mp 15} \\ x-19 \end{array}$$

$Now\ \ \frac{x-19}{x^2+2x-15} = \frac{A}{x-3} + \frac{B}{x+5} \to (i)$

$\Rightarrow x - 19 = A(x+5) + B(x-3) \to (ii)$

$put\ x - 3 = 0 \Rightarrow x = 3\ in(ii)$

$\Rightarrow 3 - 19 = A(3+5) + B(0)$

$\Rightarrow -16 = 8A \Rightarrow A = -2$

$put\ x + 5 = 0 \Rightarrow x = -5\ in(ii)$

$\Rightarrow -5 - 19 = A(0) + B(-5-3) \Rightarrow -24$
$= -8B$

$\Rightarrow B = 3$

$\because x^2 + 2x - 15$
$= x^2 + 5x - 3x - 15$
$x(x+5) - 3(x+5)$
$(x-3)(x+5)$

$$(i) \Rightarrow \frac{x-19}{x^2+2x-15} = -\frac{2}{x-3} + \frac{3}{x+5}$$

Thus,

. $\int \frac{x^2+3x-34}{x^2+2x-15} dx = \int 1dx + \int \frac{-2}{x-3} dx + \int \frac{3}{x+5} dx$

$$= x - 2\ln|x-3| + 3\ln|x+5| + c$$

Q4. $\int \boldsymbol{\frac{(a-b)x}{(x-a)(x-b)} dx, (a > b)}$

Solution: $\int \boldsymbol{\frac{(a-b)x}{(x-a)(x-b)} dx}$

Now

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\Rightarrow (a-b)x = A(x-b) + B(x-a) \to (i)$$

$$Put\ x - a = 0 \Rightarrow x = a\ in\ (i)$$

$$\Rightarrow (a-b).a = A(a-b) + B(a-a)$$

$$(a-b).b = A(0) + B(a-b) \Rightarrow A = a$$

$$Put\ x - b = 0 \Rightarrow x = b\ in\ (i)$$

$$\Rightarrow (a-b).b = A(0) + B(b-a)$$

$$(a-b).b = -B(a-b)$$

$$b = -B$$

$$B = -b$$

Thus

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$$

$$\int \frac{(a-b)x}{(x-a)(x-b)} dx$$

$$= \int \frac{a}{x-a} dx - \int \frac{b}{x-b} dx$$

$$= a\ln|x-a| - b\ln|x-b| + c$$

Q5. $\int \frac{3-x}{1-x-6x^2} dx$

Solution: $\int \frac{3-x}{1-x-6x^2} dx$

$\because 1 - x - 6x^2$
$= -6x^2 - x + 1$
-3$x(2x+1) + 1(2x+1$
$(2x+1)(1-3x)$

Now

$\frac{3-x}{1-x-6x^2} = \frac{A}{2x+1} + \frac{B}{1-3x}$

$\Rightarrow 3 - x = A(1-3x) + B(2x+1) \to (i)$

$$Put\ 2x + 1 = 0 \Rightarrow x = -\frac{1}{2} in\ (i)$$

$\Rightarrow 3 - \left(-\frac{1}{2}\right) = A\left(1 - 3\left(-\frac{1}{2}\right) + B(0)\right)$

$\Rightarrow 3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right) \Rightarrow \frac{7}{2} = A\left(\frac{5}{2}\right)$

$\Rightarrow A = \frac{7}{2}$

$Put\ 1 - 3x = 0 \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3} in\ (i)$

$$\Rightarrow 3 - \frac{1}{3} = A(0) + B\left(2\left(\frac{1}{3}\right) + 1\right)$$

$$\frac{9-1}{3} = B\left(\frac{2+3}{3}\right) \Rightarrow 8 = 5B \Rightarrow \frac{8}{5}$$

So

$$\frac{3-x}{1-x-6x^2} = \frac{7/5}{2x+1} + \frac{8/5}{1-3x}$$

: $\int \frac{3-x}{1-x-6x^2} dx = \frac{7}{5} \int \frac{1}{2x+1} dx + \frac{8}{5} \int \frac{1}{1-3x} dx$

$$= \frac{7}{10} \int \frac{2}{2x+1} dx - \frac{8}{15} \int \frac{-3}{1-3x} dx$$

$$= \frac{7}{10}\ln|2x+1| - \frac{8}{5}\ln|1-3x| + C$$

Q.6 $\int \frac{2x}{x^2-a^2} dx$

Solution: $\int \frac{2x}{x^2-a^2} dx$

Now

$$\frac{2x}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} \quad \because x^2-a^2$$

$$= (x-a)(x+a)$$

$$\Rightarrow 2x = A(x+a) + B(x-a) \rightarrow (i)$$

Put $x - a = 0 \Rightarrow x = a \; in \; (i)$

$\Rightarrow 2a = A(a+a) + B(0) \Rightarrow 2a = 2A \Rightarrow A = 1$

Put $x + a = 0 \Rightarrow x = -a \; in \; (i)$

$\Rightarrow 2(-a) = A(0) + B(-a-a) \Rightarrow -2a = -2aB$

$$\Rightarrow B = 1$$

So $\frac{2x}{x^2-a^2} = \frac{1}{x-a} + \frac{1}{x+a}$

$$\int \frac{2x}{x^2-a^2} dx = \int \frac{1}{x-a} dx + \int \frac{1}{x+a} dx$$

$$= \ln|x-a| + \ln|x+a| + c$$

$$= \ln|(x-a)(x+a)| + c$$

$$= \ln|x^2-a^2| + c$$

Q.7 $\int \frac{1}{6x^2+5x-4} dx$

Solution: $\int \frac{1}{6x^2+5x-4} dx$

Now

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$

$\because 6x^2 + 5x - 6$
$= 6x^2 - 3x + 8x - 4$
$3x(2x-1) + 4(2x-1)$
$(2x-1)(3x+4)$

$\Rightarrow 1 = A(3x+4) + B(2x-1) \rightarrow (i)$

Put$2x - 1 = 0 \Rightarrow x = \frac{1}{2} in(i)$

$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right) + B(0) \Rightarrow 1 = A\left(\frac{3+8}{2}\right)$

$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$

$$Put \; 3x+4 = 0 \Rightarrow x = -\frac{4}{3} in \; (i)$$

$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right)-1\right) \Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$

$$1 = A(3x+4) + B(2x-1) \rightarrow (i)$$

$$Put \; 2x-1 = 0 \Rightarrow x = \frac{1}{2} \; put \; in \; (i)$$

$$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right) \Rightarrow 1 = A\left(\frac{3}{2}+4\right)$$

$$\Rightarrow 1 = A\left(\frac{3+8}{2}\right) \Rightarrow 1 = A\left(\frac{11}{2}\right) \Rightarrow A = \frac{2}{11}$$

Put $3x + 4 = 0 \Rightarrow x = -\frac{4}{3}$ put in (i)

$$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right)-1\right) \Rightarrow 1$$

$$= B\left(\frac{-8-3}{3}\right)$$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So, $\frac{1}{6x^2+5x-4} = \frac{\frac{2}{11}}{2x-1} + \frac{-3/11}{3x+4}$

$\Rightarrow \int \frac{1}{6x^2+5x-4} dx = \frac{1}{11}\int \frac{2}{2x-1} dx - \frac{1}{11}\int \frac{3}{3x+4} dx$

$$= \frac{1}{11}\ln|2x-1| - \frac{1}{11} ln|3x+4| + c$$

$$= \frac{1}{11}\ln\left|\frac{2x-1}{3x+4}\right| + c$$

Q.8 $\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx$

$$\begin{array}{r l} & x \\ 2x^2-3x-2 & \overline{)\,2x^3-3x^2-x-7} \\ & \underline{\pm 2x^3 \pm 3x^2 \mp 2x} \\ & x-7 \end{array}$$

$$\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx = \int \left(x + \frac{x-7}{2x^2-3x-2}\right) dx$$

$$= \int x dx + \int \frac{x-7}{2x^2-3x-2} dx$$

NOw

$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1}$

$\because 2x^2 - 3x - 2$
$= 2x^2 - 4x + x - 2$
$3x(x-2) + 1(x-2))$
$(x-2)(3x+1)$

$x - 7 = A(2x+1) + B(x-2) \rightarrow (i)$

$\Rightarrow Put \; x - 2 = 0 \Rightarrow x = 2 \; in \; (i)$

$\Rightarrow 2 - 7 = A(2(2)+1) + B(0) \Rightarrow -5 = 5A \Rightarrow A = -\frac{5}{5} = -1$

$$A = -1$$

$$Put \; 2x+1 = 0 \Rightarrow x = -\frac{1}{2} \; in \; (i)$$

$\Rightarrow -\frac{1}{2} - 7 = A(0) + B\left(-\frac{1}{2}-2\right) \Rightarrow \frac{-1-14}{2} = B\left(\frac{-1-4}{2}\right)$

$\Rightarrow -15 = -5B \Rightarrow B = 3$

So

$$\frac{x-7}{2x^2-3x-2} = \frac{-1}{x-2} + \frac{3}{2x+1}$$

Thus $\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx = \int x dx = \int \frac{1}{x-2} dx + 3\int \frac{1}{2x+!} dx$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\int \frac{2}{2x+1} dx$$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\ln|2x+1| + c$$

Q.9 $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$

Solution: $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$

Now

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)$$
$$+C(x-1)(x-2)\to(i)$$

Put $x-1=0\Rightarrow x=1\ in\ (i)$

$$\Rightarrow 3(1)^2-12(1)+11=A(1-2)(1-3)+B(0)+C(0)$$
$$=3-12+11=A(-1)(-2)$$
$$\Rightarrow 2=2A\Rightarrow A=1$$
$$Put\ x-2=0\Rightarrow x=2in(i)$$
$$\Rightarrow 3(2)^2-12(2)+11=A(0)+B(2-1)(2-3)+C(0)$$
$$\Rightarrow 12-24+11=-B$$
$$\Rightarrow -1=-B\Rightarrow B=1$$
$$Put\ x-3=0\Rightarrow x=3\ in(i)$$
$$\Rightarrow 3(3)^2-12(3)+11=A(0)+B(0)+C(3-1)(3-2)$$
$$\Rightarrow 3(9)-36+11=C(2)(1)$$
$$\Rightarrow 27-36+11=2C$$
$$\Rightarrow 2=2C\Rightarrow C=1$$

So

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

$$\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$$
$$=\int\frac{1}{x-1}dx+\int\frac{1}{x-2}dx$$
$$+\int\frac{1}{x-3}dx$$
$$=\ln|x-1|+\ln|x-2|+\ln|x-3|+c$$

Q10. $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Solution: $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Now

$$\frac{2x-1}{x(x-1)(x-3)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3}$$

$$2x-1=A(x-1)(x-3)+B(x)(x-3)$$
$$+C(x)(x-1)\to(i)$$
$$Putx=0\ in\ (i)$$
$$2(0)-1=A(0-1)(0-3)+B(0)(C(0)$$
$$\Rightarrow -1=A(-1)(-3)\Rightarrow -1=3A\Rightarrow A=-\frac{1}{3}$$

Put $x-1=0\Rightarrow x=1\ in\ (i)$

$$\Rightarrow 2(1)-1=A(0)+B(1)(1-3)+C(0)$$
$$\Rightarrow 1=B(-2)\Rightarrow B=-\frac{1}{2}$$

Put $x-3=0\Rightarrow x=3\ in\ (i)$

$$\Rightarrow 2(3)-1=A(0)+B(0)+C(3)(3-1)$$
$$\Rightarrow 5=6C\Rightarrow C=\frac{5}{6}$$

So

$$\frac{2x-1}{x(x-1)(x-3)}=\frac{-1}{3x}+\frac{-1}{2(x-1)}+\frac{5}{6(x-3)}$$
$$\frac{2x-1}{x(x-1)(x-3)}=-\frac{1}{3x}-\frac{1}{2(x-1)}+\frac{5}{6(x-3)}$$

$$\int\frac{x-1}{x(x-1)(x-3)}dx$$
$$=-\frac{1}{3}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x-1}$$
$$+\frac{5}{6}\int\frac{1}{x-3}$$
$$=-\frac{1}{3}\ln|x|+\frac{1}{2}\ln|x-1|+\frac{5}{6}\ln|x-3|+c$$

Q.11 $\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Solution: $\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Now

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$
$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3}$$

$$5x^2+9x+6=A(x+1)(2x+3)$$
$$+B(x-1)(2x+3)+C(x+1)(x$$
$$-1)$$

Put $x+1=0\Rightarrow x=-1\ in(i)$

$$5(-1)^2+9(-1)+6$$
$$=A(0)+B(-1-1)(2(-1)+3)$$
$$+C(0)$$
$$\Rightarrow 5-9+6=B(-2)(1)$$
$$\Rightarrow 2=-2B\Rightarrow B=-1$$

Put $x-1=0\Rightarrow x=1\ in(i)$

$$\Rightarrow 5(1)^2+9(1)+6$$
$$=A(1+1)(2(1)+3)+B(0)$$
$$+C(0)$$
$$\Rightarrow 5+9+6=A(2)(5)\Rightarrow 20=A10\Rightarrow A=2`$$

Put $2x+3=0\Rightarrow x=-\frac{3}{2}\ in\ (i)$

$$\Rightarrow 5\left(-\frac{3}{2}\right)^2+9\left(-\frac{3}{2}\right)+6$$
$$=A(0)+B(0)$$
$$+C\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}-1\right)$$
$$5\left(\frac{9}{4}\right)+\left(-\frac{27}{2}\right)+6=C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$
$$\frac{45}{4}-\frac{27}{2}+6=C\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$
$$\frac{45-54+24}{4}=C\frac{5}{4}$$
$$\Rightarrow 15=5C\Rightarrow C=3$$

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$
$$=\frac{2}{x-1}+\frac{-1}{x+1}+\frac{3}{2x+3}$$

$$:\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx=2\int\frac{1}{x-1}dx-1\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{2}{2x+3}dx$$
$$=2\ln|x-1|-ln|x+1|+\frac{3}{2}|2x+3|+c$$

Q12. $\int\frac{4+7x}{(1+x)^2(2+3x)}dx$

Solution:

$$\int \frac{\mathbf{4+7x}}{\mathbf{(1+x)^2(2+3x)}}\mathbf{dx}$$

Now

$$\frac{\mathbf{4+7x}}{\mathbf{(1+x)^2(2+3x)}} = \frac{\mathbf{A}}{\mathbf{1+x}} + \frac{\mathbf{B}}{\mathbf{(1+x)^2}} + \frac{\mathbf{C}}{\mathbf{2x+3}}$$

$$\Rightarrow 4+7x = A(1+x)(2x+3) + B(2x+3)$$
$$+ C(1+x)^2 \to (i)$$

Put $1+x=0 \Rightarrow x=-1$ in (i)

$$\Rightarrow 4+7(-1) = A(0) + B(-2+3) + C(0)$$
$$\Rightarrow -3 = B \Rightarrow B = -3$$
$$Put\ 2+3x=0 \Rightarrow x = -\frac{2}{3}\ in(i)$$
$$4+7\left(-\frac{2}{3}\right) = A(0) + B(0) + C\left(1-\frac{2}{3}\right)^2$$
$$4-\frac{14}{3} = C\left(\frac{3-2}{3}\right)^2$$
$$\frac{12-14}{3} = C\left(\frac{1}{9}\right)$$
$$\Rightarrow -\frac{2}{3} = \frac{1}{9}C \ \Rightarrow -\frac{2}{3}\times\frac{9}{1} = C \Rightarrow C = -6$$

From (i)

$$4+7x = A(2+3x+2x+3x^2) + 2B + 3Bx$$
$$+ C(1+2x+x^2)$$
$$\Rightarrow 4+7x = 2A + 5Ax + 3x^2A + 2B + 3Bx + C$$
$$+ 2Cx + cx^2$$

Equating coefficient of x^2

$$0 = 3A + C \Rightarrow 3A = -C \Rightarrow 3A = -(-6)$$
$$\Rightarrow 3A = 6 \Rightarrow A = \frac{6}{3} = 2 \Rightarrow A = 2$$

So,

$$\frac{\mathbf{4+7x}}{\mathbf{(1+x)^2(2+3x)}} = \frac{\mathbf{2}}{\mathbf{1+x}} + \frac{\mathbf{3}}{\mathbf{(1+x)^2}} - \frac{\mathbf{6}}{\mathbf{2x+3}}$$

$$\int \frac{4+7x}{(1+x)^2(2+3x)}dx$$
$$= 2\int \frac{1}{1+x}dx + 3\int (1+x^2)^{-2}dx$$
$$+\frac{6}{3}\int \frac{3}{2+3x}dx$$
$$= 2\ln|1+x| + \frac{3(1+x)^{-1}}{-1} - 2\ln|2+3x| + c$$
$$\ln|1+x|^2 - \frac{3}{1+x} - \ln|2+3x|^2 + c$$

Q.13 $\int \frac{\mathbf{2x^2}}{\mathbf{(x-1)^2(x+1)}}\mathbf{dx}$

Solution:

Now

$$\frac{2x^2}{(x-1)^2(x+1)}$$
$$= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$\Rightarrow 2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

Put$x-1=0 \Rightarrow x = 1\ in(i)$

$$\Rightarrow 2(1)^2 = A(0) + B(1+1) + C(0)$$
$$\Rightarrow 2 = 2B \Rightarrow B = 1$$

Put $x+1=0 \Rightarrow x=-1$ in (i)

$$\Rightarrow 2(-1)^2 = A(0) + B(o) + C(-1-1)^2$$
$$2 = 4C \Rightarrow C = \frac{1}{2}$$

From (i)

$$2x^2 = A(x^2-1) + Bx + B + C(x^2+1-2x)$$
$$\Rightarrow 2x^2 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$$

Equating coefficients of x^2, $we\ have$

$$\Rightarrow 2 = A + C \Rightarrow 2 = A + \tfrac{1}{2} \Rightarrow A = 2 - \tfrac{1}{2}$$
$$\Rightarrow A = \frac{3}{2}$$

So,

$$\frac{2x^2}{(x-1)^2(x+1)}$$
$$= \frac{3/2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{1/2}{(x+1)}$$
$$\int \frac{\mathbf{2x^2}}{\mathbf{(x-1)^2(x+1)}}\mathbf{dx}$$
$$= \frac{\mathbf{3}}{\mathbf{2}}\int \frac{\mathbf{1}}{\mathbf{x-1}}\mathbf{dx} + \int \mathbf{(x-1)^{-2}dx}$$
$$+ \frac{\mathbf{1}}{\mathbf{2}}\int \frac{\mathbf{1}}{\mathbf{x+1}}\mathbf{dx}$$
$$= \frac{3}{2}\ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2}\ln|x+1| + c$$
$$= \frac{3}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + c$$

Q.14 $\int \frac{\mathbf{1}}{\mathbf{(x-1)(x+1)^2}}\mathbf{dx}$

Solution: $\int \frac{\mathbf{1}}{\mathbf{(x-1)(x+1)^2}}\mathbf{dx}$

Now

$$\frac{\mathbf{1}}{\mathbf{(x-1)(x+1)^2}} = \frac{\mathbf{A}}{\mathbf{x-1}} + \frac{\mathbf{B}}{\mathbf{x+1}} + \frac{\mathbf{C}}{\mathbf{(x+1)^2}}$$

$\Rightarrow 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \to (i)$

Put $x-1=0 \Rightarrow x = 1$ in (i)

$$1 = A(1+1)^2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Put $x+1=0 \Rightarrow x=-1$ in (i)

$$1 = C(-1-1)$$
$$\Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$$

From (i)

$$\Rightarrow 1 = A(x^2+2x+1) + B(x^2-1) + Cx - C$$
$$\Rightarrow 1 = Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

Equating coefficient of x^2, *we have*

$$0 = A + B \Rightarrow 0 = \frac{1}{4} + B \Rightarrow B = -\frac{1}{4}$$
$$\frac{1}{(x-1)(x+1)^2} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$$

$$\int \frac{\mathbf{1}}{(\boldsymbol{x}-\mathbf{1})(\boldsymbol{x}+\mathbf{1})^{2}} \boldsymbol{dx}$$

$$=\frac{1}{4}\int\frac{1}{x-1}dx-\frac{1}{4}\int\frac{1}{x+1}dx-\frac{1}{2}\int(x+1)^{-2}dx$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\frac{(x+1)^{-1}}{-1}+C$$

$$=\{\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|\}+\frac{1}{2(x+1)}+C$$

Q.15 $\int \frac{\boldsymbol{x+4}}{\boldsymbol{x^3-3x^2+4}}\boldsymbol{dx}$

Solution: $\int \frac{\boldsymbol{x+4}}{\boldsymbol{x^3-3x^2+4}}\boldsymbol{dx}$

Now

$$\because \boldsymbol{x^3-3x^2+4=x^3+x^2-4x^2+4}$$

$$=x^2(x+1)-1(x^2-1)$$

$$=x^2(x+1)-4(x-1)(x+1)$$

$$=(x+1)(x^2-4x+4)$$

$$\Rightarrow x^3-3x^2+4=(x+1)(x-2)^2$$

Now

$$\frac{\boldsymbol{x+4}}{\boldsymbol{x^3-3x^2+4}}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$$\Rightarrow x+4=A(x-2)^2+B(x+1)(x-2)$$
$$+C(x+1)\rightarrow(i)$$

Put $x+1=0\Rightarrow x=-1\ in(i)$

$$\Rightarrow -1+4=A(-1-2)^2+B(0)+C(0)$$

$$\Rightarrow 3=9A\Rightarrow A=\frac{1}{3}$$

Put $x-2=0\Rightarrow x=2\ in\ (i)$

$$\Rightarrow 2+4=A(0)+B(0)+C(2+1)$$

$$\Rightarrow 6=3C\Rightarrow C=2$$

From (i)

$$x+4=A(x^2-4x+4)+B(x^2-2x+x-2)$$
$$+Cx+C$$

$$\Rightarrow x+4=Ax^2-4Ax+4A+Bx^2-Bx-2B$$
$$+Cx+C$$

Equating coefficients of x^2

$$\Rightarrow 0=A+B\Rightarrow 0=\frac{1}{3}+B\Rightarrow B=-\frac{1}{3}$$

$$\frac{\boldsymbol{x+4}}{\boldsymbol{x^3-3x^2+4}}=\frac{1/3}{x+1}+\frac{-1/3}{x-2}+\frac{2}{(x-2)^2}$$

$$\int\frac{\boldsymbol{x+4}}{\boldsymbol{x^3-3x^2+4}}\boldsymbol{dx}$$

$$=\frac{\mathbf{1}}{\mathbf{3}}\int\frac{\mathbf{1}}{\boldsymbol{x+1}}\boldsymbol{dx}$$
$$-\frac{\mathbf{1}}{\mathbf{3}}\int\frac{\mathbf{1}}{\boldsymbol{x-2}}+\mathbf{2}\int(\boldsymbol{x-2})^{-2}\boldsymbol{dx}$$

$$=\frac{1}{3}ln|x+1|-\frac{1}{3}\ln|x-2|+2\frac{(x-2)^{-1}}{-1}+c$$

$$=\frac{1}{3}ln|x+1|-\frac{1}{3}\ln|x-2|-\frac{2}{x-2}+c$$

$$=\frac{1}{3}\{\ln|x+1|-\ln|x-2|\}-\frac{2}{x-2}+c$$

Q16. $\int\frac{\boldsymbol{x^3-6x^2+25}}{\boldsymbol{(x+1)^2(x-2)^2}}\boldsymbol{dx}$

Solution:

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x-2)}$$
$$+\frac{D}{(x-2)^2}$$

$$\Rightarrow x^3-6x^2+25$$
$$=A(x+1)(x-2)^2+B(x-2)^2$$
$$+C(x+1)^2(x-2)+D(x+1)^2$$
$$\rightarrow(i)$$

Put $x+1=0\Rightarrow x=-1\ in\ (i)$

$$\Rightarrow(-1)^3-6(-1)^2+25$$
$$=A(0)+B(-1-2)^2+C(0)$$
$$+D(0)$$

$$-1-6+25=9B$$

$$9B=18\Rightarrow B=2$$

Put $x-2=0\Rightarrow x=2\ in\ (i)$

$$\Rightarrow(2)^3-6(2)^2+25=D(2+1)^2$$

$$\Rightarrow 8-24+25=9D$$

$$9=9D\Rightarrow D=1$$

From (i)

$$x^3-6x^2+25=A(x+1)(x^2-4x+4)$$
$$+B(x^2-4x+4)$$

$$+C(x^2+1+2x)(x-2)+D(x^2+1+2x)$$

$$=A(x^3-4x62+4x+x62-4x+4)+Bx^2-4Bx$$
$$+4B$$

$$+C(x^3-2x^2+x-2+2x^2-4x)+Dx^2+D+2Dx$$

$$=Ax^3-3Ax^2+4A+Bx^2-4Bx+4B+Cx^3$$
$$-3Cx+Dx^2+D+2Dx$$

Equating coefficients of $x^3\ and\ x^2$

$FOr\ x^3$

$$1=A+C\rightarrow(ii)$$

For x^2 $\quad -6=-3A+B+D$

$$-6=-3A+2+1$$

$$-6-3=-3A\Rightarrow-9=-3A\Rightarrow A$$
$$=3\ put\ in\ (ii)$$

$$1=3+C\Rightarrow C=1-3=-2\Rightarrow C=-2$$

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{(x-2)}$$
$$+\frac{1}{(x-2)^2}$$

$$\int\frac{\boldsymbol{x^3-6x^2+25}}{\boldsymbol{(x+1)^2(x-2)^2}}\boldsymbol{dx}$$

$$=\mathbf{3}\int\frac{\mathbf{1}}{\boldsymbol{x+1}}\boldsymbol{dx}+\mathbf{2}\int(\boldsymbol{x+1})^{-2}\boldsymbol{dx}-\mathbf{2}\int\frac{\mathbf{1}}{\boldsymbol{x-2}}\boldsymbol{dx}+\int(\boldsymbol{x-2})^{-2}\boldsymbol{dx}$$

$$=\mathbf{3}\boldsymbol{ln}|\boldsymbol{x+1}|+\mathbf{2}\frac{(\boldsymbol{x+1})^{-1}}{-\mathbf{1}}-\mathbf{2}\boldsymbol{ln}|\boldsymbol{x-2}|+\frac{(\boldsymbol{x-2})^{-1}}{-\mathbf{1}}+\boldsymbol{C}$$

$$=\mathbf{3}\boldsymbol{ln}|\boldsymbol{x+1}|-\frac{\mathbf{2}}{\boldsymbol{x+1}}-\mathbf{2}\boldsymbol{ln}|\boldsymbol{x-2}|-\frac{\mathbf{1}}{\boldsymbol{x-2}}+\boldsymbol{C}$$

Q.17 $\int\frac{\boldsymbol{x^3+22x^2+14x-17}}{\boldsymbol{(x-3)(x+2)^3}}\boldsymbol{dx}$

Solution:

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}$$

$$=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$+\frac{D}{(x+2)^3}$$

$\Rightarrow x^3+22x^2+14x-17=A(x+2)^3+B(x-3)(x+2)^2+C(x-3)(x+2)+D(x-3)\rightarrow(i)$

$$Put\ x-3=0\Rightarrow x=3\ in\ (i)$$

$$\Rightarrow(3)^3+22(3)^2+14(3)-17=A(3+2)^3$$

$\Rightarrow 27+198+42-17=$125A

$$\Rightarrow 250=125A\Rightarrow A=2$$

$$Put\ x+2=0\rightarrow x=-2\ in(i)$$

$$\Rightarrow(-2)^3+22(-2)^2+14(-2)-17=D(-2-3)$$

$$-8+88-28-17=-5D\Rightarrow 35=-5D$$

$$D=-7$$

From (i)

$x^3+22x^2+14x-17$

$=A[x^3+6x^2+12x+8]+B(2-3)(x^2+4x+4)$

$+C(x^2+2x-3x-6)+Dx-3D$

$=Ax^3+6Ax^2+12Ax+8A$

$+B(x^3+4x^2+4x-3x^2-12x$

$-12)+Cx^2-Cx-6c+Dx-3D$

Equating coefficients of $x^2 and\ x^3$

$$For\ x^3;1=A+B\Rightarrow 1=2+B\Rightarrow B=-1$$

$$For\ x^2;\quad 22=6A+B+C\Rightarrow 22=6(2)-1+C$$

$$\Rightarrow C=22-12+1=11\Rightarrow C=11$$

So

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}$$

$$=\frac{2}{x-3}+\frac{1}{x+2}+\frac{11}{(x+2)^2}$$

$$-\frac{7}{(x+2)^3}$$

$$\int\frac{\boldsymbol{x^3+22x^2+14x-17}}{\boldsymbol{(x-3)(x+2)^3}}\boldsymbol{dx}$$

$$=\boldsymbol{2\int\frac{1}{x-3}-\int\frac{1}{x+3}dx+\int(x+2)^{-2}dx-7\int(x+2)^{-3}dx}$$

$$=\boldsymbol{2ln|x-3|-ln|x+2|+11\frac{(x+2)^{-1}}{-1}-7\frac{(x+2)^{-2}}{-2}}$$

$$\boldsymbol{+c}$$

$$=\boldsymbol{2ln|x-3|-ln|x+2|-\frac{11}{x+2}+\frac{7}{2}\frac{1}{(x+2)^2}+c}$$

Q.18 $\int\frac{x-2}{(x+1)(x^2+1)}dx$

Solution:

$$\frac{x-2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$\Rightarrow x-2=A(x^2+1)+(Bx+C)(x+1)\rightarrow(i)$$

$$Put\ x+1=0\Rightarrow x=-1\ in(i)$$

$$\Rightarrow -1-2=A((-1)^2+1)$$

$$-3=2A\Rightarrow A=-\frac{3}{2}$$

From (i)

$$\Rightarrow x-2=Ax^2+A+Bx^2+Bx+Cx+C$$

Equating coefficients of $x^2\ and\ x$

$$For\ x^2;\quad 0=A+B\Rightarrow 0=-\frac{3}{2}+B\Rightarrow B=\frac{3}{2}$$

$$For\ x;1=B+C\Rightarrow 1=\frac{3}{2}+C$$

$$\Rightarrow C=1-\frac{3}{2}=-\frac{1}{2}\Rightarrow C=-\frac{1}{2}$$

So

$$\frac{x-2}{(x+1)(x^2+1)}=\frac{-3/2}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

$$\int\frac{\boldsymbol{x-2}}{\boldsymbol{(x+1)(x^2+1)}}\boldsymbol{dx}=-\frac{\boldsymbol{3}}{\boldsymbol{2}}\int\frac{\boldsymbol{1}}{\boldsymbol{x+1}}\boldsymbol{dx}+\frac{\boldsymbol{1}}{\boldsymbol{2}}\int\frac{\boldsymbol{3x-1}}{\boldsymbol{x^2+1}}\boldsymbol{dx}$$

$$=-\frac{3}{2}ln|x+1|$$

$$+\frac{1}{2}\int\frac{3x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$=-\frac{3}{2}\ln|x+1|+\frac{3}{2}.\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x$$

$$=-\frac{3}{2}\ln|x+!|+\frac{3}{4}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x+c$$

Q.19 $\int\frac{x}{(x-1)(x^2+1)}\boldsymbol{dx}$

Solution: $\int\frac{x}{(x-1)(x^2+1)}\boldsymbol{dx}$

Now

$$\frac{x}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$\Rightarrow x=A(x^2+1)+(Bx+C)(x-1)\rightarrow(i)$$

Put$x-1=0\Rightarrow x=1\ in(i)$

$$\Rightarrow 1=A((1)^1+1)$$

$$\Rightarrow A=\frac{1}{2}$$

From (i)

$$\Rightarrow x=Ax^2+A+Bx^2-Bx+Cx-C$$

Equating coefficients of $x^2 and\ x\ we\ have$

$$For\ x^2\ ;\quad \Rightarrow 0=A+B\Rightarrow 0=\frac{1}{2}+B\Rightarrow B$$

$$=-\frac{1}{2}$$

$$For\ x;\quad 1=-B+C\Rightarrow 1=-\left(-\frac{1}{2}\right)+C$$

$$\Rightarrow 1=\frac{1}{2}+C\Rightarrow 1-\frac{1}{2}=C\Rightarrow C=\frac{1}{2}$$

So,

$$\frac{x}{(x-1)(x^2+1)}=\frac{1/2}{x-1}+\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}$$

$$\int\frac{x}{(x-1)(x^2+1)}dx$$

$$=\frac{1}{2}\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{x-1}{x^2+!}dx$$

$$=\frac{1}{2}ln|x-1|-\frac{1}{4}\int\frac{2x-2}{x^2+1}dx$$

$$\frac{1}{2}ln|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}ln|x-1| - \frac{1}{4}ln|x^2+1| + \frac{1}{2}tan^{-1}x + c$$

Q.20 $\int \frac{9x-7}{(x+3)(x^2+1)} dx$

Solution: $\int \frac{9x-7}{(x+3)(x^2+1)} dx$

Now

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 9x-7 = A(x^2+1) + (Bx+C)(x+3) \rightarrow (i)$$

Put$x+3=0 \Rightarrow x=-3\ in\ (i)$

$$\Rightarrow 9(-3)-7 = A((-3)^2+1)$$

$$-27-7 = 10A \Rightarrow -34 = 10A \Rightarrow A = -\frac{34}{10}$$

$$\Rightarrow A = -\frac{17}{5}$$

From (i)

$$9x-7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$$

Equating coefficients of x^3 *and* x

$$For\ x^2\ \ , \Rightarrow 0 = A+B \Rightarrow 0 = -\frac{17}{5} + B \Rightarrow B = \frac{17}{5}$$

$$and for\ x; 3B + C = 9 \Rightarrow 3\left(\frac{17}{5}\right) + C = 9 \Rightarrow \frac{51}{5} + C = 9$$

$$C = 9 - \frac{51}{5} = \frac{45-51}{5} \Rightarrow C = -\frac{6}{5}$$

So

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17/5}{x+3} + \frac{\frac{17}{5}x - \frac{6}{5}}{x^2+1}$$

$\int \frac{9x-7}{(x+3)(x^2+1)} dx$

$$= -\frac{17}{5}\int \frac{1}{x+3}dx$$

$$+ \frac{17}{5}\int \frac{x}{x^2+1}dx - \frac{6}{5}\int \frac{1}{x^2+1}dx$$

$$= -\frac{17}{5}\ln|x+3| + \frac{17}{10}\ln|x^2+1| - \frac{6}{5}\tan^{-1}x + c$$

Q21. $\int \frac{1+4x}{(x-3)(x^2+4)} dx$

Solution: $\int \frac{1+4x}{(x-3)(x^2+4)} dx$

Now

$$\boldsymbol{\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}}$$

$$\boldsymbol{\Rightarrow 1+4x = A(x^2+4) + (Bx+C)(x-3) \rightarrow (i)}$$

$$\boldsymbol{Put\ x-3=0 \Rightarrow x=3\ in\ (i)}$$

$$\boldsymbol{1+4(3) = A((3)^2+4) + B(3) + C(0)}$$

$$\Rightarrow 13 = A(9+4) \Rightarrow 13 = 13A \Rightarrow A = 1$$

From (i)

$$1+4x = Ax^2 + 4a + Bx^2 - 3Bx + Cx - 3C$$

Equating Coefficients of x^2 *and* x

$$\Rightarrow 0 = A+B \quad for\ x^2$$

$$0 = 1+B \Rightarrow B = -1$$

$$\Rightarrow 4 = -3B + C \Rightarrow 4-3 = C \Rightarrow C = 1$$

So

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{(-)x+1}{x^2+4}$$

$$\int \frac{1+4x}{(x-3)(x^2+4)}dx$$

$$= \int \frac{1}{x-3}dx - \int \frac{x}{x^2+4}dx$$

$$+ \int \frac{1}{x^2+4}dx$$

$$= \ln|x-3| - \frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$$

$$= \ln|x-3| - \frac{1}{2}|x^2+4| + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Q.22

$$\boldsymbol{\int \frac{12}{x^3+8}dx}$$

Solution:

$\int \frac{12}{x^3+8}dx \quad \because a^3 - b^3 = (a+b)(a^2 - ab + b^2)$

Now

$$\frac{12}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$\Rightarrow 12 = A(x^2-2x+4) + (Bx+C)(x+2) \rightarrow (i)$$

Put $x+2=0 \Rightarrow x=-2\ in\ (i)$

⇒12= $A(4+4+4) \Rightarrow 12 = 12A \Rightarrow A = 1$

From (i)

12= $Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$

Equating coefficients of x^2 *and* x we have

$$for\ x^2;\ \ 0 = A+B \Rightarrow 0 = 1+B \Rightarrow B = -1$$

$for\ x;\ \ 0 = -2(1) + 2(-1) + C \Rightarrow 0 = -2-2+C$

⇒$C = 4$

So

$$\frac{12}{x^3+8} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$

$$\int \frac{12}{x^3+8}dx = \int \frac{1}{x+2}dx - \int \frac{x-4}{x^2-2x+4}dx$$

$$= \int \frac{1}{x+2}dx - \frac{1}{2}\int \frac{2x-8}{x^2-2x+4}dx$$

$$= ln|x+2| - \frac{1}{2}\int \frac{2x-2-6}{x^2-2x+4}dx$$

$$= ln\ |x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}dx$$

$$+ \frac{6}{2}\int \frac{1}{x^2-2x+4}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4|$$

$$+ 3\int \frac{1}{x^2-2x+1+3}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4|$$

$$+ 3\int \frac{1}{(x-1)^2+\sqrt{3}}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) +$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) +$$

Q23. $\int \frac{9x^2+6}{x^3-8} dx$

Solution:

$$\int \frac{\mathbf{9x^2+6}}{\mathbf{x^3-8}}\,\boldsymbol{dx}$$

Now

$$\frac{9x^2+6}{x^3-8} = \frac{9x^2+6}{(x-2)(x^2+2x+4)}$$

$$= \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$\Rightarrow 9x+6 = A(x^3+2x+4)+(Bx+C)(x-2)$$
$$\rightarrow (i)$$

Put $= x-2=0 \Rightarrow x=2\ in\ (i)$

$$9(2)+6 = A[(2)^2+2(x)+4]+B(2)+C(0)$$

$\Rightarrow 24 = 12A \Rightarrow A = 2$

From (i)

$$9x+6 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$$

Equating coefficient of $x^2\ \ and\ x$

For $x^2; 0 = A+B \Rightarrow 0 = 2+B \Rightarrow B=-2$

$FOr\ x; 9 = 2A-2B+C \Rightarrow 9 = 2(2)-2(-2)+C$

$$\Rightarrow 9 = 4+4+C \Rightarrow 9-8 = C \Rightarrow C = 1$$

So

$$\frac{9x^2+6}{(x-2)(x^2+2x+4)} = \frac{2}{x-2} + \frac{-2x+1}{x^2+2x+4}$$

$$\int \frac{9x^2+6}{x^3-8}dx = 2\int \frac{1}{x-2}dx - \int \frac{2x-1}{x^2+2x+4}dx$$

$$= -2\int \frac{1}{x-2}dx - \int \frac{2x+2-2-1}{x^2+2x+4}dx$$

$$= -2\int \frac{1}{x-2}dx - \int \frac{2x+2}{x^2+2x+4}dx + 3\int \frac{1}{x^2+2x+4}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int \frac{1}{(x+1)^2+\left(\sqrt{3}\right)^2}dx$$

$$2\ln|x-2| - \ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$2\ln|x-2| - \ln|x^2+2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$$

Q.24 $\int \frac{\mathbf{2x^2+5x+3}}{\mathbf{(x-1)^2(x^2+4)}}\boldsymbol{dx}$

Solution: $\int \frac{\mathbf{2x^2+5x+3}}{\mathbf{(x-1)^2(x^2+4)}}\boldsymbol{dx}$

Now

$$\frac{\mathbf{2x^2+5x+3}}{\mathbf{(x-1)^2(x^2+4)}} = \frac{\boldsymbol{A}}{\mathbf{x-1}} + \frac{\boldsymbol{B}}{\mathbf{(x-1)^2}} + \frac{\boldsymbol{Cx+D}}{\mathbf{x^2+4}}$$

$\Rightarrow \mathbf{2x^2+5x+3 = A(x-1)(x^2+4)+B(x^2+4)+}$
$\mathbf{(Cx+D)(x-1)^2 \rightarrow (i)}$

Put $\mathbf{x-1=0 \Rightarrow x=1\ in\ (i)}$

$\Rightarrow \mathbf{2(1)^2+5(1)+3 = B(1+4)}$

$$2+5+3 = 5B \Rightarrow 10 = 5B \Rightarrow B = 2$$

From (i)

$$2x^2+5x+3 = A(x^3+4x-x^2-4)+Bx^2$$
$$+4B+(Cx+D)(x^2+1+2x)$$

$$Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3+Cx$$
$$-2Cx^2+Dx^2+D-2Dx$$

$Equating\ coefficient\ of\ x^3, x^2\ and\ x\ we\ get$

For $x^3\ \ \Rightarrow 0 = A+C \Rightarrow C=-A \rightarrow (ii)$

For $x^2; 2 = -A+B-2C+D$

Put $B = 2\ and\ C = -A$

$$2 = -A+2-2(-A)+D$$

$$\Rightarrow 2-2 = -A+2A+D \Rightarrow 0 = A+D$$

$$\Rightarrow D = -A \rightarrow (iii)$$

$$For\ x; 5 = 4A+C-2D\ \ putC = -A\ and\ D = -A$$

$$\Rightarrow 5 = 4A-A-2(-A)$$

$$5 = 3A+2A \Rightarrow 5 = 5A \Rightarrow A = 1$$

So (ii)$\Rightarrow C = -1\ and\ (iii) \Rightarrow D = -1$

Thus

$$\frac{\mathbf{2x^2+5x+3}}{\mathbf{(x-1)^2(x^2+4)}} = \frac{\mathbf{1}}{\mathbf{x-1}} + \frac{\mathbf{2}}{\mathbf{(x-1)^2}} + \frac{\mathbf{(-1)x \pm 1}}{\mathbf{x^2+4}}$$

⇨ $\int \frac{\mathbf{2x^2+5x+3}}{\mathbf{(x-1)^2(x^2+4)}}\boldsymbol{dx} = \int \frac{\mathbf{1}}{\mathbf{x-1}}\boldsymbol{dx} + \mathbf{2}\int \mathbf{(x-1)^{-2}} -$
$\int \frac{\mathbf{x+1}}{\mathbf{x^2+4}}\boldsymbol{dx}$

$$= \int \frac{1}{x-1}dx + 2\int (x-1)^{-2}dx - \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ln|x-1| + \frac{2(x-1)^{-1}}{-1} - \frac{1}{2}\int \frac{2x}{x^2+4}dx - \int \frac{1}{(x)^2+(2)^2}dx$$

$$= \ln|x-1| - \frac{2}{x-1}$$
$$-\frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Q25.

$$\int \frac{\mathbf{2x^2-x-7}}{\mathbf{(x+2)^2(x^2+x+1)}}\boldsymbol{dx}$$

Solution:

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}$$
$$= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1}$$

$$\Rightarrow 2x^2-x-7 = A(x+2)(x^2+x+1)$$
$$+B(x^2+x+1)$$
$$+(Cx+D)(x+2)^2$$

$$Put\ x+2 = 0 \Rightarrow x = -2$$

$$\Rightarrow 2(-2)^2 — 2(-2) - 7 = B((-2)^2+(-2)+1)$$

$$\Rightarrow 8+2-7 = B(4-2+1)$$

$$\Rightarrow 3 = 3B \Rightarrow B = 1$$

From (i)

$$2x^2-x-7 = A(x^3+x^2+x+2x^2+2)+Bx^2$$
$$+Bx+B+C(Cx+D)(x^2+4x$$
$$+4)$$

$$= Ax^3+3Ax^2+3A+Bx^2+Bx+B+Cx^3+4Cx^2$$
$$+4Cx+Dx^2+4Dx+4D$$

Equating coefficients of $x^3, x^2\ and\ x$

$$for\ x^3;\ \ 2+3A+B+4C+D$$

Put $B = 1,\ \ C = -A \rightarrow (ii)$

For $x^2;\ \ 2 = 3A+B+4C+D \Rightarrow 2-1 = -A+D$

$$\Rightarrow D = A+1 \rightarrow (iii)$$

$$For\ x;\ \ -1 = 3A+B+4C+4D$$

Put $B = 1, C = -A, D = A+1$

$$\Rightarrow -1 = 3A+1-4A+4A+4$$

$$-1-1-4 = 3A \Rightarrow -6 = 3A \Rightarrow A = -2$$

So $(ii) \Rightarrow C = 2\ \ and\ (iii) \Rightarrow B = -1$

Thus,

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}$$
$$=\frac{-2}{x+2}+\frac{1}{(x+2)^2}+\frac{2x-1}{x^2+x+1}$$

$$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$
$$=-2\int\frac{1}{x+2}dx$$
$$+\int(x+2)^{-2}dx+\int\frac{2x+1-2}{x^2+x+1}dx$$
$$=-2\ln|x+2|+\frac{(x+2)^{-1}}{-1}$$
$$+\int\frac{2x+1}{x^2+x+1}dx-2\int\frac{1}{x^2+x+1}dx$$

$$=-2ln|x+2|-\frac{1}{x+2}+ln|x^2+x+1|$$
$$-2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$
$$=-2ln|x+2|-\frac{1}{x+2}+ln|x^2+x+1|$$
$$-2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$
$$-2ln|x+2|-\frac{1}{x+2}+ln|x^2+x+1|$$
$$-2\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=-2ln|x+2|-\frac{1}{x+2}ln|x^2+x+1|-\frac{2}{\frac{\sqrt{3}}{2}}tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$
$$+c$$
$$=-2ln|x+2|-\frac{1}{x+2}+ln|x^3+x+1|-\frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Q.26 $\int\frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

Solution: $\int\frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

$$\because\frac{3x+1}{(4x^2+1)(x^2-x+1)}$$
$$=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{(x^2-x+1)}$$
$$\Rightarrow 3x+1=(Ax+B)(x^2-x+1)+(Cx+D)(4x^2+1)$$
$$3x+1=Ax^3-Ax^2+Ax+Bx^2-Bx+B+4Cx^3$$
$$+Cx+4Dx^2+D$$

Equating coefficients of x^3, x^2, x *and constants terms.*

Forx^3; $0=A+4C\rightarrow(i)$

For $x^2; 0=-A+B+4D\rightarrow(ii)$

for $x\,; 3=A+B+C\rightarrow(iii)$

For constant term $;1=B+D\rightarrow(iv)$

From (i) $A=-4C$ *and* $(iv)\Rightarrow B=1-D$

Put in (ii)*and* (iii)

$$\Rightarrow 0=-(-4C)+(1-D)+4D \text{ and } 3$$
$$=-4C-(1-D)+C$$
$$0=4C+1+4D \qquad 3=-4C-1+D+C$$
$$0=4C+3D+1\rightarrow(v) \qquad 0=-3C+D-4$$
$$\Rightarrow D=3C+4 \text{ put in } (iv)$$
$$\Rightarrow 0=4C+3(3C+4)+1$$
$$0=4C+3(3C+4)+1$$
$$0=4C+9C+12+1\Rightarrow 0=13C+13$$
$$\Rightarrow -13C=12\Rightarrow C=-1$$
$$As\ A=-4C\Rightarrow A=-4(-1)\Rightarrow A=4\because C$$
$$=-1$$
$$As\ D=3C+4\Rightarrow D=3(-1)+4=-3+4$$
$$\Rightarrow D=1$$
$$As\ B=1-D=1-1=0\Rightarrow B=0$$

Thus

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)}$$
$$=\frac{4x+0}{4x^2+1}+\frac{(-1)x+1}{(x^2-x+1)}$$
$$\frac{3x+1}{(4x^2+1)(x^2-x+1)}$$
$$=\frac{1}{2}\frac{8x}{4x^2+1}+\frac{(-1)(x-1)}{x^2-x+1}$$
$$\int\frac{3x+1}{(4x^2+1)(x^2-x+1)}dx=\frac{1}{2}\int\frac{8x}{4x^3+1}-\frac{1}{2}\int\frac{2x-2}{x^2-x+1}dx$$
$$=\frac{1}{2}ln|4x^2+1|-\frac{1}{2}\int\frac{2x-1-1}{x^2-x+1}dx$$
$$=\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int\frac{1}{x^2-x+1}dx$$

$$=\frac{1}{2}ln|4x^2+1|-\frac{1}{2}ln|x^2-x+1|$$
$$=\frac{1}{2}\int\frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}}dx$$

$$=\frac{1}{2}ln|4x^2+1|-\frac{1}{2}ln|x^2-x+1|$$
$$+\frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^3}$$
$$=\frac{1}{2}ln|4x^2+1|-\frac{1}{2}ln|x^2-x+1|+\frac{1}{2}.\frac{1}{\frac{\sqrt{3}}{2}}tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}+c$$
$$\frac{1}{2}ln|4x^2+1|-\frac{1}{2}ln|x^2-x+1|+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$
$$+c$$

Q27. $\int\frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Solution:

$$\int\frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$$
$$\because\frac{4x+1}{(x^2+4)(x^2+4x+5)}$$
$$=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+4x+5}$$
$$\Rightarrow 4x+1=(Ax+B)(x^2+4x+5)+(Cx+D)(x^2$$
$$+4)$$
$$\Rightarrow 4x+1=Ax^3+4Ax^2+5Ax+Bx^2+4Bx$$
$$+5B+Cx^3+4cx+Dx^2+4D$$

Equating coefficients of x^3, x^2 , x *and constant term.*

$$Put\ x^3; 0 = A + C \to (i)$$

$$for\ x^2; 0 = 4A + B + D \to (ii)$$

$$for\ x; 4 = 5A + 4B + 4C \to (iiii)$$

For constant term$1 = 5B + 4D \to (iv)$

From $(i) \Rightarrow A = -C\ and\ \ (iv) \Rightarrow 5B = 1 - 4D$

$$B = \frac{1-4D}{5} put\ in\ (ii) and\ (iii)$$

$$so\ (ii) \Rightarrow 0 = 4(-C) + \frac{1-4D}{5} + D\ and\ (iii) \Rightarrow 4$$

$$= 5(-C) + 4\left(\frac{-4D}{5}\right) + 4C$$

$$\Rightarrow 0 = -4C + \frac{1-4D}{5} + D \quad \Rightarrow 20$$

$$= -25C + 4 - 16D + 20C$$

$$0 = -20C + 1 - 4D + 5D \quad \Rightarrow 16D$$

$$= -5C + 4 - 20$$

$$\Rightarrow 0 = -20C + D + 1 \qquad \Rightarrow D = \frac{-5C - 16}{16}$$

$$\to (vi)$$

$\Rightarrow D = 20C - 1 \to (v)$

$By\ (v) and\ (vi) \Rightarrow 20C - 1 = \frac{-5C-16}{16}$

$$\Rightarrow 320C - 16 = -5C - 16 \Rightarrow 320C + 5C = 0$$

$$\Rightarrow 320C = 0 \Rightarrow C = 0$$

As $a = -C \Rightarrow A = 0$

$$As\ D = 20C - 1 \Rightarrow D = 20(0) - 1 \Rightarrow D = -1$$

$$As\ B = \frac{1-4D}{5} \Rightarrow B = \frac{1-4(-1)}{5} = \frac{5}{3} = 1$$

$$B = 1$$

So

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x+(-1)}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}$$

$$= \int \frac{1}{x^2+4} dx$$

$$- \int \frac{1}{x^2+4x+4+1} dx$$

$$= \frac{1}{2} tan^{-1} \frac{x}{2} - \int \frac{1}{(x-2)^2 + (1)^2} dx$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} - \tan^{-1}(x-2) + c$$

Q28. $\displaystyle\int \frac{\boldsymbol{6a^2}}{\boldsymbol{(x^2+a^2)(x^2+4a^2)}} \boldsymbol{dx}$

$$\because \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2}$$

$$\Rightarrow 6a^2 = (Ax+B)(x^2+4a^2) + (Cx+D)(x^2 + a^2)$$

$$\Rightarrow 6a^2 = Ax^3 + 4a^2Ax + Bx^2 + 4Ba^2 + Cx^3 + Ca^2x + Dx^2 + Da^2$$

Equating coefficients of $x^3 + x^2, x\ and\ constants\ term.$

$$Put\ x^3; 0 = A + C \to (i)$$

$$for\ x^2; 0 = B + D \to (ii)$$

$$for\ x; 0 = 4a^2A + a^2C \Rightarrow 0 = (4A + C)a^2$$

$$\Rightarrow 4A + C \to (iiii)$$

For constant term $1 = 5B + 4D \to (iv)$

From $(i) \Rightarrow A = -C\ and\ \ (iv) \Rightarrow B = -D$

$$Put\ in\ (iii)\ and\ (iv) so$$

$$(iii)\ 4(-C) + C = 0 \Rightarrow -4C + c = 0 \Rightarrow -3C = 0$$

$$\Rightarrow C = 0$$

$$(iv) 4(-D) + D = 6 \Rightarrow -4D + d \Rightarrow -3D = 6$$

$$D = -2$$

As $A = -C \ \Rightarrow A = 0 \ \because C = 0$

$As\ B = -D \Rightarrow B = -(-2) \Rightarrow B = 2 \ \because D = -2$

So

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x+(-2)}{x^2+4a^2}$$

$$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} dx = 2\int \frac{1}{x^2+a^2} dx - 2\int \frac{1}{x^2+(2a)^2} dx$$

$$= \frac{2}{a} tan^{-1} \frac{x}{a} - \frac{1}{a} tan^{-1}\ \frac{x}{2a} + c$$

$$= \frac{2}{a} tan^{-1} \frac{x}{a} - \frac{1}{a} tan^{-1} \frac{x}{2a} + c$$

Q29. $\displaystyle\int \frac{\boldsymbol{2x^2-2}}{\boldsymbol{(x^4+x^2+1)(x^2-x+1)}} \boldsymbol{dx}$

Solution:

$$\int \frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)}$$

$$= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

$$2x^2 - 2 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2 + x + 1)$$

$$= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

Equating coefficients of $x^3, x^2, x\ and\ constant\ term.$

$$for\ x^3; \ \ 0 = A + C \to (i)$$

$$for\ x^2; 0 = A - B + C + D \to (ii)$$

$$for\ x; 2 = -A + B + C + D \to (iii)$$

$$for\ constant\ term;\ -2 = -A + C - 2$$

$$\Rightarrow 2 + 2 = -A + C \Rightarrow -A + C = 4 \to (v)$$

Put $A + C = 0\ in(ii) \Rightarrow 0 = -B + D \to (vi)$

$$Now\ by\ (i) + (v) \Rightarrow 2C = 4 \Rightarrow C = 2$$

$$as\ A + C = 0 \Rightarrow A + 2 = 0 \Rightarrow A = -2$$

Now by (iv)+$(vi) \Rightarrow 2D = -D \Rightarrow D = -1$

$As\ B + D = -2 \ \Rightarrow B - 1 = -2 \Rightarrow B = -1$

So;

$$\frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)}$$

$$= \frac{-2x-1}{x^2+x+1} + \frac{2x-1}{x^2-x+1}$$

$$\int \frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}$$

$$= -\int \frac{2x+1}{x^2+x+1} dx$$

$$+ \int \frac{2x-1}{x^2-x+1} dx$$

$$= -ln|x^2+x+1| + ln|x^2-x+1| + c$$

$$= ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$$

Q 30. $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)} dx$

Solution: $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)} dx$

$$\because \frac{3x-8}{(x^2-x+2)(x^2+x+2)}$$

$$= \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2}$$

$$3x-8 = (Ax+B)(x^2+x+2) + (Cx+D)(x^2-x+2)$$

$$= Ax^3 + Ax^2 + 2Ax + Bx^2 + Bx + 2B + Cx^3 - Cx^2$$

$$+ 2Cx + Dx^2 - Dx + 2D$$

Equating coefficients of x^3, x^2, x *and constant term*.

$$for\ x^3;\ \ 0 = A + C \to (i)$$

$$for\ x^2; 0 = A + B - C + D \to (ii)$$

$$for\ x; 3 = 2A + B + 2C - D \to (iii)$$

$$for\ constant\ term;\ -8 = 2B + 2D \Rightarrow B + D = -4$$

$$\to (iv)$$

From $(i) \Rightarrow A = -C$ *and from*$(iv) \Rightarrow B = -4 - D$

Put *in* (ii)*and* (iii)*so*

$$(ii) \Rightarrow 0 = -C + (-4 - D) - C + D$$

$$0 = -C - 4 - B - C + D$$

$$0 = -2C - 4$$

$$\Rightarrow 2C = -4 \Rightarrow C = -2\ \ as\ \ A = -C \Rightarrow A = 2$$

$$(ii) \Rightarrow 3 = 2(-C) - 4 - D + 2C - D$$

$$3 = -2C - 4 + 2C - 2D$$

$$\Rightarrow 3 + 4 = -2D \Rightarrow D = -\frac{7}{2}$$

As $B = -4 - D = -4\left(-\frac{7}{2}\right) = -4 + \frac{7}{2} = \frac{-8+7}{2}$=-

$1/2 \Rightarrow B = -\frac{1}{2}$

So

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)}$$

$$= \frac{2x-1/2}{x^2-x+2} + \frac{-2x+(-7/2)}{x^2+x+2}$$

$$\int \frac{3x-8}{(x^2-x+2)(x^2-x+2)}$$

$$= -\int \frac{2x+1-1-1/2}{x^2-x+2} dx$$

$$+ \int \frac{2x+1-1+7/2}{x^2+x+2} dx$$

$$= \int \frac{2x-1+\frac{1}{2}}{x^2-x+2} dx + \int \frac{2x+1+\frac{5}{2}}{x^2+x+2} dx$$

$$= \int \frac{2x-1}{x^2-x+2} dx + \frac{1}{2} \int \frac{1}{x^2-x+2} dx - \int \frac{2x+1}{x^2+x+2} dx$$

$$- \int \frac{5/2}{x^2+x+2} dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - ln|x^2+x+2|$$

$$-\frac{5}{2}\int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2} dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} - ln|x^2+x+2|$$

$$-\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}} dx$$

$$= \boldsymbol{ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} dx}$$

$$\boldsymbol{- ln|x^2+x+2|}$$

$$\boldsymbol{-\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}} dx}$$

$$\boldsymbol{= ln|x^2-x+1| + \frac{1}{2}.\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} dx}$$

$$\boldsymbol{- ln|x^2+x+1| - \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}}$$

$$\boldsymbol{= ln|x^2-x+1| + \frac{1}{2}.\frac{1}{\frac{\sqrt{7}}{2}} tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) - ln|x^2+x+1|}$$

$$\boldsymbol{-\frac{5}{2}.\frac{1}{\frac{\sqrt{7}}{2}} tan^{-1}\left(\frac{x+1/2}{\frac{\sqrt{7}}{2}}\right)}$$

$$\boldsymbol{ln|x^2-x+1| + \frac{1}{\sqrt{7}} tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) - ln|x^2+x+1|}$$

$$\boldsymbol{-\frac{5}{\sqrt{7}} tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + c}$$

Q31. $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} dx$

Solution: $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} dx$

$$\because \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+3}$$

$$\Rightarrow 3x^3 + 4x^2 + 9x + 5$$

$$= (Ax+B)(x^2+2x+3) + (Cx+D)(x^2+x+1)$$

$$ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx^3 + Cx^2 + Cx$$

$$+ Dx^2 + Dx + D$$

Equation coefficients of x^3, x^2, x *and constant term*.

$$\boldsymbol{for\ x^3; 3 = A + C \to (i)}$$

For $x^2; 4 = 2A + B + C + D \to (ii)$

For $x; 9 = 3A + 2B + C + D \to (iii)$

For constant term; $5 = 3B + 2B + C + D \to (iv)$

From $(i) \Rightarrow A = 3 - C$ *and from* $(iv) \Rightarrow D = 5 - 3B$

Put in (ii) and (iii)

$$(ii) \Rightarrow 4 = 2(3-C) + B + C + 5 - 3B$$

$$4 = 6 - 2C + B + C + 5 - 3B$$

$$4 - 6 - 5 = -C - 2B \Rightarrow -7 = -(C + 2B)$$

$$\Rightarrow C + 2B = 7 \to (v)$$

$$(iii) \Rightarrow 9 = 3(3-C) + 2B + C + 5 - 3B$$

$$\Rightarrow 9 = 9 - 3C + 2B + C + 5 - 3B$$

$$\Rightarrow 9 - 9 - 5 = -2C - B \Rightarrow B = -2C + 5\ put\ in(v)$$

$$\Rightarrow C + 2(-2C+5) = 7 \Rightarrow C - 4C + 10 = 7$$

$$\Rightarrow -3C + 10 + 7 \Rightarrow -3C = 7 - 10$$

$$\Rightarrow -3C = -3 \Rightarrow C = 1$$

As B= $5 - 2C = 5 - 2(1) = 3 \Rightarrow B = 3$

$$As\ D = 5 - 3B = 5 - 3(3) = 5 - 9 = -4 \Rightarrow D$$

$$= -4$$

$$As\ A = 3 - C = 3 - 1 = 2 \Rightarrow A = 2$$

So

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{2x+3}{x^2+x+1} + \frac{x-4}{x^2+2x+3}$$

$$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$$

$$= \int \frac{2x+1+2}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x-8}{x^2+2x+3}dx$$

$$= \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{1}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x+2-10}{x^2+2x+3}dx$$

$$= ln|x^2+x+1| + 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$+\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx$$

$$-5\int \frac{1}{x^2+2x+3}dx$$

$$= ln|x^2+x+1| + 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$+\frac{1}{2}ln|x^2+2x+3|$$

$$-5\int \frac{1}{x^2+2x+1+2}dx$$

$$= ln|x^2+x+1| + 2.\frac{1}{\frac{\sqrt{3}}{2}}tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{1}{2}ln|x^2+2x+3|$$

$$-5\int \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}dx$$

$$= ln|x^2+x+1| + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}ln|x^2+2x+3|$$

$$-\frac{5}{\sqrt{2}}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$$= ln|x^2+x+1| + ln|x^2+2x+3|^{\frac{1}{2}} + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{5}{2}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$$= ln|x^2+x+1|\sqrt{x^2+2x+3} + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{5}{2}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

The Definite integrals:

If $\int f(x)dx = \emptyset(x) + c, then\ the\ integral\ of\ f(x)$ $from\ a\ to\ b\ is\ denoted\ by\ \int_a^b f(x)dx$

And read $a + cs$ definite integral of $f(x)$ here a is called lower limit and b is called upper limit.

*the interval $[a, b]$ is called range of integration.

We evaluate $\int_a^b f(x)dx$ as;

Consider $\int f(x)dx = \emptyset(x) + c$

$$\Rightarrow \int_a^b f(x)dx \ = |\emptyset(x) + c|_a^b$$

$$= [\emptyset(b) + c] - [\emptyset(a) + c]$$

$$= \emptyset(b) + c - \emptyset(a) - c$$

$$\Rightarrow \int_a^b f(x)dx \ = \emptyset(b) - \emptyset(a)$$

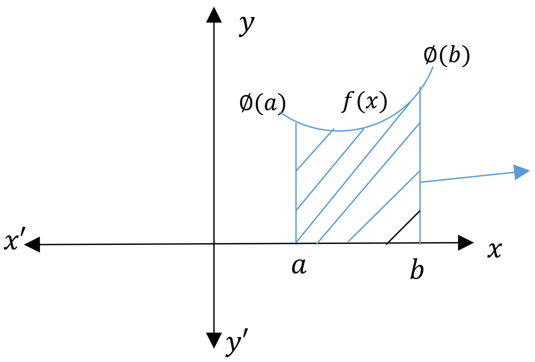
Note: if the lower limit is a constant and upper limit is a variable, then the integral is a function of the upper limit.

$$\int_a^x f(t)dt \ = |\emptyset(t)|_a^x = \emptyset(x) - \emptyset(a)$$

The area under the curve

$$\int_a^b f(x)dx \ = \emptyset(b) - \emptyset(a)$$

Represented the "area of region" bounded under the curve of function $f(x)$ the $x - axis$ and between two ordinates $x = a, x = b\ as\ shown\ in\ figure.$



Fundamental theorem of calculus:

If $f(x)$is continuous $\forall\ x \in [a, b]\ and\ \emptyset'(x) = f(x)$

$$\int_a^b f(x)dx \ = \emptyset(b) - \emptyset(a)$$

Is called fundamental theorem of integral calculus.

Properties of Definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$= \emptyset(b) - \emptyset(a)$$

$$= -[\emptyset(a) - \emptyset(b)]$$

$$= -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\textbf{(b)} \int_a^b \boldsymbol{f(x)dx} = \int_a^c \boldsymbol{f(x)dx} + \int_c^b \boldsymbol{f(x)dx} \;\; \boldsymbol{a < c < b}$$

Proof:

$$\int_a^c f(x)dx = \emptyset(c) - \emptyset(a)$$

$$\int_c^b f(x)dx = \emptyset(b) - \emptyset(c)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx =$$

$$= \emptyset(c) - \emptyset(a) + \emptyset(b) - \emptyset(c)$$

$$= \emptyset(b) - \emptyset(a)$$

$$\int_a^b f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\textbf{(c)} \int_a^a \boldsymbol{f(x)dx} = \mathbf{0}$$

Proof:

$$\int_a^a f(x)dx \;\; = \emptyset(a) - \emptyset(a)$$

$$= 0$$

$$\Rightarrow \int_a^a f(x)dx = 0$$

Also member $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

and $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Exercise 3.6

Evaluate the following indefinate integrls:

$\boldsymbol{Q.1:} \int_1^2 (\boldsymbol{x^2+1})\, \boldsymbol{dx}$

SOLUTION:

$$\int_1^2 (x^2+1)\, dx$$

$$= \left|\frac{x^3}{3} + x\right|_1^2$$

$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{1^3}{3} + 1\right)$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)$$

$$= \left(\frac{8+6}{3}\right) - \left(\frac{1+3}{3}\right)$$

$$= \frac{14}{3} - \frac{4}{3}$$

$$= \frac{14-4}{3} = \frac{10}{3}$$

$\boldsymbol{Q.2:} \int_{-1}^1 \left(\boldsymbol{x^{\frac{1}{3}} + 1}\right) \boldsymbol{dx}$

SOLUTION:

$$= \int_{-1}^1 \left(x^{\frac{1}{3}}.1 + 1\right) dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + x\right|_{-1}^1$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x\right|_{-1}^1$$

$$= \left|\frac{3}{4}x^{\frac{4}{3}} + x\right|_{-1}^1$$

$$= \left(\frac{3}{4}(1)^{\frac{4}{3}} + 1\right) - \left(\frac{3}{4}(-1)^{\frac{4}{3}} + (-1)\right)$$

$$= \left(\frac{3}{4}.1 + 1\right) - \left(\frac{3}{4}.1 - 1\right)$$

$$= \left(\frac{3+4}{4}\right) - \left(\frac{3-4}{4}\right)$$

$$= \frac{7}{4} - \frac{-1}{4} = \frac{7}{4} + \frac{1}{4}$$

$$= \frac{7+1}{4} = \frac{8}{4} = 2$$

$\boldsymbol{Q.3:} \int_{-2}^0 \frac{\mathbf{1}}{\boldsymbol{(2x-1)^2}} \boldsymbol{dx}$

SOLUTION:

$$= \int_{-2}^0 (2x-1)^{-2}\, dx$$

$$= \frac{1}{2}\int_{-2}^0 (2x-1)^{-2}.2\, dx$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-2+1}}{-2+1}\right|_{-2}^0$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-1}}{-1}\right|_{-2}^0$$

$$= -\frac{1}{2}\left|\frac{1}{2x-1}\right|_{-2}^0$$

$$= -\frac{1}{2}\left[\left(\frac{1}{2(0)-1}\right) - \left(\frac{1}{2(-2)-1}\right)\right]$$

$$= -\frac{1}{2}\left[\left(\frac{1}{-1}\right) - \left(\frac{1}{-5}\right)\right]$$

$$= -\frac{1}{2}\left[-1 + \frac{1}{5}\right] = -\frac{1}{2}\left[\frac{-5+1}{5}\right]$$

$$= -\frac{1}{2}\left[\frac{-4}{5}\right] = \frac{2}{5}$$

$\boldsymbol{Q.4:} \int_{-6}^2 \sqrt{\boldsymbol{3-x}}\, \boldsymbol{dx}$

SOLUTION:

$$= \int_{-6}^{2} (3-x)^{\frac{1}{2}}\, dx$$

$$= (-1)\int_{-6}^{2} (3-x)^{\frac{1}{2}}\, (-1)dx$$

$$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-6}^{2}$$

$$= -\left|\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-6}^{2}$$

$$= -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^{2}$$

$$= -\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right)-\left((3-(-6))^{\frac{3}{2}}\right)\right]$$

$$= -\frac{2}{3}\left[\left((1)^{\frac{3}{2}}\right)-\left((9)^{\frac{3}{2}}\right)\right]$$

$$= -\frac{2}{3}\left[1-\left((3^2)^{\frac{3}{2}}\right)\right]$$

$$= -\frac{2}{3}[1-27] = \frac{52}{3}$$

$Q.5$: $\int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\, dt$

***SOLUTION*:**

$$= \int_{1}^{\sqrt{5}} (2t-1)^{\frac{3}{2}}\, dt$$

$$= \frac{1}{2}\int_{1}^{\sqrt{5}} (2t-1)^{\frac{3}{2}}.2\, dt$$

$$= \frac{1}{2}\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right|_{1}^{\sqrt{5}} = \frac{1}{2}\left|\frac{(2t-1)^{\frac{5}{2}}}{\frac{5}{2}}\right|_{1}^{\sqrt{5}}$$

$$= \frac{1}{2}.\frac{2}{5}\left[\left((2\sqrt{5}-1)^{\frac{5}{2}}\right)-\left((2(1)-1)^{\frac{5}{2}}\right)\right]$$

$$= \frac{1}{5}\left[(2\sqrt{5}-1)^{\frac{5}{2}}-1\right]$$

$Q.6$: $\int_{2}^{\sqrt{5}} x\sqrt{x^2-1}\, dx$

***SOLUTION*:**

$$= \int_{2}^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\, x\, dx$$

$$= \frac{1}{2}\int_{2}^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\ .2x\, dx$$

$$= \frac{1}{2}\left|\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{\sqrt{5}} = \frac{1}{2}\left|\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{\sqrt{5}}$$

$$= \frac{1}{2}.\frac{2}{3}\left[\left(\left((\sqrt{5})^2-1\right)^{\frac{3}{2}}\right)-\left(\left((2)^2-1\right)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((4)^{\frac{3}{2}}\right)-\left(3^{\frac{3}{2}}\right)\right] = \frac{1}{3}\left[\left((2^2)^{\frac{3}{2}}\right)-\left(3^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}[8-3\sqrt{3}] = \frac{8}{3}-\sqrt{3}$$

$Q.7$: $\int_{1}^{2} \frac{x}{x^2+2}\, dx$

***SOLUTION*:**

$$= \frac{1}{2}\int_{1}^{2} \frac{2x}{x^2+2}\, dx$$

$$= \frac{1}{2}|\ln(x^2+2)|_{1}^{2}$$

$$= \frac{1}{2}[\ln(2^2+2)-\ln(1^2+2)]$$

$$= \frac{1}{2}[\ln(6)-\ln(3)]$$

$$= \frac{1}{2}\left[\ln\left(\frac{6}{3}\right)\right] = \frac{1}{2}\ln 2$$

$$= \ln 2^{\frac{1}{2}} = \ln\sqrt{2}$$

Properties of natural log

$\mathbf{\ln(AB) = \ln A + \ln B}$

$\mathbf{\ln\left(\frac{A}{B}\right) = \ln A - \ln B}$

$\mathbf{\ln A^B = B\ln A}$

$Q.8$: $\int_{2}^{3} \left(x-\frac{1}{x}\right)^2\, dx$

***SOLUTION*:**

$$= \int_{2}^{3} \left(x^2+\frac{1}{x^2}-2x.\frac{1}{x}\right)\, dx$$

$$= \int_{2}^{3} (x^2+x^{-2}-2)\, dx$$

$$= \left|\frac{x^{2+1}}{2+1}+\frac{x^{-2+1}}{-2+1}-2x\right|_{2}^{3}$$

$$= \left|\frac{x^3}{3}+\frac{x^{-1}}{-1}-2x\right|_{2}^{3} = \left|\frac{x^3}{3}-\frac{1}{x}-2x\right|_{2}^{3}$$

$$= \left(\frac{3^3}{3}-\frac{1}{3}-2(3)\right)-\left(\frac{2^3}{3}-\frac{1}{2}-2(2)\right)$$

$$= \left(\frac{27}{3}-\frac{1}{3}-6\right)-\left(\frac{8}{3}-\frac{1}{2}-4\right)$$

$$= \left(\frac{27-1-18}{3}\right)-\left(\frac{16-3-24}{6}\right)$$

$$= \left(\frac{8}{3}\right)-\left(\frac{-11}{6}\right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$$

$Q.9$: $\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\, dx$

***SOLUTION*:**

$$= \int_{-1}^{1} (x^2+x+1)^{\frac{1}{2}}\left(\frac{2x+1}{2}\right)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2+x+1)^{\frac{1}{2}}(2x+1)\, dx$$

$$= \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-1}^{1} \quad = \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{1}{2}.\frac{2}{3}\left[\left((1^2+1+1)^{\frac{3}{2}}\right) - \left(((-1)^2+(-1)+1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right) - \left((1-1+1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right) - \left((1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}[3\sqrt{3}-1] = \sqrt{3} - \frac{1}{3} \quad ANS.$$

$$3^{\frac{3}{2}} = \left(3^{\frac{1}{2}}\right)^3 = (\sqrt{3})^3 = (\sqrt{3})^2(\sqrt{3})^1 = 3\sqrt{3}$$

$Q.10$: $\int_0^3 \frac{dx}{x^2+9}$

SOLUTION:

$$= \int_0^3 \frac{1}{3^2+x^2}\ dx = \left|\frac{1}{3}\tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}\left[\left(\tan^{-1}\frac{3}{3}\right) - \left(\tan^{-1}\frac{0}{3}\right)\right]$$

$$= \frac{1}{3}[(\tan^{-1}1) - (\tan^{-1}0)]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$$

$Q.11$: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t\ dt$

SOLUTION:

$$= |\sin t|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\sin\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

$Q.12$: $\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)\ dx$

SOLUTION:

$Here\ f(x) = \ x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$\left|\frac{\left(x+\frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2 = \left|\frac{\left(x+\frac{1}{x}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^2$$

$$= \frac{2}{3}\left[\left(\left(2+\frac{1}{2}\right)^{\frac{3}{2}}\right) - \left(\left(1+\frac{1}{1}\right)^{\frac{3}{2}}\right)\right]$$

$$= \frac{2}{3}\left[\left(\left(\frac{5}{2}\right)^{\frac{3}{2}}\right) - \left((2)^{\frac{3}{2}}\right)\right]$$

$$= \frac{2}{3}\left[\frac{5}{2}\sqrt{\frac{5}{2}} - 2\sqrt{2}\right]$$

$$= \frac{2}{3}.\frac{5}{2}\frac{\sqrt{5}}{\sqrt{2}} - \frac{2}{3}.2\sqrt{2}$$

$$= \frac{5\sqrt{5}}{3\sqrt{2}} - \frac{4\sqrt{2}}{3} = \frac{5\sqrt{5}-4(2)}{3\sqrt{2}}$$

$$= \frac{5\sqrt{5}-8}{3\sqrt{2}}$$

$Q.13$: $\int_1^2 \ln x\ dx$

SOLUTION:

Consider

$\int \ln x\ \ dx = \int \ln x\ .1\ dx$

$Here\ \ U = \ln x\ \ \ ,\ V = 1$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x\ .\int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx$

$= \ln x\ .x\ - \int\left[\frac{1}{x}.x\right]\ dx$

$= \ln x\ .x\ - \int 1\,dx$

$= x\ \ln x\ \ - x + c$

Taking limits

$\int_1^2 \ln x\ dx = |x\ \ln x\ \ - x|_1^2$

$= (2\ \ln 2\ \ - 2) - (1\ \ln 1\ \ - 1)$

$= (2\ \ln 2\ \ - 2) - (1(0)\ \ - 1)$

$= (2\ \ln 2\ \ - 2)$

$Q.14$: $\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)\ dx$

SOLUTION:

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)\ dx = \left|\frac{e^{\frac{x}{2}}}{\frac{1}{2}} - \frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right|_0^2 = \left|2e^{\frac{x}{2}} + 2e^{-\frac{x}{2}}\right|_0^2$$

$$= 2\left[\left(e^{\frac{2}{2}} + e^{-\frac{2}{2}}\right) - \left(e^{\frac{0}{2}} + e^{-\frac{0}{2}}\right)\right]$$

$$= 2[(e^1 + e^{-1}) - (e^0 + e^0)]$$

$$= 2\left[e + \frac{1}{e} - 1 - 1\right] = 2\left[e + \frac{1}{e} - 2\right]$$

$$= 2\left[\frac{e^2+1-2e}{e}\right] = \frac{2}{e}(e^2 + 1^2 - 2e)$$

$Q.15$: $\int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\ d\theta$

SOLUTION:

$$= \int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{1+\cos 2\theta}\ dx = \int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{2\cos^2\theta}\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\right] dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta.\cos\theta}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}[\sec\theta + \sec\theta\tan\theta]\, d\theta$$

$$= \frac{1}{2}|\ln|\sec\theta + \tan\theta| + \sec\theta|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| + \sec\frac{\pi}{4}\right) - (\ln|\sec 0 + \tan 0| + \sec 0)\right]$$

$$= \frac{1}{2}\left[\left(\ln\left|\sqrt{2} + 1\right| + \sqrt{2}\right) - (\ln|1 + 0| + 1)\right]$$

$$= \frac{1}{2}\left[\left(\ln\left|\sqrt{2} + 1\right| + \sqrt{2}\right) - (0 + 1)\right]$$

$$= \frac{1}{2}\left[\ln\left|\sqrt{2} + 1\right| + \sqrt{2} - 1\right]$$

$Q.16$: $\int_0^{\frac{\pi}{6}} \cos^3\theta\ d\theta$

***SOLUTION*:**

$$= \int_0^{\frac{\pi}{6}} \cos\theta\ \cos^2\theta\ d\theta = \int_0^{\frac{\pi}{6}} \cos\theta\ (1 - \sin^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (\cos\theta - \sin^2\theta\cos\theta)\ d\theta = \left|\sin\theta - \frac{\sin^3\theta}{3}\right|_0^{\frac{\pi}{6}}$$

$$= \left(\sin\frac{\pi}{6} - \frac{\sin^3\frac{\pi}{6}}{3}\right) - \left(\sin 0 - \frac{\sin^2 0}{3}\right)$$

$$= \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3}\right) - \left(0 - \frac{0}{3}\right) = \left(\frac{1}{2} - \frac{1}{24}\right) - (0) = \frac{12-1}{24} = \frac{11}{24}$$

$Q.17$: $\int_0^{\frac{\pi}{6}} \cos^2\theta\ \cot^2\theta\, d\theta$

***SOLUTION*:**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta\ (\mathrm{cosec}^2\theta - 1)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos^2\theta\, \mathrm{cosec}^2\theta - \cos^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\frac{\cos^2\theta}{\sin^2\theta} - \cos^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cot^2\theta - \cos^2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\mathrm{cosec}^2\theta - 1 - \frac{1+\cos 2\theta}{2})\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\frac{2\mathrm{cosec}^2\theta - 2 - 1 - \cos 2\theta}{2})\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\mathrm{cosec}^2\theta - 3 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[-2\cot\theta - 3\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(-2\cot\frac{\pi}{4} - 3\frac{\pi}{4} - \frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right) - 2\cot\frac{\pi}{6} - 3\frac{\pi}{6} - \frac{\sin 2\left(\frac{\pi}{6}\right)}{2}\right)$$

$$= \frac{1}{2}\left[\left(-2(1) - \frac{3\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(-2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(-2 - \frac{3\pi}{4} - \frac{1}{2}.1\right) - 2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2}\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left[-2 - \frac{3\pi}{4} - \frac{1}{2} + 2\sqrt{3} + \frac{\pi}{2} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{2}\left[-2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[-2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[\frac{-8-2+8\sqrt{3}+\sqrt{3}-3\pi+2\pi}{4}\right]$$

$$= \frac{1}{2}\left[\frac{-10+9\sqrt{3}-\pi}{4}\right]$$

$$= \frac{-10+9\sqrt{3}-\pi}{8}$$

$Q.18$: $\int_0^{\frac{\pi}{4}} \cos^4 t\ dt$

***SOLUTION*:**

$$\int_0^{\frac{\pi}{4}} \cos^4 t\ dt = \int_0^{\frac{\pi}{4}} (\cos^2 t)^2\ dt = \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2t}{2}\right)^2\ dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\cos^2 2t+2\cos 2t}{4}\ dt$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(1 + \frac{1+\cos 4t}{2} + 2\cos 2t\right)\ dt$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\frac{2+1+\cos 4t+4\cos 2t}{2}\right)\ dt = \frac{1}{8}\int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t)\, dt$$

$$= \frac{1}{8}\left[3t + \frac{\sin 4t}{4} + 4\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\left[\left(3.\frac{\pi}{4} + \frac{\sin 4\left(\frac{\pi}{4}\right)}{4} + 4\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right) - \left(3(0) + \frac{\sin 4(0)}{4} + 4\frac{\sin 2(0)}{2}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + \frac{\sin\pi}{4} + 2\sin\left(\frac{\pi}{2}\right)\right) - \left(0 + \frac{\sin 0}{4} + 4\frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + \frac{0}{4} + 2.1\right) - \left(0 + \frac{0}{4} + 4.\frac{0}{2}\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi}{4} + 0 + 2\right) - 0\right]$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + 2\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi+8}{4}\right)\right] = \frac{3\pi+8}{32}$$

$Q.19$: $\int_0^{\frac{\pi}{3}} \cos^2\theta\ \sin\theta\, d\theta$

SOLUTION:

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\ \sin\theta\, d\theta = -\int_0^{\frac{\pi}{3}} \cos^2\theta\,(-\ \sin\theta)\, d\theta = -\left|\frac{\cos^3\theta}{3}\right|_0^{\frac{\pi}{3}} = -\frac{1}{3}\left[\left(\cos^3\frac{\pi}{3}\right) - (\cos^3 0)\right] = -\frac{1}{3}\left[\cos^3\frac{\pi}{3} - \cos^3 0\right]$$

$$= -\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - (1)^3\right] = -\frac{1}{3}\left[\frac{1}{8} - 1\right] = -\frac{1}{3}\left[\frac{1-8}{8}\right] = -\frac{1}{3}\left[\frac{-7}{8}\right] = \frac{7}{24}$$

$Q.20$: $\int_0^{\frac{\pi}{4}} (1 + \cos^2\theta)\tan^2\theta\ d\theta$

SOLUTION:

$$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta = \int_0^{\frac{\pi}{4}} (\tan^2\theta + \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \sin^2\theta)\, dt = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right] d\theta$$

$$= \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\tan\frac{\pi}{4} - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right) - \left(\tan 0 - \frac{1}{2}0 - \frac{1}{4}\sin 2(0)\right)\right]$$

$$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(0 - 0 - \frac{1}{4}\sin(0)\right)\right]$$

$$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0)\right] = \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}(1)\right) - (0)\right]$$

$Q.21$: $\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta}\ d\theta$

SOLUTION:

Divide uo and down by $\cos\theta$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta}\ d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sec\theta}{\cos\theta}}{\frac{\sin\theta + \cos\theta}{\cos\theta}}\ d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta\ \sec\theta}{\tan\theta + 1}\ d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan\theta + 1}\ d\theta$$

Here $f(x) = \tan\theta + 1 \quad \Rightarrow f'(x) = \sec^2\theta$

$$= [\ln|\tan\theta + 1|]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\ln\left|\tan\frac{\pi}{4} + 1\right|\right) - (\ln|\tan 0 + 1|)\right]$$

$$= [(\ln|1 + 1|) - (\ln|0 + 1|)]$$

$$= \ln|2| - \ln|1| = \ln 2 - 0 = \ln 2$$

$Q.22$: $\int_{-1}^{5} |x - 3|\ dx$

SOLUTION:

$$\int_{-1}^{5} |x - 3|\ dx$$

$$= \int_{-1}^{3} |x - 3|\ dx + \int_{3}^{5} |x - 3|\ dx$$

$$= \int_{-1}^{3} -(x - 3)\ dx + \int_{3}^{5} (x - 3)\ dx$$

$$= -\int_{-1}^{3} (x - 3).1\ dx + \int_{3}^{5} (x - 3).1\ dx$$

$$= -\left|\frac{(x-3)^2}{2}\right|_{-1}^{3} + \left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$= -\frac{1}{2}[((3-3)^2) - ((-1-3)^2)] + \frac{1}{2}[((5-3)^2) - ((3-3)^2)]$$

$$= -\frac{1}{2}[0 - 16] + \frac{1}{2}[4 - 0]$$

$$= -\frac{1}{2}[0 - 16] + \frac{1}{2}[4 - 0] = 8 + 2 = 10$$

$Q.23$: $\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\ dx$

$$\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\ dx\ = 3\int_{\frac{1}{8}}^{1} \left(x^{\frac{1}{3}} + 2\right)^2 .\frac{1}{3}x^{-\frac{2}{3}}\, dx = 3\left|\frac{\left(x^{\frac{1}{3}}+2\right)^{2+1}}{2+1}\right|_{\frac{1}{8}}^{1} = \frac{3}{3}\left|\left(x^{\frac{1}{3}} + 2\right)^3\right|_{\frac{1}{8}}^{1} = \left(\left((1)^{\frac{1}{3}} + 2\right)^3\right) - \left(\left(\left(\frac{1}{8}\right)^{\frac{1}{3}} + 2\right)^3\right)$$

$$= ((1 + 2)^3) - \left(\left((2^{-3})^{\frac{1}{3}} + 2\right)^3\right) = (3)^3 - \left(\frac{1}{2} + 2\right)^3 = 27 - \left(\frac{1+4}{2}\right)^3 = 27 - \frac{125}{8} = \frac{216-125}{8} = \frac{91}{8}$$

$$\int_1^3 \frac{x^2-2}{x+1}\ dx \quad (\textit{Improper fraction})$$

$$\begin{array}{r} x - 1 \\ x + 1 \overline{\smash{)}\ x^2 - 2} \\ \underline{\pm x^2 \pm x} \\ -x - 2 \\ \underline{\mp x \mp 1} \\ -1 \end{array}$$

$$= \int_1^3 \left(Q + \frac{R}{D}\right)\ dx$$

$$= \int_1^3 \left(x - 1 - \frac{1}{1+x}\right)\ dx$$

$$= \left|\frac{x^2}{2} - x - \ln|x + 1|\right|_1^3$$

$$= \left(\frac{3^2}{2} - 3 - \ln|3 + 1|\right) - \left(\frac{1}{2} - 1 - \ln|1 + 1|\right)$$

$$= \left(\frac{9}{2} - 3 - \ln 4\right) - \left(\frac{1}{2} - 1 - \ln 2\right)$$

$$= \frac{9}{2} - 3 - \ln 4 - \frac{1}{2} + 1 + \ln 2$$

$$= \frac{9}{2} - 2 - \frac{1}{2} - \ln 4 + \ln 2$$

$$= \frac{9-4-1}{2} - \ln 2^2 + \ln 2$$

$$= 2 - 2\ln 2 + \ln 2 = 2 - \ln 2$$

$\boldsymbol{Q.25}$: $\boldsymbol{\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx}$

SOLUTION:

$$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx = \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}\, dx = |\ln|x^3 - x^2 + x - 1||_2^3$$

$$= (\ln|3^3 - 3^2 + 3 - 1|) - (\ln|2^3 - 2^2 + 2 - 1|)$$

$$= (\ln|27 - 9 + 3 - 1|) - (\ln|8 - 4 + 2 - 1|)$$

$$= \ln 20 - \ln 5 = \ln \frac{20}{5} = \ln 4$$

$\boldsymbol{Q.26}$: $\boldsymbol{\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx}$

SOLUTION:

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x)\, dx$$

$$= |\sec x - \tan x|_0^{\frac{\pi}{4}}$$

$$= \left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)$$

$$= \left(\sqrt{2} - 1\right) - (1 + 0)$$

$$= \sqrt{2} - 1 - 1 = \sqrt{2} - 2$$

$\boldsymbol{Q.27}$: $\boldsymbol{\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx}$

SOLUTION:

$$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}\, dx = -\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$$

$$= -\int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= -\int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x}\right) dx$$

$$= -\int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x)\, dx$$

$$= -|\sec x - \tan x|_0^{\frac{\pi}{4}}$$

$$= -\left[\left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)\right]$$

$$= -\left[\left(\sqrt{2} - 1\right) - (1 + 0)\right]$$

$$= -\sqrt{2} + 1 + 1 = 2 - \sqrt{2}$$

$\boldsymbol{Q.28}$: $\boldsymbol{\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx}$

SOLUTION:

$$\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{-3x}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{4-3x}{\sqrt{4-3x}}\, dx - \int_0^1 \frac{-4}{\sqrt{4-3x}}\, dx = -\int_0^1 \sqrt{4-3x}\, dx + 4\int_0^1 \frac{1}{\sqrt{4-3x}}\, dx$$

$$= -\frac{1}{-3}\int_0^1 (4-3x)^{\frac{1}{2}}\,(-3)dx + \frac{4}{-3}\int_0^1 (4-3x)^{-\frac{1}{2}}\,(-3)dx$$

$$= \frac{1}{3}\left|\frac{(4-3x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_0^1 = \frac{1}{3}\left|\frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^1 = \frac{1}{3}.\frac{2}{3}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{4}{3}.\frac{2}{1}\left|(4-3x)^{\frac{1}{2}}\right|_0^1$$

$$= \frac{2}{9}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{8}{3}\left|(4-3x)^{\frac{1}{2}}\right|_0^1 = \frac{2}{9}\left[\left((4-3(1))^{\frac{3}{2}}\right) - \left((4-3(0))^{\frac{3}{2}}\right)\right] - \frac{8}{3}\left[\left((4-3(1))^{\frac{1}{2}}\right) - \left((4-3(0))^{\frac{1}{2}}\right)\right]$$

$$= \frac{2}{9}\left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}} - (4)^{\frac{1}{2}}\right] = \frac{2}{9}\left[1 - (2^2)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}} - (2^2)^{\frac{1}{2}}\right] = \frac{2}{9}[1 - 2^3] - \frac{8}{3}[1 - 2^1]$$

$$= \frac{2}{9}[1-8] - \frac{8}{3}[1-2] = \frac{2}{9}[-7] - \frac{8}{3}[-1] = \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}$$

$\boldsymbol{Q.29}$: $\boldsymbol{\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(2+\sin x)}\, dx}$

SOLUTION:

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2}{\sin x(2+\sin x)}\,.\cos x\;dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(2+\sin x)-\sin x}{\sin x(2+\sin x)}\,.\cos x\;dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{(2+\sin x)}{\sin x(2+\sin x)} - \frac{\sin x}{\sin x(2+\sin x)}\right] \cos x\;dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{1}{\sin x} - \frac{1}{2+\sin x}\right] \cos x\;dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{\cos x}{\sin x} - \frac{\cos x}{2+\sin x}\right] dx$$

$$=\frac{1}{2}\left|ln|\sin x|-ln|2+\sin x|\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\left(ln\left|\sin\frac{\pi}{2}\right|-ln\left|2+\sin\frac{\pi}{2}\right|\right)-\left(ln\left|\sin\frac{\pi}{6}\right|-ln\left|2+\sin\frac{\pi}{6}\right|\right)\right]$$

$$=\frac{1}{2}\left[(\ln(1)-ln|2+1|)-\left(ln\frac{1}{2}-ln\left|2+\frac{1}{2}\right|\right)\right]$$

$$=\frac{1}{2}\left[0-\ln 3-ln\frac{1}{2}+ln\frac{5}{2}\right]=\frac{1}{2}[-\ln 3-(ln1-ln2)+(\ln 5-ln2)]$$

$$=\frac{1}{2}[-\ln 3-ln1+ln2+\ln 5-ln2]=\frac{1}{2}[-\ln 3-0+\ln 5]$$

$$=\frac{1}{2}[\ln 5-\ln 3]=\frac{1}{2}\left[\ln\frac{5}{3}\right]=\frac{1}{2}\ln\frac{5}{3}$$

$Q.30$: $\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$

***SOLUTION*:**

$$\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx=$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{(1+\cos x)(2+\cos x)}\,\sin x\,dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{(2+\cos x)-(1+\cos x)}{(1+\cos x)(2+\cos x)}\,\sin x\,dx$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{(2+\cos x)}{(1+\cos x)(2+\cos x)}-\frac{(1+\cos x)}{(1+\cos x)(2+\cos x)}\right]\sin x\,dx$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{1}{(1+\cos x)}-\frac{1}{(2+\cos x)}\right]\sin x\,dx$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{\sin x}{(1+\cos x)}-\frac{\sin x}{(2+\cos x)}\right]dx$$

$$=\frac{-1}{-1}\int_0^{\frac{\pi}{2}}\left[\frac{\sin x}{(1+\cos x)}-\frac{\sin x}{(2+\cos x)}\right]dx$$

$$=\frac{1}{-1}\int_0^{\frac{\pi}{2}}\left[\frac{-\sin x}{(1+\cos x)}-\frac{-\sin x}{(2+\cos x)}\right]dx$$

$$=-[\ln|1+\cos x|-\ln|2+\cos x|]_0^{\frac{\pi}{2}}$$

$$=-\left[\left(\ln\left|1+\cos\frac{\pi}{2}\right|-\ln\left|2+\cos\frac{\pi}{2}\right|\right)-(\ln|1+\cos 0|-\ln|2+\cos 0|)\right]$$

$$=-[(\ln|1+0|-\ln|2+0|)-(\ln|1+1|-\ln|2+1|)]$$

$$=-[\ln(1)-\ln(2)-\ln(2)+\ln(3)]$$

$$=-[0-2\ln(2)+\ln 3]=2\ln(2)-\ln(3)$$

$$=\ln(2)^2-\ln(3)=\ln 4-\ln(3)=\ln\frac{4}{3}$$

$$\because\left(\begin{aligned}\boldsymbol{alnb}&=\boldsymbol{lnb^a}\\ \because \boldsymbol{lna}-\boldsymbol{lnb}&\\ &=\boldsymbol{ln\frac{a}{b}}\end{aligned}\right)$$

***Application of definate integral*: *Area under the curve*:**

Case1. *if* $f(x)\geq 0\ \forall x\in[a,b]$ *then curve lies above* $x-axis$.

$$A=\int_a^b f(x)dx\ \ where\ a<b$$

$$A\ is\ area\ of\ region\ above\ x-axis.under\ the\ curve\ of\ function\ y=f(x)from\ a\ to\ b$$

Case II $if\ f(x)\leq 0\ \forall x\in[a,b]$ $then\ curve\ lies\ below$

$$x-axis.so\ A=-\int_a^b f(x)dx\ \ where\ a<b$$

$A\ is\ area\ of\ region\ below\ x-axis,under$ The curve of function $y=f(x)from\ a\ to\ b$

Exercise 3.7

$Q.1$: *Find the area between the* $\boldsymbol{x-axis\ and\ the\ curve\ y=x^2+1\ from\ x=1\ to\ x=2}$

***SOLUTION*:**

$Given\ \ y=x^2+1$ $\qquad As\ y=x^2+1>0\ in\ [1,2], therefore\ curve\ is\ above\ x-axis$

$$Required\ Area=\int_1^2 y\ dx=\int_1^2(x^2+1)\ dx$$

$$=\left|\frac{x^3}{3}+x\right|_1^2=\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)$$

$$=\left(\frac{2^3}{3}+2\right)-\left(\frac{1^3}{3}+1\right)$$

$$=\left(\frac{8+6}{3}\right)-\left(\frac{1+3}{3}\right)=\frac{14}{3}-\frac{4}{3}$$

$$=\frac{14-4}{3}=\frac{10}{3}\quad square\ unit$$

$Q.2$: *Find the area above the* $\boldsymbol{x-axis\ and\ under\ the\ curve\ y=5-x^2\ from\ x=-1\ to\ x=2}$

***SOLUTION*:**

$Given\ \ y=5-x^2\ \ As\ y=5-x^2>0\ in\ [-1,2]$

$, therefore\ curve\ is\ above\ x-axisRequired\ Area$

$$=\int_{-1}^2 y\ dx=\int_{-1}^2(5-x^2)\ dx=\left|5x-\frac{x^3}{3}\right|_{-1}^2$$

$$=\left(5.2-\frac{2^3}{3}\right)-\left(5(-1)-\frac{(-1)^3}{3}\right)$$

$$=\left(10-\frac{8}{3}\right)-\left(-5-\frac{-1}{3}\right)$$

$$= \left(\frac{30-8}{3}\right) - \left(\frac{-15+1}{3}\right)$$

$$= \frac{22}{3} - \frac{-14}{3} = \frac{22+14}{3} = \frac{36}{3} = 12 \quad sq.unit$$

$Q.3$: *Find the area below the curve $y = 3\sqrt{x}$ and above the $x -$ axis between $x = 1$ to $x = 4$*

***SOLUTION*:**

Given $y = 3\sqrt{x}$ *As* $y = 3\sqrt{x} > 0$

in $[1,4]$, *therefore curve is above* $x -$ *axis*

$$Required\ Area = \int_1^4 y\ dx = 3\int_{-1}^{2} \sqrt{x}\ dx$$

$$= 3\int_1^4 (x)^{\frac{1}{2}}.1\ dx = 3\left|\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4$$

$$= 3\left|\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^4 = 3.\frac{2}{3}\left|(x)^{\frac{3}{2}}\right|_1^4$$

$$= 2\left((4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right) = 2\left((2^2)^{\frac{3}{2}} - 1\right)$$

$$= 2(2^3 - 1) = 2(8-1) = 2(7)$$

$$= 14 \quad sq.unit$$

$Q.4$: *Find the area bounded by* cos *function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$*

***SOLUTION*:**

Given $y = \cos x$ *As* $y = \cos x \geq$
0 *in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *therefore curve is above* $x -$ *axis*

$$Required\ Area = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\ dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$$

$$= |\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)$$

$$= 1 - (-1) = 1 + 1 = 2 \quad sq.unit$$

$Q.5$: *Find the area betweeen the $x -$ axis and the curve $y = 4x - x^2$.*

Put $y = 0$ $4x - x^2 = 0$

***SOLUTION*:**

$x(4 - x) = 0$

Given $y = 4x - x^2$ $x = 0$ *or* $4 - x = 0$

$x = 4$

As $y = 4x - x^2 \geq$
0 *in* $[0,4]$, *therefore curve is above* $x -$ *axis*

$$Required\ Area = \int_0^4 y\ dx = \int_0^4 (4x - x^2)\ dx$$

$$= \left|4\frac{x^2}{2} - \frac{x^3}{3}\right|_0^4$$

$$= \left(2.4^2 - \frac{4^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right)$$

$$= \left(32 - \frac{64}{3}\right) - (0 - 0) = \frac{96-64}{3}$$

$$= \frac{32}{3} \ sq.unit$$

$Q.6$: *Determine the area bounded by the parabola $y = x^2 + 2x - 3$ and the $x -$ axis.*

***SOLUTION*:**

Given $y = x^2 + 2x - 3$
As $y =$ $y = x^2 + 2x - 3 \leq$
0 *in* $[-3,1]$, *therefore curve is below* $x -$ *axis*

$$Required\ Area = -\int_{-3}^{1} y\ dx$$

$$= -\int_{-3}^{1} (x^2 + 2x - 3)\ dx$$

$$= -\left|\frac{x^3}{3} + 2\frac{x^2}{2} - 3x\right|_{-3}^{1}$$

$$= -\left|\frac{x^3}{3} + x^2 - 3x\right|_{-3}^{1}$$

$$= -\left[\left(\frac{1^3}{3} + 1^2 - 3.1\right) - \left(\frac{(-3)^3}{3} + (-3)^2 - 3(-3)\right)\right]$$

$$= -\left[\left(\frac{1}{3} + 1 - 3\right) - \left(\frac{-27}{3} + 9 + 9\right)\right]$$

$$= -\left[\frac{1}{3} + 1 - 3 + \frac{27}{3} - 9 - 9\right]$$

$$= -\left[\frac{1}{3} + \frac{27}{3} - 20\right]$$

$$= -\left[\frac{1+27-60}{3}\right]$$

$$= -\left[\frac{-32}{3}\right] = \frac{32}{3} \ sq.unit$$

Put $y = o$ $x^2 + 2x - 3 = 0$
$x^2 + 3x - x - 3 = 0$
$x(x + 3) - 1(x + 3) = 0$
$(x + 3) - (x - 1) = 0$
$x + 3 = 0$ *or* $x - 1 = 0$
$x = -3$, $x = 0$

$Q.7$: *Find the area bounded by the curve $y = x^3 + 1$, the $x -$ axis and line $x = 2$*

***SOLUTION*:**

Given $y = x^3 + 1$ *Put* $y = 0$ $x^3 + 1 = 0$
$(x + 1)(x^2 - x + 1) = 0$
$x + 1 = 0$ *or* $x^2 - x + 1 = 0$ *(Solve itself)*
$x = -1$

After solving this equation gives imaginary roots so neglect.

As $y = x^3 + 1 \geq$
0 *in* $[-1,2]$, *therefore curve is above* $x -$ *axis*

$$Required\ Area=\int_{-1}^{2} y\ dx=\int_{-1}^{2}(x^3+1)\ dx$$
$$=\left|\frac{x^4}{4}+x\right|_{-1}^{2}$$
$$=\left(\frac{2^4}{4}+2\right)-\left(\frac{(-1)^4}{4}+(-1)\right)$$
$$=(4+2)-\left(\frac{1}{4}-1\right)$$
$$=(6)-\left(\frac{1-4}{4}\right)=6-\frac{-3}{4}=6+\frac{3}{4}$$
$$=\frac{24+3}{4}=\frac{27}{4}\qquad square\ unit.$$

Q.8: *Find the area bounded by the curve $y=x^3-4x$, and the $x-$ axis.*

***SOLUTION*:**

$Put\ y=0\quad x^3-4x=0$
$x(x^2-4)=0$
$x(x-2)(x+2)=0$
$x=0\ \ or\ \ x-2=0\ \ ,or\ \ x+2=0$
$x=0\ \ or\ \ x=2\ \ ,or\ \ x=-2$

Given $y=x^3-4x$

As $y=x^3-4x\geq 0$ *in* $[-2,0]$, *therefore the curve is above* $x-$ *axis*

As $y=x^3-4x\leq 0$ *in* $[0,2]$, *therefore the curve is below* $x-$ *axis*

$$Required\ Area=\int_{-2}^{0} y\ dx-\int_{0}^{2} y\ dx$$
$$=\int_{-2}^{0}(x^3-4x)\ dx-\int_{0}^{2}(x^3-4x)\ dx$$
$$=\left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{-2}^{0}-\left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{0}^{2}$$
$$=\left[\left(\frac{0^4}{4}-2(0)^2\right)-\left(\frac{(-2)^4}{4}-2(-2)^2\right)\right]-\left[\left(\frac{(2)^4}{4}-2(2)^2\right)-\left(\frac{(0)^4}{4}-2(0)^2\right)\right]$$
$$=[(0-0)-(4-8)]-[(4-8)-(0-0)]$$
$$=0+4+4+0$$
$$=8\qquad square\ unit.$$

Q.9: *Find the area between the curve $y=x(x-1)(x+1)$, and the $x-$ axis.*

***SOLUTION*:**

Given $y=x(x-1)(x+1)=x^3-x$

As $y=x^3-x\geq 0$ *in* $[-1,0]$, *therefore the curve is above* $x-$ *axis*

As $y=x^3-x\leq 0$ *in* $[0,1]$, *therefore the curve is below* $x-$ *axis*

$$Required\ Area=\int_{-1}^{0} y\ dx-\int_{0}^{1} y\ dx$$
$$=\int_{-1}^{0}(x^3-x)\ dx-\int_{0}^{1}(x^3-x)\ dx$$
$$=\left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{-1}^{0}-\left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{0}^{1}$$
$$=\left[\left(\frac{0^4}{4}-\frac{(0)^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)\right]-\left[\left(\frac{(1)^4}{4}-\frac{(1)^2}{2}\right)-\left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)\right]$$
$$=\left[(0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]-\left[\left(\frac{1}{4}-\frac{1}{2}\right)-(0-0)\right]$$
$$=0-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}+\frac{1}{2}+0$$
$$=\frac{-1+2-1+2}{4}=\frac{1}{2}$$

Q.10: *Find the area above the $x-$ axis, bounded by the curve $y^2=3-x$ from $x=-1$ to $x=2$*

***SOLUTION*:**

Given $y^2=3-x\ \Longrightarrow y=\sqrt{3-x}$ *As* $y=\sqrt{3-x}\ \geq 0$ *in* $[-1,2]$, *therefore curve is above* $x-$ *axis*

$$Required\ Area=\int_{-1}^{2} y\ dx=\int_{-1}^{2}\sqrt{3-x}\ \ dx$$
$$=-\int_{-1}^{2}(3-x)^{\frac{1}{2}}(-1)\ dx$$
$$=-\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{2}$$
$$=-\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-1}^{2}$$
$$=-\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right)-\left((3-(-1))^{\frac{3}{2}}\right)\right]$$
$$=-\frac{2}{3}\left[1-\left((4)^{\frac{3}{2}}\right)\right]=-\frac{2}{3}[1-8]$$
$$=\frac{14}{3}\quad sq.unit$$

Q.11: *Find the area betweeen the $x-$ axis and the curve $y=\cos\frac{1}{2}x$ from $x=-\pi$ to π.*

***SOLUTION*:**

Given $y = \cos\frac{1}{2}x$ *As* $y = \cos\frac{1}{2}x \geq$ 0 *in* $[-\pi, \pi]$*, therefore curve is above* $x -$ *axis*

Required Area $= \int_{-\pi}^{\pi} y\ dx = \int_{-\pi}^{\pi} \cos\frac{1}{2}x\ dx$

$$= \left|\frac{\sin\frac{1}{2}x}{\frac{1}{2}}\right|_{-\pi}^{\pi} = 2\left[\left(\sin\frac{1}{2}(\pi)\right) - \left(\sin\frac{1}{2}(-\pi)\right)\right]$$

$= 2[1 - (-1)] = 4$

Q. 12: Find the area betweeen the x − axis and the curve $y = \sin 2x$ ***from*** $x = 0$ ***to*** $\frac{\pi}{3}$.

SOLUTION:

Given $y = \sin 2x$ *As* $y = \sin 2x \geq 0$ *in* $\left[0, \frac{\pi}{3}\right]$*, therefore curve is above* $x -$ *axis*

Required Area $= \int_0^{\frac{\pi}{3}} y\ dx = \int_0^{\frac{\pi}{3}} \sin 2x\ dx$

$= \left|\frac{-\cos 2x}{2}\right|_0^{\frac{\pi}{3}} = -\frac{1}{2}\left[\left(\cos\frac{2\pi}{3}\right) - (\cos 2(0))\right]$

$= -\frac{1}{2}\left[-\frac{1}{2} - 1\right] = \frac{3}{4}$ *sq. unit*

Q. 13: Find the area betweeen the x − axis and the curve $y = \sqrt{2ax - x^2}$ ***when*** $a >$ **0.**

SOLUTION:

Given $y = \sin 2x$ *As* $y = \sin 2x \geq$ 0 *in* $\left[0, \frac{\pi}{3}\right]$*, therefore curve is above* $x -$ *axis*

Required Area $= \int_0^{2a} y\ dx = \int_0^{2a} \sqrt{2ax - x^2}\ dx =$ $\int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2}\ dx$

$= \int_0^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)}\ dx$

$= \int_0^{2a} \sqrt{a^2 - (x - a)^2}\ dx$

Using formula $\int \sqrt{a^2 - x^2}\, dx$

$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + c$

$$= \left[\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + \left(\frac{x-a}{2}\right)\sqrt{a^2 - (x-a)^2}\right]_0^{2a}$$

$$= \left(\frac{a^2}{2}\sin^{-1}\left(\frac{2a-a}{a}\right) + \left(\frac{2a-a}{2}\right)\sqrt{a^2 - (2a-a)^2}\right) - \left(\frac{a^2}{2}\sin^{-1}\left(\frac{0-a}{a}\right) + \left(\frac{0-a}{2}\right)\sqrt{a^2 - (0-a)^2}\right)$$

$= \left(\frac{a^2}{2}\sin^{-1}(1) + \left(\frac{a}{2}\right)\sqrt{a^2 - a^2}\right) - \left(\frac{a^2}{2}\sin^{-1}(-1) + \left(\frac{-a}{2}\right)\sqrt{a^2 - a^2}\right) = \left(\frac{a^2}{2}.\frac{\pi}{2} + 0\right) - \left(\frac{a^2}{2}\left(-\frac{\pi}{2}\right) - 0\right) = \frac{a^2\pi}{4} + \frac{a^2\pi}{4}$

$= \frac{a^2\pi + a^2\pi}{4} = \frac{2a^2\pi}{4} = \frac{a^2\pi}{2}$ *square unit.*

Differential equation:
An equation containing atleast one derivative of a dependent variable with respect to an independent variable is called differential equation. e. g.

$y\frac{dy}{dx} + 2x = 0$ *and* $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$

Order of Differential equation:
The order of the differential equation is the order of the highest derivative in the equation.

$y\frac{dy}{dx} + 2x = 0$ (1*st order differential equation*)

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x =$ 0(2*nd order differential equation*)

Degree of Differential equation:
The degree of a differential equation is the greatest power of the highest order derivative in the equation.

$x\frac{d^4y}{dx^4} + \frac{dy}{dx} - x\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + 2x = 0$

(1*st degree differential equation*)

$x\left(\frac{d^4y}{dx^4}\right)^3 + \frac{dy}{dx} - x\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + 2x = 0$

(3*rd degree differential equation*)

General solution:
The solution of differential equation which contains arbitrary constants is called general solution.

Particular solution:
The solution obtained from general solution by applying the initial conditions is called particular solution.

Initial value conditions:
The arbitrary constants involving in the solution of a differential equation can be determine by the someven conditions. such conditions are called Initial value c

TIT BIT: *General soltion of the differential equation of order n contains n arbitrary constants which can be determined by n initial values condtions.*

Exercise 3.8

Q. 1: Check that each of the following equations written against the differential equation is its solution.

i) $x\frac{dy}{dx} = 1 + y$

Prove that $y = cx - 1$

SOLUTION:

$x\frac{dy}{dx} = 1 + y$

Separating the variables:

$x\,dy = (1 + y)dx$

$\frac{1}{1+y}dy = \frac{1}{x}\ dx$

Integrating both sides

$\int \frac{1}{1+y}dy = \int \frac{1}{x}\ dx$

$ln|1 + y| = ln|x| + ln|c|$

$ln|1 + y| = ln|cx|$

$1 + y = cx$

$y = cx - 1$

ii) $\boldsymbol{x^2(2y+1)\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y+1)\frac{dy}{dx} = 1$

$(2y+1)dy = \frac{1}{x^2}\ dx$

Integrating both sides

$\int(2y+1)dy = \int x^{-2}\ dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

iii) $\boldsymbol{y\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y+1)\frac{dy}{dx} = 1$

$(2y+1)dy = \frac{1}{x^2}\ dx$

Integrating both sides

$\int(2y+1)dy = \int x^{-2}\ dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

iv) $\boldsymbol{\frac{1}{x}\frac{dy}{dx} - 2y = 0}$

Prove that $\boldsymbol{y = ce^{x^2}}$

SOLUTION:

$\frac{1}{x}\frac{dy}{dx} = 2y$

Separating the variables:

$\frac{1}{y}\ dy = 2x\ dx$

Integrating both sides

$\int\frac{1}{y}dy = \int 2x\ dx$

$ln|y| = 2\frac{x^2}{2} + c_1$

$ln|y| = x^2 + c_1$

$e^{ln|y|} = e^{x^2+c_1}$

$y = e^{x^2} + e^{c_1}$

$y = e^{x^2} + c$

v) $\boldsymbol{\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}}$

Prove $\boldsymbol{y = \tan(e^x + c)}$

SOLUTION:

$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

Separating the variables:

$\frac{1}{y^2+1}\ dy = \frac{1}{e^{-x}}dx$

$\frac{1}{1+y^2}\ dy = e^x dx$

Integrating both sides

$\int\frac{1}{1+y^2}dy = \int e^x\ dx$

$\tan^{-1} y = e^x + c$

$y = \tan(e^x + c)$

Q.2: $\boldsymbol{\frac{dy}{dx} = -y}$

SOLUTION:

$\frac{dy}{dx} = -y$

Separating the variables:

$\frac{1}{y}dy = -\ dx$

Integrating both sides

$\int\frac{1}{y}dy = -\int 1\ dx$

$\ln y = -x + c_1$

$e^{\ln y} = e^{-x+c_1}$

$y = e^{-x}\ e^{c_1}$

$y = c\ e^{-x}$

Q.3: $\boldsymbol{y\ dx + x\ dy = 0}$

SOLUTION:

$y\ dx + x\ dy = 0$

Separating the variables:

$x\ dy = -y\ dx$

$\frac{1}{y}\ dy = -\frac{1}{x}\ dx$

Integrating both sides

$\int\frac{1}{y}dy = -\int\frac{1}{x}\ dx$

$\ln y = -\ln x + \ln c$

$\ln y + \ln x = +\ln c$

$\ln(xy) = +\ln c$

$xy = c$

Q.4: $\boldsymbol{\frac{dy}{dx} = \frac{1-x}{y}}$

SOLUTION:

$\frac{dy}{dx} = \frac{1-x}{y}$

Separating the variables:

$y\ dy = (1-x)\ dx$

Integrating both sides

$\int y\ dy = \int(1-x)\ dx$

$\frac{y^2}{2} = x - \frac{x^2}{2} + c_1$

$y^2 = 2x - x^2 + 2c_1$

$y^2 = x(2-x) + c$

Q.5: $\boldsymbol{\frac{dy}{dx} = \frac{y}{x^2}}$

SOLUTION:

$\frac{dy}{dx} = \frac{y}{x^2}$

Separating the variables:

$\frac{1}{y}\ dy = \frac{1}{x^2}\ dx$

Integrating both sides

$\int\frac{1}{y}dy = \int x^{-2}\ dx$

$\ln y = \frac{x^{-2+1}}{-2+1} + c_1$

$\ln y = \frac{x^{-1}}{-1} + c_1$

$$\ln y = -\frac{1}{x} + c_1$$

$$e^{\ln y} = e^{-\frac{1}{x}+c_1}$$

$$y = e^{-\frac{1}{x}}\ e^{c_1}$$

$$y = c\, e^{-\frac{1}{x}}$$

$Q.6$: $\sin y \csc x \frac{dy}{dx} = 1$

***SOLUTION*:**

$$\sin y \csc x \frac{dy}{dx} = 1$$

Separating the variables:

$$\sin y \frac{1}{\sin x}\ dy = dx$$

$$\sin y\ \ dy = \sin x\, dx$$

Integrating both sides

$$\int \sin y\ \ dy = \int \sin x\ \ dx$$

$$-\cos y = -\cos x + c_1$$

$$-\cos y = -(\cos x - c_1)$$

$$\cos y = \cos x - c_1$$

$$\cos y = \cos x + c$$

$Q.7$: $x\,dy + y(x-1)dx = 0$

***SOLUTION*:**

$$x\,dy + y(x-1)dx = 0$$

Separating the variables:

$$x\,dy = -y(x-1)dx$$

$$\frac{1}{y}\ dy = -\left(\frac{x-1}{x}\right)\ dx$$

$$\frac{1}{y}\ dy = -\left(1-\frac{1}{x}\right)\ dx$$

Integrating both sides

$$\int \frac{1}{y} dy = \int -1 + \frac{1}{x}\ dx$$

$$\ln y = -x + \ln x + \ln c$$

$$\ln y = \ln(xc) - x$$

$$\ln y - \ln(xc) = -x$$

$$\ln\left(\frac{y}{xc}\right) = -x$$

$$e^{\ln\left(\frac{y}{xc}\right)} = e^{-x}$$

$$\frac{y}{xc} = e^{-x}$$

$$y = cx\, e^{-x}$$

$Q.8$: $\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$

***SOLUTION*:**

$$\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$$

Separating the variables:

$$(x^2+1)y\,dx = x(y+1)dy$$

$$x(y+1)\,dy = (x^2+1)y\,dx$$

$$\frac{y+1}{y}\ dy = \frac{x^2+1}{x}\ dx$$

$$1 + \frac{1}{y}\ dy = x + \frac{1}{x}\ dx$$

Integrating both sides

$$\int \left(1+\frac{1}{y}\right) dy = \int \left(x + \frac{1}{x}\right)\ dx$$

$$y + \ln y = \frac{x^2}{2} + \ln(x) + \ln c$$

$$y + \ln y = \frac{x^2}{2} + \ln(xc)$$

$$\ln y - \ln(xc) = \frac{x^2}{2} - y$$

$$\ln\left(\frac{y}{xc}\right) = \frac{x^2}{2} - y$$

$$e^{\ln\left(\frac{y}{xc}\right)} = e^{\frac{x^2}{2}-y}$$

$$\frac{y}{xc} = e^{\frac{x^2}{2}}\ e^{-y}$$

$$\frac{y}{e^{-y}} = xc\, e^{\frac{x^2}{2}}$$

$$y\, e^{y} = cx\, e^{\frac{x^2}{2}}$$

$Q.9$: $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}\ (1+y^2)$

***SOLUTION*:**

$$\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}\ (1+y^2)$$

Separating the variables:

$$\frac{1}{1+y^2} dy = \frac{1}{2}\ (x)dx$$

Integrating both sides

$$\int \frac{1}{1+y^2}\ dy = \frac{1}{2}\int\ x\,dx$$

$$\tan^{-1} y = \frac{1}{2}\frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{4} + c\right)$$

$Q.10$: $2x^2y\frac{dy}{dx} = x^2 - 1$

***SOLUTION*:**

$$2x^2y\frac{dy}{dx} = x^2 - 1$$

Separating the variables:

$$y\,dy = \frac{x^2-1}{2x^2} dx$$

$$y\,dy = \frac{1}{2}\left(\frac{x^2-1}{x^2}\right) dx$$

Integrating both sides

$$\int y\,dy = \frac{1}{2}\int\ \left(1 - \frac{1}{x^2}\right)\ dx$$

$$\int y\,dy = \frac{1}{2}\int\ (1 - x^{-2})\ dx$$

$$\frac{y^2}{2} = \frac{1}{2}\ \left(x - \frac{x^{-1}}{-1}\right) + c$$

$$y^2 =\ x + \frac{1}{x} + c$$

$Q.11$: $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

***SOLUTION*:**

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Separating the variables:

$$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x(2y+1)-2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{2xy+x-2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

$$(2y+1)dy = xdx$$

Integrating both sides

$$\int(2y+1)\ dy = \int\ x\,dx$$

$$2\frac{y^2}{2} + y = \frac{x^2}{2} + c$$

$$y(y+1) = \frac{x^2}{2} + c$$

$Q.12$: $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2$

***SOLUTION*:**

$$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2$$

Separating the variables:

$$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$x^2(1-y)dy = -y^2(1+x)dx$$

$$\frac{1-y}{y^2}\ dy = -\frac{1+x}{x^2}dx$$

$$\frac{1}{y^2} - \frac{y}{y^2}\ dy = -\left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$$

$$\frac{1}{y^2} - \frac{1}{y}\ dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$$

$$\left(y^{-2} - \frac{1}{y}\right)\ dy = -\left(x^{-2} + \frac{1}{x}\right)dx$$

Integrating both sides

$$\int \left(y^{-2} - \frac{1}{y}\right)\ dy = -\int\ \left(x^{-2} + \frac{1}{x}\right)\ dx$$

$$\frac{y^{-2+1}}{-2+1} - \ln y = -\left(\frac{x^{-2+1}}{-2+1} + \ln x\right) + c_1$$

$$\frac{y^{-1}}{-1} - \ln y = -\left(\frac{x^{-1}}{-1} + \ln x\right) + c_1$$

$$-\frac{1}{y} - \ln y = -\left(-\frac{1}{x} + \ln x\right) + c_1$$

$$\ln y + \frac{1}{y} = \left(-\frac{1}{x} + \ln x\right) - c_1$$

$$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c$$

$Q.13$: $\sec^2 x\ \tan y\ dx + \sec^2 y\ \tan x\ dy = 0$

SOLUTION:

$$\sec^2 x\ \tan y\ dx + \sec^2 y\ \tan x\ dy = 0$$

Separating the variables:

$$\sec^2 y\ \tan x\ dy = \sec^2 x\ \tan y\ dx$$

$$\frac{\sec^2 y}{\tan y}\ dy = \frac{\sec^2 x}{\tan x}\ dx$$

Integrating both sides

$$\int \frac{\sec^2 y}{\tan y}\ dy = -\int \frac{\sec^2 x}{\tan x}\ dx$$

$$\ln(\tan y) = -\ln(\tan x) + \ln c$$

$$\ln(\tan y) + \ln(\tan x) = \ln c$$

$$\ln(\tan y \tan x) = \ln c$$

$$\tan y \tan x = c$$

$Q.14$: $\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

SOLUTION:

$$\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

Separating the variables:

$$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$$

$$-x\frac{dy}{dx} - 2\frac{dy}{dx} = 2y^2 - y$$

Multiplying both sides by -1

$$x\frac{dy}{dx} + 2\frac{dy}{dx} = y - 2y^2$$

$$(x+2)\frac{dy}{dx} = y - 2y^2$$

$$(x+2)dy = y(1-2y)\ dx$$

$$\frac{1}{y(1-2y)}dy = \frac{1}{x+2}\ dx$$

Integrating both sides

$$\int \frac{(1-2y)+2y}{y(1-2y)}\ dy = \int\ \frac{1}{x+2}\ dx$$

$$\int \left[\frac{(1-2y)}{y(1-2y)} + \frac{2y}{y(1-2y)}\right]\ dy = \int\ \frac{1}{x+2}\ dx$$

$$\int \left[\frac{1}{y} + \frac{2}{(1-2y)}\right]\ dy = \int\ \frac{1}{x+2}\ dx$$

$$\int \frac{1}{y}\ dy - \int \frac{2}{2y-1}\ dy = \int\ \frac{1}{x+2}\ dx$$

$$\ln(y) + \ln(2y-1) = \ln(x+2) + \ln(c)$$

$$\ln\left(\frac{y}{2y-1}\right) = \ln\ c(x+2)$$

$$\frac{y}{2y-1} = c(x+2)$$

$Q.15$: $1 + \cos x \tan y\frac{dy}{dx} = 0$

SOLUTION:

$$1 + \cos x \tan y\frac{dy}{dx} = 0$$

Separating the variables:

$$\cos x \tan y\frac{dy}{dx} = -1$$

$$\tan y\, dy = -\frac{1}{\cos x}\ dx$$

Integrating both sides

$$\int \frac{-\sin y}{\cos y}\ dy = \int\ \sec x\ dx$$

$$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$$

$$\ln(\cos y) = \ln[c(\sec x + \tan x)]$$

$$\cos y = c(\sec x + \tan x)$$

$Q.16$: $y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

SOLUTION:

$$y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$$

Separating the variables:

$$y - x\frac{dy}{dx} = 3 + 3x\frac{dy}{dx}$$

$$y - 3 = 3x\frac{dy}{dx} + x\frac{dy}{dx}$$

$$y - 3 = 4x\frac{dy}{dx}$$

$$4x\frac{dy}{dx} = y - 3$$

$$\frac{1}{y-3}\ dy = \frac{1}{4x}dx$$

Integrating both sides

$$\int \frac{1}{y-3}\ dy = \frac{1}{4}\int\ \frac{1}{x}\ dx$$

$$\ln(y-3) = \frac{1}{4}\ln x + \ln c$$

$$\ln(y-3) = \ln x^{\frac{1}{4}} + \ln c$$

$$\ln(y-3) = \ln\left(cx^{\frac{1}{4}}\right)$$

$$y - 3 = cx^{\frac{1}{4}}$$

$$y = 3 + cx^{\frac{1}{4}}$$

$Q.17$: $\sec x + \tan y\frac{dy}{dx} = 0$

SOLUTION:

$$\sec x + \tan y\frac{dy}{dx} = 0$$

Separating the variables:

$$\tan y\frac{dy}{dx} = -\sec x$$

$$\tan y\ dy = -\sec x\ dx$$

Integrating both sides

$$\int \tan y\ dy = -\int\ \sec x\ dx$$

$$\int \frac{-\sin y}{\cos y}\ dy = \int\ \sec x\ dx$$

$$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$$

$$\ln(\cos y) = \ln[c(\sec x + \tan x)]$$

$$\cos y = c(\sec x + \tan x)$$

$Q.18$: $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

SOLUTION:

$(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$ *Separating the variables*:

$dy = \frac{e^x-e^{-x}}{e^x+e^{-x}}\ dx$

Integrating both sides

$\int 1\ \ dy = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}\ dx$

$y = \ln(e^x + e^{-x}) + c$

Q.19: Find the general solution of the following equation $\frac{dy}{dx} - x = xy^2$. Also find the Particular solution if $y = 1$ when $x = 0$

SOLUTION:

$\frac{dy}{dx} - x = xy^2$

Separating the variables:

$\frac{dy}{dx} = xy^2 + x$

$\frac{dy}{dx} = x(y^2 + 1)$

$\frac{1}{y^2+1} dy = x\ dx$

Integrating both sides

$\int \frac{1}{1+y^2}\ dy = \int\ x\ dx$

$\text{Tan}^{-1} y = \frac{x^2}{2} + c$

(General solution) (1)

At $x = 0\ \ ,\ \ y = 1$

$\tan^{-1}(1) = \frac{(0)^2}{2} + c$

$\frac{\pi}{4} = c$ *(Put in* 1)

$\tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4}$

(Particular solution)

Q.20: Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x = 4$ when $t = 0$

SOLUTION:

$\frac{dx}{dt} = 2x$

Separating the variables:

$dx = 2x\ dt$

$\frac{1}{x}\ dx = 2\ dt$

Integrating both sides

$\int \frac{1}{x}\ dy = 2\int\ 1\ dx$

$\ln x = 2t + c_1$

$e^{\ln x} = e^{2t+c_1}$

$x = e^{2t}\ \ e^{c_1}$

$x = ce^{2t}$ *Where* $e^{c_1} = c$

(General solution) (1)

At $x = 4\ \ ,\ \ t = 0$

$4 = ce^{2(0)}$

$4 = ce^0$

$4 = c$ *Put in* (1) $\because\ e^0 = 1$

$x = 4e^{2t}$

(Particular solution)

Q.21: Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the Particular solution if $s = 4e$ when $t = 0$

SOLUTION:

$\frac{ds}{dt} + 2st = 0$

Separating the variables:

$\frac{ds}{dt} = -2st$

$\frac{1}{s} ds = -2t\ dt$

Integrating both sides

$\int \frac{1}{s} ds = -\int\ 2t\ dx$

$\ln s = -2\frac{t^2}{2} + c_1$

$\ln s = -t^2 + c_1$

$s = e^{-t^2+c_1}$

$s = e^{-t^2}\ e^{c_1}$

$s = ce^{-t^2}$ *Where* $e^{c_1} = c$

(General solution) (1)

At $s = 4e\ \ ,\ \ t = 0$

$4e = ce^{-(0)^2}$

$4e = ce^0$

$4e = c$ *Put in* (1) $\because\ e^0 = 1$

$s = 4e\ .\ e^{-t^2}$

$s = 4\ e^{1-t^2}$

(Particular solution)

Q22. In a culture, bacteria increase number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:

Let P be numbers of bacteria then

$$\frac{dP}{dt} \propto P$$

⇨ $\frac{dP}{dt} = kP$

⇨ $\frac{1}{P}\ dP = kdt$

take integral

⇨ $\int \frac{1}{P} dP = k\int dt$

⇨ $lnP = kt + lnc$

⇨ $lnp - lnc = kt$

⇨ $\ln\frac{p}{c} = kt$

⇨ $\frac{P}{c} = e^{kt}$

⇨ $p = ce^{kt} \to (i)$

put p=200 , t=0 (condition1)

$200 = ce^{k(0)} = ce^0$

⇨ $c = 200\ \ \because e^0 = 1$

So $(i)p = 200e^{kt} \to (ii)$

Put$p = 400\ when\ t = 2\ (conditionII)$

$so(i)\ \ \Rightarrow\ \ 400 = 200e^{kt}$

⇨ $2 = e^{kt} \Rightarrow ln2 = lne^{2k}$

⇨ $2k = ln2$

⇨ $k = \frac{1}{2}ln2$

So $(ii) \Rightarrow p = 200e^{\frac{1}{2}ln2}$

⇨ $p = 200e^{\frac{ln2}{2}(4)}$ $for\ t = 4$

⇨ $p = 200^{2ln2} = 200e^{ln2^2} = 200e^{ln4}$

⇨ $p = 200(4) \Rightarrow p = 800$

Which is required number of bacteria present four latter.

Q.23 a ball is thrown vertically upward with a velocity of 2450cm/sec neglecting air resistance, find

i. **Velocity of ball at any time t**
ii. **Distance traveled in any time t**
iii. **Maximum height attained by the ball**

Solution:

Let $v\ is\ velocity\ and\ g\ is\ acceleration, so$

$$i)\ \frac{dv}{dt} = -g \quad for\ upward$$

⇨ $dv = -gdt$

⇨ $\int dv = -g \int dt$

⇨ $v = -gt + c_1$

$Put\ v = 2450\ , t = 0\ so$

$$2450 = -g(0) + c_1 \Rightarrow c_1 = 2450$$

$$v = -gt + 2450 \quad \because g = 9.8m/sec$$

Thus $v = -980t + 2450 \quad \Rightarrow g = 980cm/sec$

ii) $let\ h\ be\ height\ so$

$$v = \frac{dh}{dt}$$

⇨ $\frac{dh}{dt} = v$

⇨ $\frac{dh}{dt} = -980 + 2450$

⇨ $dh = -980tdt + 2450dt$

⇨ $\int dh = -980 \int tdt + 2450 \int dt$

⇨ $h = -980\frac{t^2}{2} + 2450t + c_2$

$$put\ h = 0\ , t = 0$$

$$0 = -490(0)^2 + 2450(0) + c_2$$

$$\Rightarrow c_2 = 0$$

$$so\ \ h = -490t^2 + 2450$$

$$(iii)\ \ For\ max. hight, v = 0$$

So $0 = -980t^2 + 2450\ from\ (i)$

⇨ $980t = \frac{2450}{980}$

⇨ $t = \frac{5}{2}$

So $h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{5}{2}\right)^2$

$$= 6125 - 30625$$

$$\Rightarrow h = 3062.5$$

$$So\ max. hight = 3062.5cm$$

$$\max hight = 30.6m\ \ (\div\ by\ 100)$$