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Chapter 2.

DIFFERENTIATION

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Average rate of change

Let f be a real valued function the (difference quotient) $\frac{f(x_1)-f(x)}{x_1-x}$ is called average rate of cahnge.

Derivative:

let $f(x)$ be a function, then its derivative is denoted by $f'(x)$ or $\frac{df}{dx}$ and defined as;

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

- The process of finding f' is called differentiation.

Notation for Derivative:

For the function $y = f(x)$

$$\Rightarrow y + \delta y = f(x + \delta x) \Rightarrow \delta y = f(x + \delta x) - f(x)$$

Where δy is the increment of y (change in the valve of y corresponding to δx (increment of x) dividing (i) by δx on both sides, we get

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit on both sides as $\delta x \rightarrow 0$ we get

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \rightarrow (ii)$$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$, so (ii) is written as $\frac{dy}{dx} = f'(x)$

Note: the symbol $\frac{dy}{dx}$ is used for the derivative of y with replace to x and here it is not a quotient of dy and dx is also denoted by y'

*Different mathematician used symbols given below.

Leibniz $\frac{dy}{dx}$ or $\frac{df}{dx}$ **Newton** $f'(x)$

Lagrange $f'(x)$ **Cauchy** $D(fx)$

Finding $f'(x)$ from definition of derivative:

Given a function

$f, f(x)$ if it exist can be found by the following four Steps.

Step1. Find $f(x + \delta x)$

Step2. Simplify $f(x + \delta x) - f(x)$

Step3. Divide $f(x + \delta x) - f(x)$ by δx to get

$$\frac{f(x + \delta x) - f(x)}{\delta x}$$

And simplify it.

Step4. Find $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ the method of finding

derivatives by this process is called differentiation by definition or by from first principle.

Derivation of x^n when $n \in Z$

a) Let $y = x^n$ when n is + ve integer.

$$\Rightarrow y = x^n \rightarrow (i), y + \delta y = (x + \delta x)^n \rightarrow (ii)$$

$$\text{by (ii) - (i)} \Rightarrow y + \delta y - y = (x + \delta x)^n - x^n$$

$$\Rightarrow \delta y = (x + \delta x)^n - x^n$$

Using binomial theorem, we have

$$= \delta x \left[nx^{n-1} + \frac{(n-1)}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^n - x^n \right]$$

Dividing both side by δx

$$\Rightarrow \delta y = \left[x^n + nx^{n-1} \cdot \delta x + \frac{n(n-1)(n-2)}{2!} x^{n-2} \cdot (\delta x)^2 + \dots + (\delta x)^{n-1} \right]$$

$$\Rightarrow x^n + nx^{n-1} \cdot \delta x + \frac{n(n-1)(n-2)}{2!} x^{n-2} \cdot (\delta x)^2 + \dots + (\delta x)^{n-1}$$

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} \left[nx^{(n-1)} + \frac{(n-1)}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^{n-1} \right]$$

applying both sides by $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\delta x}{\delta x} \left[nx^{(n-1)} + \frac{(n-1)}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^{n-1} \right]$$

$$\frac{dy}{dx} = nx^{n-1} + \frac{n-1}{2!} x^{n-1}(0) + \dots + (0)^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = nx^{(n-1)}$$

b)

Let $y = x^n$ where n is - ve integer.

let $n = -m$ (m is + ve integer). then

$$y = x^{-m} \Rightarrow y = \frac{1}{x^m} \rightarrow (i)$$

$$\Rightarrow y + \delta y = \frac{1}{(x + \delta x)^m} \rightarrow (ii)$$

$$\text{by (ii) - (i)} \Rightarrow y + \delta y - y = \frac{1}{(x + \delta x)^m} - \frac{1}{x^m}$$

$$\Rightarrow \frac{x^m - (x + \delta x)^m}{x^m(x + \delta x)^m} \text{ using binomial theorem}$$

$$= \frac{x^m - (x^m + mx^{m-1} \cdot \delta x + \frac{m(m-1)}{2!} x^{m-2} \cdot (\delta x)^2 + \dots + (\delta x)^2)}{x^m(x + \delta x)^m}$$

$$= \frac{x^m - x^m - mx^{m-1} \cdot \delta x - \frac{mx^{m-1}}{2!} x^{m-2} \cdot (\delta x)^2 - \dots - (\delta x)^2}{x^m(x + \delta x)^m}$$

$$\delta y = \frac{-\delta x}{x^m(x + \delta x)^m} \left(mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \cdot (\delta x) + \dots + (\delta x)^{m-1} \right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{-\delta x}{x^m(x + \delta x)^m} \left(\frac{1}{\delta x} \right) \left(mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \cdot (\delta x) + \dots + (\delta x)^{m-1} \right)$$

Applying limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{-1}{x^m(x + \delta x)^m} \left(mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \cdot (\delta x) + \dots + (\delta x)^{m-1} \right) \right]$$

$$\frac{dy}{dx} = -\frac{1}{x^m \cdot x^m} \left(mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \cdot (0) + \dots + (0)^{m-1} \right)$$

$$\frac{dy}{dx} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{-m-1}$$

$$\Rightarrow \frac{dy}{dx} = -mx^{m-1} \text{ or } \frac{dy}{dx} = nx^{n-1}$$

So for we have proved that $\frac{dy}{dx} = nx^{n-1}, n \in Z$

Note that $\frac{d}{dx}(x^n) = nx^{n-1}$ is called power rule and holds if $n \in Q - Z$

Exercise 2.1

Question # 1.

Find the definition, the derivatives w.r.t 'x' of the following functions defined as:

(i). $2x^2 + 1$

Solution.

Let $y = 2x^2 + 1 \rightarrow (i)$

$y + \delta y = 2(x + \delta x)^2 + 1 \rightarrow (ii)$

by (ii) - (i)

$\delta y = 2(x + \delta x)^2 + 1 - y$

$\delta y = 2(x + \delta x)^2 + 1 - 2x^2 - 1$

$\delta y = 2(x^2 + \delta x^2 + 2x\delta x) - 2x^2$

$\delta y = 2x^2 + 2\delta x^2 + 4x\delta x - 2x^2$

$\delta y = 2\delta x^2 + 4x\delta x$

$\delta y = \delta x(2\delta x + 4x)$

Dividing both sides by δx

$\frac{\delta y}{\delta x} = (2\delta x + 4x)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (2\delta x + 4x)$

$\frac{dy}{dx} = 2(0) + 4x$

$\frac{dy}{dx} = 4x$

Which is the required.

(ii). $2 - \sqrt{x}$

Solution.

Let $y = 2 - \sqrt{x} \rightarrow (i)$

$y + \delta y = 2 - \sqrt{x + \delta x} \rightarrow (ii)$

eq(ii) - eq(i)

$\delta y = 2 - \sqrt{x + \delta x} - y$

$\delta y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x}$

$\delta y = \sqrt{x} - \sqrt{x + \delta x}$

$\delta y = x^{\frac{1}{2}} - (x + \delta x)^{\frac{1}{2}}$

$\delta y = x^{\frac{1}{2}} - x^{\frac{1}{2}} \left(1 + \frac{\delta x}{x} \right)^{\frac{1}{2}}$

$\delta y = x^{\frac{1}{2}} - x^{\frac{1}{2}} \left(1 + \frac{1}{2} \cdot \frac{\delta x}{x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \left(\frac{\delta x}{x} \right)^2 + \dots \right)$

$\delta y = x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{1}{2}} \left(\frac{1}{2} \cdot \frac{\delta x}{x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \left(\frac{\delta x}{x} \right)^2 + \dots \right)$

$\delta y = -x^{\frac{1}{2}} \delta x \left(\frac{1}{2x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$

Dividing both sides by δx , we have

$\frac{dy}{dx} = -x^{\frac{1}{2}} \left(\frac{1}{2x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} -x^{\frac{1}{2}} \left(\frac{1}{2x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$

$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} -x^{\frac{1}{2}} \left(\frac{1}{2x} + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \frac{(0)}{x^2} + \dots \right)$

$\frac{dy}{dx} = -x^{\frac{1}{2}} \left(\frac{1}{2x} \right)$

$\frac{dy}{dx} = -\frac{1}{2} x^{\frac{1}{2}-1}$

$\frac{dy}{dx} = -\frac{1}{2} x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2x^{\frac{1}{2}}}$

Which is the required.

(iii). $\frac{1}{\sqrt{x}}$

Solution.

Let $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{-\frac{1}{2}} \rightarrow (ii)$

eq(ii) - eq(i)

$\delta y = (x + \delta x)^{-\frac{1}{2}} - y$

$\delta y = (x + \delta x)^{-\frac{1}{2}} - x^{-\frac{1}{2}}$

$\delta y = (x + \delta x)^{-\frac{1}{2}} - x^{-\frac{1}{2}}$

$\delta y = x^{-\frac{1}{2}} \left(1 + \frac{\delta x}{x} \right)^{-\frac{1}{2}} - x^{-\frac{1}{2}}$

$\delta y = x^{-\frac{1}{2}} \left(1 - \frac{1}{2} \cdot \frac{\delta x}{x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right) - x^{-\frac{1}{2}}$

$\delta y = x^{-\frac{1}{2}} + x^{-\frac{1}{2}} \left(-\frac{1}{2} \cdot \frac{\delta x}{x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right) - x^{-\frac{1}{2}}$

$\delta y = x^{-\frac{1}{2}} \delta x \left(-\frac{1}{2x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \frac{\delta x}{x^2} + \dots \right)$

Dividing both sides by δx , we have

$\frac{dy}{dx} = x^{-\frac{1}{2}} \left(-\frac{1}{2x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \frac{\delta x}{x^2} + \dots \right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} x^{-\frac{1}{2}} \left(-\frac{1}{2x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \frac{\delta x}{x^2} + \dots \right)$

$\frac{dy}{dx} = x^{-\frac{1}{2}} \left(-\frac{1}{2x} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \frac{(0)}{x^2} + \dots \right)$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} \left(-\frac{1}{2x} \right)$$

$$\frac{dy}{dx} = -\frac{1}{2} x^{-\frac{1}{2}-1}$$

$$\frac{dy}{dx} = -\frac{1}{2} x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2x^{\frac{3}{2}}}$$

Which is the required.

(iv) $\frac{1}{x^3}$

Solution.

Let $y = \frac{1}{x^3} = x^{-3} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{-3} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x)^{-3} - y$$

$$\delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\delta y = x^{-3} \left(1 + \frac{\delta x}{x} \right)^{-3} - x^{-3}$$

$$\delta y = x^{-3} \left(\left(1 + \frac{\delta x}{x} \right)^{-3} - 1 \right)$$

$$\delta y = x^{-3} \left(1 - \frac{3\delta x}{x} + \frac{-3(-3-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right)$$

$$\delta y = x^{-3} \left(-\frac{3\delta x}{x} + \frac{-3(-3-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right)$$

$$\delta y = x^{-3} \delta x \left(-\frac{3}{x} + \frac{-3(-3-1)\delta x}{2! x^2} + \dots \right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{-3} \left(-\frac{3}{x} + \frac{-3(-3-1)\delta x}{2! x^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{-3} \left(-\frac{3}{x} + \frac{-3(-3-1)\delta x}{2! x^2} + \dots \right)$$

$$\frac{dy}{dx} = x^{-3} \left(-\frac{3}{x} + \frac{-3(-3-1)(0)}{2! x^2} + \dots \right)$$

$$\frac{dy}{dx} = -\frac{3}{x^{1+3}}$$

$$\frac{dy}{dx} = -\frac{3}{x^4}$$

Which is the required.

(v). $\frac{1}{x-a}$

Solution.

Let $y = \frac{1}{x-a} = (x-a)^{-1} \rightarrow (i)$

$y + \delta y = (x + \delta x - a)^{-1} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x - a)^{-1} - y$$

$$\delta y = (x - a + \delta x)^{-1} - (x - a)^{-1}$$

$$\delta y = (x - a)^{-1} \left(1 + \frac{\delta x}{x - a} \right)^{-1} - (x - a)^{-1}$$

$$\delta y = (x - a)^{-1} \left(\left(1 + \frac{\delta x}{x - a} \right)^{-1} - 1 \right)$$

$$\delta y = (x - a)^{-1} \left(1 - \frac{\delta x}{x - a} + \frac{-1(-1-1)}{2!} \left(\frac{\delta x}{x - a} \right)^2 + \dots - 1 \right)$$

$$\delta y = (x - a)^{-1} \left(-\frac{\delta x}{x - a} + \frac{-1(-1-1)}{2!} \left(\frac{\delta x}{x - a} \right)^2 + \dots \right)$$

$$\delta y = (x - a)^{-1} \delta x \left(-\frac{1}{x - a} + \frac{-1(-1-1)}{2!} \frac{\delta x}{(x - a)^2} + \dots \right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (x - a)^{-1} \left(-\frac{1}{x - a} + \frac{-1(-1-1)}{2!} \frac{\delta x}{(x - a)^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x - a)^{-1} \left(-\frac{1}{x - a} + \frac{-1(-1-1)}{2!} \frac{\delta x}{(x - a)^2} + \dots \right)$$

$$\frac{dy}{dx} = (x - a)^{-1} \left(-\frac{1}{x - a} + \frac{-1(-1-1)}{2!} \frac{(0)}{(x - a)^2} + \dots \right)$$

$$\frac{dy}{dx} = -\frac{1}{(x - a)^{1+1}}$$

$$\frac{dy}{dx} = \frac{1}{(x - a)^2}$$

Which is the required.

(vi). $x(x - 3)$

Solution.

Let $y = x(x - 3) = x^2 - 3x$

$y + \delta y = (x + \delta x)^2 - 3(x + \delta x)$

$\delta y = (x + \delta x)^2 - 3x - 3\delta x - y$

$\delta y = x^2 + \delta x^2 + 2x\delta x - 3x - 3\delta x - x^2 + 3x$

$\delta y = \delta x^2 + 2x\delta x - 3\delta x$

$\delta y = \delta x(\delta x + 2x - 3)$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (\delta x + 2x - 3)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (\delta x + 2x - 3)$$

$$\frac{dy}{dx} = (0 + 2x - 3)$$

$$\frac{dy}{dx} = 2x - 3$$

Which is the required.

(vii) $\frac{2}{x^4}$

Solution:

Let $y = \frac{2}{x^4} = 2x^{-4} \rightarrow (i)$

$y + \delta y = 2(x + \delta x)^{-4} \rightarrow (ii)$

$e(ii) - e(i)$

$\delta y = 2(x + \delta x)^{-4} - y$

$\delta y = 2(x + \delta x)^{-4} - 2x^{-4}$

$$\delta y = x^{-4} \left(1 + \frac{\delta x}{x}\right)^{-4} - 2x^{-4}$$

$$\delta y = 2x^{-4} \left(\left(1 + \frac{\delta x}{x}\right)^{-4} - 1 \right)$$

$$\delta y = 2x^{-4} \left(1 - \frac{4\delta x}{x} + \frac{-4(-4-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1 \right)$$

$$\delta y = 2x^{-4} \left(-\frac{4\delta x}{x} + \frac{-4(-4-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots \right)$$

$$\delta y = 2x^{-4} \delta x \left(-\frac{4}{x} + \frac{-4(-4-1)\delta x}{2! x^2} + \dots \right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = 2x^{-4} \left(-\frac{4}{x} + \frac{-4(-4-1)\delta x}{2! x^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2x^{-4} \left(-\frac{4}{x} + \frac{-4(-4-1)\delta x}{2! x^2} + \dots \right)$$

$$\frac{dy}{dx} = 2x^{-4} \left(-\frac{4}{x} + \frac{-4(-4-1)(0)}{2! x^2} + \dots \right)$$

$$\frac{dy}{dx} = -\frac{8}{x^{1+4}}$$

$$\frac{dy}{dx} = -\frac{8}{x^5}$$

Which is the required.

(viii). $(x + 4)^{\frac{1}{3}}$

Solution.

Let $y = (x + 4)^{\frac{1}{3}} \rightarrow (i)$

$y + \delta y = (x + \delta x + 4)^{\frac{1}{3}} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x + 4)^{\frac{1}{3}} - y$$

$$\delta y = (x + 4 + \delta x)^{\frac{1}{3}} - (x + 4)^{\frac{1}{3}}$$

$$\delta y = (x + 4)^{\frac{1}{3}} \left(1 + \frac{\delta x}{x + 4} \right)^{\frac{1}{3}} - (x + 4)^{\frac{1}{3}}$$

$$\delta y = (x + 4)^{\frac{1}{3}} \left(\left(1 + \frac{1}{3} \cdot \frac{\delta x}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \left(\frac{\delta x}{x + 4} \right)^2 + \dots \right) - 1 \right)$$

$$\delta y = (x + 4)^{\frac{1}{3}} \left(1 + \frac{1}{3} \cdot \frac{\delta x}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \left(\frac{\delta x}{x + 4} \right)^2 + \dots - 1 \right)$$

$$\delta y = (x + 4)^{\frac{1}{3}} \delta x \left(\frac{1}{3} \cdot \frac{1}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \frac{\delta x}{(x + 4)^2} + \dots \right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (x + 4)^{\frac{1}{3}} \left(\frac{1}{3} \cdot \frac{1}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \frac{\delta x}{(x + 4)^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x + 4)^{\frac{1}{3}} \left(\frac{1}{3} \cdot \frac{1}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \frac{\delta x}{(x + 4)^2} + \dots \right)$$

$$\frac{dy}{dx} = (x + 4)^{\frac{1}{3}} \left(\frac{1}{3} \cdot \frac{1}{x + 4} + \frac{1}{3} \left(\frac{1}{3} - 1 \right) \frac{(0)}{(x + 4)^2} + \dots \right)$$

$$\frac{dy}{dx} = (x + 4)^{\frac{1}{3}} \cdot \frac{1}{3(x + 4)}$$

$$\frac{dy}{dx} = \frac{1}{3(x + 4)^{1-\frac{1}{3}}}$$

$$\frac{dy}{dx} = \frac{1}{3(x + 4)^{\frac{2}{3}}}$$

Which is the required.

(ix). $x^{\frac{3}{2}}$

Solution.

Let $y = x^{\frac{3}{2}} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{\frac{3}{2}} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x)^{\frac{3}{2}} - y$$

$$\delta y = (x + \delta x)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}} \left(1 + \frac{\delta x}{x} \right)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}} \left(1 + \frac{3}{2} \cdot \frac{\delta x}{x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{(\delta x)^2}{2!} + \dots \right) - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}} + x^{\frac{3}{2}} \left(\frac{3}{2} \cdot \frac{\delta x}{x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{(\delta x)^2}{2!} + \dots \right) - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}} \delta x \left(\frac{3}{2x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{\frac{3}{2}} \left(\frac{3}{2x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{\frac{3}{2}} \left(\frac{3}{2x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{\delta x}{x^2} + \dots \right)$$

$$\frac{dy}{dx} = x^{\frac{3}{2}} \left(\frac{3}{2x} + \frac{3}{2} \left(\frac{3}{2} - 1 \right) \frac{(0)}{x^2} + \dots \right)$$

$$\frac{dy}{dx} = x^{\frac{3}{2}} \left(\frac{3}{2x} \right)$$

$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{3}{2}-1}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$$

Which is the required.

(x). $x^{\frac{5}{2}}$

Solution.

Let $y = x^{\frac{5}{2}} \rightarrow (i)$

$$y + \delta y = (x + \delta x)^{\frac{5}{2}} \rightarrow (ii)$$

$$\delta y = (x + \delta x)^{\frac{5}{2}} - y$$

$$\delta y = (x + \delta x)^{\frac{5}{2}} - x^{\frac{5}{2}}$$

$$\delta y = x^{\frac{5}{2}} \left(1 + \frac{\delta x}{x}\right)^{\frac{5}{2}} - x^{\frac{5}{2}}$$

$$\delta y = x^{\frac{5}{2}} \left(1 + \frac{5}{2} \cdot \frac{\delta x}{x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{(\delta x)^2}{x^2} + \dots\right) - x^{\frac{5}{2}}$$

$$\delta y = x^{\frac{5}{2}} + x^{\frac{5}{2}} \left(\frac{5}{2} \cdot \frac{\delta x}{x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{(\delta x)^2}{x^2} + \dots\right) - x^{\frac{5}{2}}$$

$$\delta y = x^{\frac{5}{2}} \delta x \left(\frac{5}{2x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{\delta x}{x^2} + \dots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{\frac{5}{2}} \left(\frac{5}{2x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{\delta x}{x^2} + \dots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{\frac{5}{2}} \left(\frac{5}{2x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{\delta x}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = x^{\frac{5}{2}} \left(\frac{5}{2x} + \frac{5}{2} \left(\frac{5}{2} - 1\right) \frac{(0)}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = x^{\frac{5}{2}} \left(\frac{5}{2x}\right)$$

$$\frac{dy}{dx} = \frac{5}{2} x^{\frac{5}{2}-1}$$

$$\frac{dy}{dx} = \frac{5}{2} x^{\frac{3}{2}}$$

Which is the required.

(xi). x^m , $m \in N$

Solution.

$$\text{Let } y = x^m \rightarrow (i)$$

$$y + \delta y = (x + \delta x)^m \rightarrow (ii)$$

eq(ii) - eq(i)

$$\delta y = (x + \delta x)^m - y$$

$$\delta y = (x + \delta x)^m - x^m$$

$$\delta y = x^m \left(1 + \frac{\delta x}{x}\right)^m - x^m$$

$$\delta y = x^m \left(\left(1 + \frac{\delta x}{x}\right)^m - 1\right)$$

$$\delta y = x^m \left(1 + \frac{m\delta x}{x} + \frac{m(m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right)$$

$$\delta y = x^m \left(\frac{m\delta x}{x} + \frac{m(m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right)$$

$$\delta y = x^m \delta x \left(\frac{m}{x} + \frac{m(m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^m \left(\frac{m}{x} + \frac{m(m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^m \left(\frac{m}{x} + \frac{m(m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = x^m \left(\frac{m}{x} + \frac{m(m-1)}{2!} \frac{(0)}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = mx^{m-1}$$

Which is the required.

(xii). $\frac{1}{x^m}$, $m \in N$

Solution.

$$\text{Let } y = x^{-m} \rightarrow (i)$$

$$y + \delta y = (x + \delta x)^{-m} \rightarrow (ii)$$

eq(ii) - eq(i)

$$\delta y = (x + \delta x)^{-m} - y$$

$$\delta y = (x + \delta x)^{-m} - x^{-m}$$

$$\delta y = x^{-m} \left(1 + \frac{\delta x}{x}\right)^{-m} - x^{-m}$$

$$\delta y = x^{-m} \left(\left(1 + \frac{\delta x}{x}\right)^{-m} - 1\right)$$

$$\delta y = x^{-m} \left(1 + \frac{-m\delta x}{x} + \frac{-m(-m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right)$$

$$\delta y = x^{-m} \left(\frac{-m\delta x}{x} + \frac{-m(-m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right)$$

$$\delta y = x^{-m} \delta x \left(\frac{-m}{x} + \frac{-m(-m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{-m} \left(\frac{-m}{x} + \frac{-m(-m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{-m} \left(\frac{-m}{x} + \frac{-m(-m-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = x^{-m} \left(\frac{-m}{x} + \frac{-m(-m-1)}{2!} \frac{(0)}{x^2} + \dots\right)$$

$$\frac{dy}{dx} = -\frac{m}{x^{1+m}}$$

$$\frac{dy}{dx} = -\frac{m}{x^{m+1}}$$

Which is the required.

(xiii). x^{40}

Solution.

Let $y = x^{40} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{40} \rightarrow (ii)$

$eq(ii) - eq(i)$

$\delta y = (x + \delta x)^{40} - y$

$\delta y = (x + \delta x)^{40} - x^{40}$

$\delta y = x^{40} \left(1 + \frac{\delta x}{x}\right)^{40} - x^{40}$

$\delta y = x^{40} \left(\left(1 + \frac{\delta x}{x}\right)^{40} - 1\right)$

$\delta y = x^{40} \left(1 + \frac{40\delta x}{x} + \frac{40(40-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right)$

$\delta y = x^{40} \left(\frac{40\delta x}{x} + \frac{40(40-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right)$

$\delta y = x^{40} \delta x \left(\frac{40}{x} + \frac{40(40-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = x^{40} \left(\frac{40}{x} + \frac{40(40-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{40} \left(\frac{40}{x} + \frac{40(40-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

$\frac{dy}{dx} = x^{40} \left(\frac{40}{x} + \frac{40(40-1)}{2!} \frac{(0)}{x^2} + \dots\right)$

$\frac{dy}{dx} = 40x^{40-1}$

$\frac{dy}{dx} = 40x^{39}$

Which is the required.

(xiv). x^{-100}

Solution.

Let $y = x^{-100} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{-100} \rightarrow (ii)$

$\delta y = (x + \delta x)^{-100} - y$

$\delta y = (x + \delta x)^{-100} - x^{-100}$

$\delta y = x^{-100} \left(1 + \frac{\delta x}{x}\right)^{-100} - x^{-100}$

$\delta y = x^{-100} \left(\left(1 + \frac{\delta x}{x}\right)^{-100} - 1\right)$

$\delta y = x^{-100} \left(1 + \frac{-100\delta x}{x} + \frac{-100(-100-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right)$

$\delta y = x^{-100} \left(\frac{-100\delta x}{x} + \frac{-100(-100-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right)$

$\delta y = x^{-100} \delta x \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = x^{-100} \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{-100} \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

$\delta y = x^{-100} \delta x \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Dividing both sides by δx , we have

$\frac{\delta y}{\delta x} = x^{-100} \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{-100} \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{\delta x}{x^2} + \dots\right)$

$\frac{dy}{dx} = x^{-100} \left(\frac{-100}{x} + \frac{-100(-100-1)}{2!} \frac{(0)}{x^2} + \dots\right)$

$\frac{dy}{dx} = -\frac{100}{x^{1+100}}$

$\frac{dy}{dx} = -\frac{100}{x^{101}}$

Which is the required.

Q2. Find $\frac{dy}{dx}$ from first principal if

(i) $\sqrt{x+2}$

Solution.

Let $y = (x+2)^{\frac{1}{2}} \rightarrow (i)$

$y + \delta y = (x + \delta x + 2)^{\frac{1}{2}} \rightarrow (ii)$

$\delta y = (x + \delta x + 2)^{\frac{1}{2}} - y$

$\delta y = (x + 2 + \delta x)^{\frac{1}{2}} - (x + 2)^{\frac{1}{2}}$

$\delta y = (x + 2)^{\frac{1}{2}} \left(1 + \frac{\delta x}{x + 2}\right)^{\frac{1}{2}} - (x + 2)^{\frac{1}{2}}$

$\delta y = (x + 2)^{\frac{1}{2}} \left(\left(1 + \frac{\delta x}{x + 2}\right)^{\frac{1}{2}} - 1\right)$

$\delta y = (x + 2)^{\frac{1}{2}} \left(1 + \frac{1}{2} \frac{\delta x}{x + 2} + \frac{1}{2} \left(\frac{1}{2} - 1\right) \frac{(\delta x)^2}{(x + 2)^2} + \dots - 1\right)$

$\delta y = (x + 2)^{\frac{1}{2}} \delta x \left(\frac{1}{2} \frac{1}{x + 2} + \frac{1}{2} \left(\frac{1}{2} - 1\right) \frac{\delta x}{(x + 2)^2} + \dots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = (x + 2)^{\frac{1}{2}} \left(\frac{1}{2} \frac{1}{x + 2} + \frac{1}{2} \left(\frac{1}{2} - 1\right) \frac{\delta x}{(x + 2)^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x + 2)^{\frac{1}{2}} \left(\frac{1}{2} \frac{1}{x + 2} + \frac{1}{2} \left(\frac{1}{2} - 1\right) \frac{\delta x}{(x + 2)^2} + \dots\right)$

$\frac{dy}{dx} = (x + 2)^{\frac{1}{2}} \left(\frac{1}{2} \frac{1}{x + 2} + \frac{1}{2} \left(\frac{1}{2} - 1\right) \frac{(0)}{(x + 2)^2} + \dots\right)$

$\frac{dy}{dx} = (x + 2)^{\frac{1}{2}} \frac{1}{2} \frac{1}{x + 2}$

$$\frac{dy}{dx} = \frac{1}{2(x+2)^{1-\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{2(x+2)^{\frac{1}{2}}}$$

Which is the required.

(ii). $\frac{1}{\sqrt{x+a}}$

Solution.

Let $y = (x + a)^{-\frac{1}{2}} \rightarrow (i)$

$y + \delta y = (x + \delta x + a)^{-\frac{1}{2}} \rightarrow (ii)$

$\delta y = (x + \delta x + a)^{-\frac{1}{2}} - y$

$\delta y = (x + a + \delta x)^{-\frac{1}{2}} - (x + a)^{-\frac{1}{2}}$

$\delta y = (x + a)^{-\frac{1}{2}} \left(1 + \frac{\delta x}{x + a}\right)^{-\frac{1}{2}} - (x + a)^{-\frac{1}{2}}$

$\delta y = (x + a)^{-\frac{1}{2}} \left(\left(1 + \frac{\delta x}{x + a}\right)^{-\frac{1}{2}} - 1\right)$

$\delta y = (x + a)^{-\frac{1}{2}} \left(1 - \frac{1}{2} \frac{\delta x}{x + a} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right)}{2!} \left(\frac{\delta x}{x + a}\right)^2 + \dots - 1\right)$

$\delta y = (x + a)^{-\frac{1}{2}} \delta x \left(-\frac{1}{2} \frac{1}{x + a} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right)}{2!} \frac{\delta x}{(x + a)^2} + \dots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = (x + a)^{-\frac{1}{2}} \left(-\frac{1}{2} \frac{1}{x + a} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right)}{2!} \frac{\delta x}{(x + a)^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x + a)^{-\frac{1}{2}} \left(-\frac{1}{2} \frac{1}{x + a} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right)}{2!} \frac{\delta x}{(x + a)^2} + \dots\right)$

$\frac{dy}{dx} = (x + a)^{-\frac{1}{2}} \left(-\frac{1}{2} \frac{1}{x + a} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1\right)}{2!} \frac{(0)}{(x + a)^2} + \dots\right)$

$\frac{dy}{dx} = (x + a)^{-\frac{1}{2}} \left(-\frac{1}{2} \frac{1}{x + a}\right)$

$\frac{dy}{dx} = -\frac{1}{2(x + a)^{1+\frac{1}{2}}}$

$\frac{dy}{dx} = -\frac{1}{2(x + a)^{\frac{3}{2}}}$

Which is the required.

Differentiation of expression of the types.

$(ax + b)^n$ and $\frac{1}{(ax+b)^n}$, $n = 1, 2, 3$

Exercise 2.2

Question # 1

Find from first principles, the derivatives of the following expansions w.r.t. their respective independent variables:

(i) $(ax + b)^3$

Solution.

Let $y = (ax + b)^3 \rightarrow (i)$

$y + \delta y = (a(x + \delta x) + b)^3 \rightarrow (ii)$

$eq(ii) - eq(i)$

$\delta y = (ax + a\delta x + b)^3 - y$

$\delta y = ((ax + b) + a\delta x)^3 - (ax + b)^3$

$\delta y = (ax + b)^3 \left(1 + \frac{a\delta x}{(ax + b)}\right)^3 - (ax + b)^3$

$\delta y = (ax + b)^3 \left(\left(1 + \frac{a\delta x}{(ax + b)}\right)^3 - 1\right)$

$\delta y = (ax + b)^3 \left(1 + \frac{3a\delta x}{(ax + b)} + \frac{3(3-1)}{2!} \left(\frac{a\delta x}{(ax + b)}\right)^2 + \dots - 1\right)$

$\delta y = ax + b^3 \delta x \left(\frac{3a}{(ax + b)} + \frac{3(3-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots\right)$

both sides by δx

$\frac{\delta y}{\delta x} = (ax + b)^3 \left(\frac{3a}{(ax + b)} + \frac{3(3-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (ax + b)^3 \left(\frac{3a}{(ax + b)} + \frac{3(3-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots\right)$

$\frac{dy}{dx} = (ax + b)^3 \left(\frac{3a}{(ax + b)} + \frac{3(3-1)}{2!} \frac{a^2(0)}{(ax + b)^2} + \dots\right)$

$\frac{dy}{dx} = (ax + b)^3 \left(\frac{3a}{(ax + b)}\right)$

$\frac{dy}{dx} = 3a(ax + b)^{3-1}$

$\frac{dy}{dx} = 3a(ax + b)^2$

Which is the required.

(ii) $(2x + 3)^5$

Solution.

Let $y = (2x + 3)^5 \rightarrow (i)$

$y + \delta y = (2(x + \delta x) + 3)^5 \rightarrow (ii)$

$eq(ii) - eq(i)$

$\delta y = (2x + 2\delta x + 3)^5 - y$

$\delta y = ((2x + 3) + 2\delta x)^5 - (2x + 3)^5$

$\delta y = (2x + 3)^5 \left(1 + \frac{2\delta x}{(2x + 3)}\right)^5 - (2x + 3)^5$

$\delta y = (2x + 3)^5 \left(\left(1 + \frac{2\delta x}{(2x + 3)}\right)^5 - 1\right)$

$\delta y = (2x + 3)^5 \left(1 + \frac{5(2)\delta x}{(2x + 3)} + \frac{5(5-1)}{2!} \left(\frac{2\delta x}{(2x + 3)}\right)^2 + \dots - 1\right)$

$$\delta y = (2x + 3)^5 \delta x \left(\frac{5(2)}{(2x+3)} + \frac{5(5-1)}{2!} \frac{2^2 \delta x}{(2x+3)^2} + \dots \right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = (2x + 3)^5 \left(\frac{5(2)}{(2x+3)} + \frac{5(5-1)}{2!} \frac{2^2 \delta x}{(2x+3)^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (2x + 3)^5 \left(\frac{5(2)}{(2x+3)} + \frac{5(5-1)}{2!} \frac{2^2 \delta x}{(2x+3)^2} + \dots \right)$$

$$\frac{dy}{dx} = (2x + 3)^5 \left(\frac{5(2)}{(2x+3)} + \frac{5(5-1)}{2!} \frac{2^2(0)}{(2x+3)^2} + \dots \right)$$

$$\frac{dy}{dx} = (2x + 3)^5 \left(\frac{5(2)}{(2x+3)} \right)$$

$$\frac{dy}{dx} = 10(2x + 3)^{5-1}$$

$$\frac{dy}{dx} = 10(2x + 3)^4$$

Which is the required.

$$(iii) \quad (3t + 2)^{-2}$$

Solution.

$$\text{Let } y = (3t + 2)^{-2} \rightarrow (i)$$

$$y + \delta y = (3(t + \delta t) + 2)^{-2} \rightarrow (ii)$$

$$eq(ii) - eq(i)$$

$$\delta y = (3t + 3\delta t + 2)^{-2} - y$$

$$\delta y = ((3t + 2) + 3\delta t)^{-2} - (3t + 2)^{-2}$$

$$\delta y = (3t + 2)^{-2} \left(1 + \frac{3\delta t}{(3t + 2)} \right)^{-2} - (3t + 2)^{-2}$$

$$\delta y = (3t + 2)^{-2} \left(\left(1 + \frac{3\delta t}{(3t + 2)} \right)^{-2} - 1 \right)$$

$$\delta y = (3t + 2)^{-2} \left(1 + \frac{-2(3)\delta t}{(3t + 2)} + \frac{-2(-2-1)}{2!} \left(\frac{3\delta t}{(3t + 2)} \right)^2 + \dots - 1 \right)$$

$$\delta y = (3t + 2)^{-2} \delta t \left(-\frac{2(3)}{(3t + 2)} + \frac{-2(-2-1)}{2!} \frac{3\delta t}{(3t + 2)^2} + \dots \right)$$

Dividing both sides by δt

$$\frac{\delta y}{\delta t} = (3t + 2)^{-2} \left(-\frac{2(3)}{(3t + 2)} + \frac{-2(-2-1)}{2!} \frac{3\delta t}{(3t + 2)^2} + \dots \right)$$

Taking limit when $\delta t \rightarrow 0$

$$\lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t} = \lim_{\delta t \rightarrow 0} (3t + 2)^{-2} \left(-\frac{2(3)}{(3t + 2)} + \frac{-2(-2-1)}{2!} \frac{3\delta t}{(3t + 2)^2} + \dots \right)$$

$$\frac{dy}{dt} = (3t + 2)^{-2} \left(-\frac{2(3)}{(3t + 2)} + \frac{-2(-2-1)}{2!} \frac{3(0)}{(3t + 2)^2} + \dots \right)$$

$$\frac{dy}{dt} = (3t + 2)^{-2} \left(-\frac{2(3)}{(3t + 2)} \right)$$

$$\frac{dy}{dt} = -\frac{6}{(3t + 2)^{1+2}}$$

$$\frac{dy}{dt} = -\frac{6}{(3t + 2)^3}$$

Which is the required.

$$(iv) \quad (ax + b)^{-5}$$

Solution.

$$\text{Let } y = (ax + b)^{-5} \rightarrow (i)$$

$$y + \delta y = (a(x + \delta x) + b)^{-5} \rightarrow (ii)$$

$$eq(ii) - eq(i)$$

$$\delta y = (ax + a\delta x + b)^{-5} - y$$

$$\delta y = ((ax + b) + a\delta x)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left(1 + \frac{a\delta x}{(ax + b)} \right)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left(\left(1 + \frac{a\delta x}{(ax + b)} \right)^{-5} - 1 \right)$$

$$\delta y = (ax + b)^{-5} \left(1 - \frac{5a\delta x}{(ax + b)} + \frac{-5(-5-1)}{2!} \left(\frac{a\delta x}{(ax + b)} \right)^2 + \dots - 1 \right)$$

$$\delta y = (ax + b)^{-5} \delta x \left(-\frac{5a}{(ax + b)} + \frac{-5(-5-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots \right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = (ax + b)^{-5} \left(-\frac{5a}{(ax + b)} + \frac{-5(-5-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (ax + b)^{-5} \left(-\frac{5a}{(ax + b)} + \frac{-5(-5-1)}{2!} \frac{a^2 \delta x}{(ax + b)^2} + \dots \right)$$

$$\frac{dy}{dx} = (ax + b)^{-5} \left(-\frac{5a}{(ax + b)} + \frac{-5(-5-1)}{2!} \frac{a^2(0)}{(ax + b)^2} + \dots \right)$$

$$\frac{dy}{dx} = (ax + b)^{-5} \left(-\frac{5a}{(ax + b)} \right)$$

$$\frac{dy}{dx} = -\frac{5a}{(ax + b)^{1+5}}$$

$$\frac{dy}{dx} = -\frac{5a}{(ax + b)^6}$$

Which is the required.

$$(v) \quad \frac{1}{(az - b)^7}$$

Solution.

$$\text{Let } y = (az - b)^{-7} \rightarrow (i)$$

$$y + \delta y = (a(z + \delta z) - b)^{-7} \rightarrow (ii)$$

$$eq(ii) - eq(i)$$

$$\delta y = (az + a\delta z - b)^{-7} - y$$

$$\delta y = ((az - b) + a\delta z)^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left(1 + \frac{a\delta z}{(az - b)} \right)^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left(\left(1 + \frac{a\delta z}{(az - b)} \right)^{-7} - 1 \right)$$

$$\delta y = (az - b)^{-7} \left(1 - \frac{7a\delta z}{(az - b)} + \frac{-7(-7-1)}{2!} \left(\frac{a\delta z}{(az - b)} \right)^2 + \dots - 1 \right)$$

$$\delta y = (az - b)^{-7} \delta z \left(-\frac{7a}{(az - b)} + \frac{-7(-7-1)}{2!} \frac{a^2 \delta z}{(az - b)^2} + \dots \right)$$

Dividing both sides by δz

$$\frac{\delta y}{\delta z} = (az - b)^{-7} \left(-\frac{7a}{(az - b)} + \frac{-7(-7-1)}{2!} \frac{a^2 \delta z}{(az - b)^2} + \dots \right)$$

Taking limit when $\delta z \rightarrow 0$

$$\lim_{\delta z \rightarrow 0} \frac{\delta y}{\delta z} = \lim_{\delta z \rightarrow 0} (az - b)^{-7} \left(-\frac{7a}{(az - b)} + \frac{-7(-7-1)}{2!} \frac{a^2 \delta z}{(az - b)^2} + \dots \right)$$

$$\frac{dy}{dz} = (az - b)^{-7} \left(-\frac{7a}{(az - b)} + \frac{-7(-7-1)}{2!} \frac{a^2(0)}{(az - b)^2} + \dots \right)$$

$$\frac{dy}{dz} = (az - b)^{-7} \left(-\frac{7a}{(az - b)} \right)$$

$$\frac{dy}{dz} = -\frac{7a}{(az - b)^{1+7}}$$

$$\frac{dy}{dz} = -\frac{7a}{(az - b)^8}$$

Which is the required.

Theorems on Differentiation:

We have, so far proved the following two formulas;

1. $\frac{d}{dx}(c) = 0$

i.e derivative of constant function is zero.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$

power rule when constant function is zero.

3. Derivative of $y = cf(x)$ i.e $\frac{dy}{dx} = cf'(x)$

Proof:

$$y = cf(x) \rightarrow (i) \quad y + \delta y = cf(x + \delta x) \rightarrow (ii)$$

$$\text{by } (ii) - (i) \Rightarrow y + \delta y - y = cf(x + \delta x) - cf(x)$$

$$\Rightarrow \delta y = c(f(x + \delta x) - f(x))$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} [c(f(x + \delta x) - f(x))]$$

$$\Rightarrow \frac{dy}{dx} = cf'(x)$$

Reciprocal Rule:

If $f(x)$ is differentiable at x and $g(x) \neq 0$ then $\frac{1}{g(x)}$ is differentiable at x and $\frac{d}{dx} \left(\frac{1}{g(x)} \right) = -\frac{g'(x)}{[g(x)]^2}$

Exercise 2.3`

Differentiate w. r. t "x"

Question # 1. $x^4 + 2x^3 + x^2$

Solution.

Let $y = x^4 + 2x^3 + x^2$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 2x^3 + x^2)$$

$$\frac{dy}{dx} = \frac{d}{dx}x^4 + 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2$$

$$\frac{dy}{dx} = 4x^{4-1} + 2.3x^{3-1} + 2x^{2-1}$$

$$\frac{dy}{dx} = 4x^3 + 6x^2 + 2x$$

Which is required.

Question # 2. $x^{-3} + 2x^{-\frac{3}{2}} + 3$

Solution.

Let $y = x^{-3} + 2x^{-\frac{3}{2}} + 3$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^{-3} + 2x^{-\frac{3}{2}} + 3 \right)$$

$$\frac{dy}{dx} = \frac{d}{dx}x^{-3} + 2\frac{d}{dx}x^{-\frac{3}{2}} + \frac{d}{dx}(3)$$

$$\frac{dy}{dx} = -3x^{-3-1} - 2 \cdot \frac{3}{2}x^{-\frac{3}{2}-1} + 0$$

$$\frac{dy}{dx} = -3x^{-4} - 3x^{-\frac{5}{2}}$$

$$\frac{dy}{dx} = -3 \left(\frac{1}{x^4} + \frac{1}{x^{\frac{5}{2}}} \right)$$

Which is required.

Question#3. $x^{-3} + 2x^{-\frac{3}{2}} + 3$

Solution.

Let $y = \frac{a+x}{a-x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{a+x}{a-x} \right)$$

$$\frac{dy}{dx} = \frac{(a-x) \frac{d}{dx}(a+x) - (a+x) \frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a-x+a+x}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{2a}{(a-x)^2}$$

Which is required.

Question#4. $\frac{2x-3}{2x+1}$

Solution.

Let $y = \frac{2x-3}{2x+1}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-3}{2x+1} \right)$$

$$\frac{dy}{dx} = \frac{(2x+1) \frac{d}{dx}(2x-3) - (2x-3) \frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2(2x+1-2x+3)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2(4)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{8}{(2x+1)^2}$$

Which is required.

Question#5. $(x-5)(3-x)$

Solution.

Let $y = (x-5)(3-x)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}((x-5)(3-x))$$

$$\frac{dy}{dx} = \frac{d}{dx}(3x - x^2 - 15 + 5x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(-x^2 + 8x - 15)$$

$$\frac{dy}{dx} = -\frac{d}{dx}x^2 + 8\frac{d}{dx}x - \frac{d}{dx}(15)$$

$$\frac{dy}{dx} = -\frac{d}{dx}x^2 + 8\frac{d}{dx}x - \frac{d}{dx}(15)$$

$$\frac{dy}{dx} = -2x + 8 + 0$$

$$\frac{dy}{dx} = -2x + 8$$

Which is required.

Question#5. $(\sqrt{x} - \frac{1}{\sqrt{x}})^2$

Solution.

$$\text{Let } y = (\sqrt{x} - \frac{1}{\sqrt{x}})^2$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2$$

$$\frac{dy}{dx} = \frac{d}{dx} \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}} \right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}} \right) \right)$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(x + \frac{1}{x} - 2 \right)$$

$$\frac{dy}{dx} = \frac{d}{dx} (x + x^{-1} - 2)$$

$$\frac{dy}{dx} = \frac{d}{dx}x + \frac{d}{dx}x^{-1} - \frac{d}{dx}(2)$$

$$\frac{dy}{dx} = 1 - x^{-2} - 0$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$$

Which is required.

Question#7. $\frac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$

Solution.

$$\text{Let } y = \frac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$$

$$y = \frac{(1+\sqrt{x})x(1-x^{\frac{1}{2}})}{\sqrt{x}}$$

$$y = \frac{x(1-(\sqrt{x})^2)}{\sqrt{x}}$$

$$y = \sqrt{x}(1-x)$$

$$y = \sqrt{x} - x\sqrt{x}$$

$$y = x^{\frac{1}{2}} - x^{\frac{3}{2}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (x^{\frac{1}{2}} - x^{\frac{3}{2}})$$

$$\frac{dy}{dx} = \frac{d}{dx}x^{\frac{1}{2}} - \frac{d}{dx}x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 3\sqrt{x} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1-3x}{\sqrt{x}} \right)$$

Which is required.

Question#8. $\frac{(x^2+1)^2}{x^2-1}$

Solution.

$$\text{Let } y = \frac{(x^2+1)^2}{x^2-1}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{(x^2+1)^2}{x^2-1} \right)$$

$$\frac{dy}{dx}$$

$$\frac{dy}{dx}$$

$$= \frac{(x^2-1)\frac{d}{dx}((x^2+1)^2) - (x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2(x^2+1)^1)\frac{d}{dx}(x^2+1) - (x^2+1)^2(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2(x^2+1)^1)(2x) - (x^2+1)^2(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1)(2(x^2-1) - x^2 - 1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1)(2x^2 - 2 - x^2 - 1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$$

Which is required.

Question # 9. $\frac{x^2+1}{x^2-3}$

$$\text{Let } y = \frac{x^2+1}{x^2-3}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-3} \right)$$

$$\frac{dy}{dx} = \frac{(x^2-3)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-3)(2x) - (x^2+1)(2x)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x((x^2-3) - (x^2+1))}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2-3-x^2-1)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x(-4)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{-8x}{(x^2-3)^2}$$

Which is required.

Question#10. $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

Solution.

$$\text{Let } y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \sqrt{\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+x}{1-x} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1-x) \frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1-x)}{(1-x)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \left(\frac{1-x+1+x}{(1-x)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \left(\frac{2}{(1-x)^2} \right)$$

$$\frac{dy}{dx} = \left(\frac{1}{(1-x)^{2-\frac{1}{2}}(1+x)^{\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \left(\frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}} \right)$$

Which is required.

Question # 11. $\frac{2x-1}{\sqrt{x^2+1}}$

Solution.

$$\text{Let } y = \frac{2x-1}{\sqrt{x^2+1}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-1}{\sqrt{x^2+1}} \right)$$

$$\frac{dy}{dx} = \left(\frac{(\sqrt{x^2+1}) \frac{d}{dx}(2x-1) - (2x-1) \frac{d}{dx}(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2} \right)$$

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+1})(2) - (2x-1) \frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(2) - (2x-1)(x)}{(x^2+1)(x^2+1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{2x^2+2-2x^2+x}{(x^2+1)^{1+\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{x+2}{(x^2+1)^{\frac{3}{2}}}$$

Which is required.

Question # 12. $\frac{\sqrt{a-x}}{\sqrt{a+x}}$

Solution.

$$\text{Let } y = \frac{\sqrt{a-x}}{\sqrt{a+x}} = \sqrt{\frac{a-x}{a+x}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{a-x}{a+x}} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{a-x}{a+x} \right)$$

$$= \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \left(\frac{(a+x) \frac{d}{dx}(a-x) - (a-x) \frac{d}{dx}(a+x)}{(a+x)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \left(\frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \left(\frac{-a-x-a+x}{(a+x)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{-2a}{(a+x)^{2-\frac{1}{2}}(a-x)^{\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \left(\frac{-a}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{1}{2}}} \right)$$

Which is required.

Question # 13. $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution.

$$\text{Let } y = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = \sqrt{\frac{x^2+1}{x^2-1}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{x^2+1}{x^2-1}} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2+1}{x^2-1} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2-1}{x^2+1} \right)^{\frac{1}{2}} \left(\frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2-1}{x^2+1} \right)^{\frac{1}{2}} \left(\frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2-1}{x^2+1} \right)^{\frac{1}{2}} \left(\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2-1}{x^2+1} \right)^{\frac{1}{2}} \left(\frac{2x(-2)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \left(\frac{x(-2)}{(x^2-1)^{2-\frac{1}{2}}(x^2+1)^{\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \frac{-2x}{(x^2-1)^{\frac{3}{2}}\sqrt{x^2+1}}$$

Which is required.

Question # 14. $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Solution.

$$\text{Let } y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \cdot \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$y = \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}$$

$$y = \frac{(\sqrt{1+x})^2 + (\sqrt{1-x})^2 - 2(\sqrt{1+x})(\sqrt{1-x})}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}$$

$$y = \frac{1+x+1-x-2\sqrt{1-x^2}}{1+x-1+x}$$

$$y = \frac{2-2\sqrt{1-x^2}}{2x}$$

$$y = \frac{1-\sqrt{1-x^2}}{x}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1-\sqrt{1-x^2}}{x} \right)$$

$$\frac{dy}{dx} = \frac{x \frac{d}{dx} (1-\sqrt{1-x^2}) - (1-\sqrt{1-x^2}) \frac{d}{dx} x}{x^2}$$

$$\frac{dy}{dx} = \frac{x \left(-\frac{-2x}{2\sqrt{1-x^2}} \right) - (1-\sqrt{1-x^2})}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 - \sqrt{1-x^2} + 1 - x^2}{x^2 \sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1-\sqrt{1-x^2}}{x^2 \sqrt{1-x^2}}$$

Which is required.

Question # 15. $\frac{x\sqrt{a+x}}{\sqrt{a-x}}$

Solution.

$$\text{Let } y = \frac{x\sqrt{a+x}}{\sqrt{a-x}} = x \sqrt{\frac{a+x}{a-x}}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(x \sqrt{\frac{a+x}{a-x}} \right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} \frac{d}{dx} (x) + x \frac{d}{dx} \sqrt{\frac{a+x}{a-x}}$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + x \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{a+x}{a-x} \right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}}$$

$$+ x \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}} \left(\frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2} \right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + x \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}} \left(\frac{a-x+a+x}{(a-x)^2} \right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + x \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}} \left(\frac{2a}{(a-x)^2} \right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + \left(\frac{ax}{(a-x)^{\frac{3}{2}} (a+x)^{\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \frac{(a+x)(a-x) + ax}{(a-x)^{\frac{3}{2}} (a+x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{a^2 - x^2 + ax}{(a-x)^{\frac{3}{2}} (a+x)^{\frac{1}{2}}}$$

Which is required.

Question # 16. If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, Show that $2x \frac{dy}{dx} +$

$$y = 2\sqrt{x}.$$

Solution.

$$\text{Since } y = \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$y = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

Differentiating w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}$$

Multiplying by 2x both sides

$$2x \frac{dy}{dx} = 2x \left(\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}} \right)$$

$$2x \frac{dy}{dx} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

Adding u on both sides, we have

$$2x \frac{dy}{dx} + y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + y$$

$$2x \frac{dy}{dx} + y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$2x \frac{dy}{dx} + y = 2x^{\frac{1}{2}}$$

$$2x \frac{dy}{dx} + y = 2\sqrt{x}$$

Hence Proved.

Question # 17. If $y = x^4 + 2x^2 +$

2 then prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$.

Solution.

$$\text{Since } y = x^4 + 2x^2 + 2$$

Differentiate w.r.t. "x"

$$\frac{dy}{dx} = 4x^3 + 4x$$

$$\frac{dy}{dx} = 4x(x^2 + 1)$$

$$\frac{dy}{dx} = 4x\sqrt{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = 4x\sqrt{x^4 + 2x^2 + 1}$$

$$\frac{dy}{dx} = 4x\sqrt{x^4 + 2x^2 + 2 - 1}$$

$$\frac{dy}{dx} = 4x\sqrt{y-1} \quad \because y = x^4 + 2x^2 + 1$$

Hence Proved. Which is require

The Chain Rule:

if $y = (f \circ g)(x)$ or $y = f(g(x))$ let $u = g(x)$ then $y = f(u)$ and u

$= g(x)$ so we find $\frac{dy}{du}$ and $\frac{du}{dx}$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ is called chain rule.}$$

Derivative of a function given in the form of parametric equation.

Differentiation of implicit Relations:

Exercise 2.4

Question # 1.

Find $\frac{dy}{dx}$ by making the suitable substitution in the following functions defined as:

Differentiate the following w.r.t "x".

$$(i) \quad y = \sqrt{\frac{1-x}{1+x}}$$

Solution.

$$\text{Given } y = \sqrt{\frac{1-x}{1+x}}$$

$$\text{Put } u = \frac{1-x}{1+x}$$

$$\text{So } y = \sqrt{u}$$

$$y = u^{\frac{1}{2}}$$

Differentiate "u" w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{1-x}{1+x} \right)$$

$$\frac{du}{dx} = \frac{(1+x) \frac{d}{dx}(1-x) - (1-x) \frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du} u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} \cdot \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}(1+x)^{\frac{3}{2}}}$$

Which is required.

$$(ii) \quad y = \sqrt{x + \sqrt{x}}$$

Solution.

$$\text{Given } y = \sqrt{x + \sqrt{x}}$$

$$\text{Put } u = x + \sqrt{x}$$

$$\text{So } y = \sqrt{u}$$

$$y = u^{\frac{1}{2}}$$

Differentiate "u" w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} (x + \sqrt{x})$$

$$\frac{du}{dx} = 1 + \frac{1}{2} x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du} u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} (x + \sqrt{x})^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x + \sqrt{x}}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x} + 1}{4\sqrt{x} \cdot \sqrt{x + \sqrt{x}}}$$

Which is required.

$$(iii) \quad y = x \sqrt{\frac{a+x}{a-x}}$$

Solution.

$$\text{Given } y = x \sqrt{\frac{a+x}{a-x}}$$

$$\text{Put } u = \frac{a+x}{a-x}$$

$$\text{So } y = x \sqrt{u}$$

$$y = x \cdot u^{\frac{1}{2}}$$

Differentiate "u" w.r.t. "x"

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{a+x}{a-x} \right)$$

$$= \frac{(a-x) \frac{d}{dx}(a+x) - (a+x) \frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{a-x+a+x}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{2a}{(a-x)^2}$$

Now differentiate y w.r.t. x

$$\frac{dy}{du} = \frac{d}{du} (xu^{\frac{1}{2}})$$

$$\frac{dy}{du} = x \cdot \frac{d}{du} (u^{\frac{1}{2}}) + u^{\frac{1}{2}} \frac{d}{du} (x)$$

$$\frac{dy}{du} = x \cdot \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{dx} + u^{\frac{1}{2}} (1)$$

$$\frac{dy}{du} = x \cdot \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{dx} + u^{\frac{1}{2}}$$

Now Using Above values

$$\frac{dy}{dx} = x \cdot \frac{1}{2} \left(\frac{a+x}{a-x}\right)^{-\frac{1}{2}} \cdot \frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{ax(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}(a-x)^2} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^1(a-x)^{\frac{1}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{ax + (a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^1(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{ax + a^2 - x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

Which is required.

(iv) $y = (3x^2 - 2x + 7)^6$

Solution.

Given $y = (3x^2 - 2x + 7)^6$

Put $u = 3x^2 - 2x + 7$

So $y = u^6$

Differentiate "u" w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}(3x^2 - 2x + 7)$$

$$\frac{du}{dx} = 3(2x) - 2(1) + 0$$

$$\frac{du}{dx} = 6x - 2$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du} u^6$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{du} = 6(3x^2 - 2x + 7)^5$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 6(3x^2 - 2x + 7)^5 \cdot (6x - 2)$$

$$\frac{dy}{dx} = 12(3x - 1)(3x^2 - 2x + 7)^5$$

Which is required.

(v) $y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$

Solution.

Given $y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$

Put $u = \frac{a^2+x^2}{a^2-x^2}$

So $y = \sqrt{u}$

$$y = u^{\frac{1}{2}}$$

Differentiate "u" w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{a^2+x^2}{a^2-x^2} \right)$$

$$\frac{du}{dx}$$

$$= \frac{(a^2-x^2) \frac{d}{dx}(a^2+x^2) - (a^2+x^2) \frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{(a^2-x^2)(2x) - (a^2+x^2)(-2x)}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(a^2-x^2+a^2+x^2)}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{2a^2x}{(a^2-x^2)^2}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du} u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{a^2+x^2}{a^2-x^2} \right)^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{a^2-x^2}{a^2+x^2} \right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2-x^2}{a^2+x^2} \right)^{\frac{1}{2}} \cdot \frac{2a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{dx} = \frac{a^2x}{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{a^2x}{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{3}{2}}}$$

Which is required.

Question # 2 Find $\frac{dy}{dx}$ if :

(i) $3x + 4y + 7 = 0$

Solution:

Given $3x + 4y + 7 = 0$

Differentiate w.r.t. "x"

$$\frac{d}{dx}(3x + 4y + 7) = \frac{d}{dx}(0)$$

$$3 + 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3}{4}$$

Which is required.

(ii) $xy + y^2 = 2$

Solution:

Given $xy + y^2 = 2$

Differentiate w.r.t. "x"

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$y \cdot (1) + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$$

$$(x + 2y) \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{(x + 2y)}$$

Which is required.

(iii) $x^2 - 4xy - 5y = 0$

Solution:

Given $x^2 - 4xy - 5y = 0$

Differentiate w.r.t. "x"

$$\frac{d}{dx}(x^2 - 4xy - 5y) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) - 4 \frac{d}{dx}(xy) - 5 \frac{d}{dx}(y) = 0$$

$$2x - 4 \left(y \cdot (1) + x \cdot \frac{dy}{dx} \right) - 5 \frac{dy}{dx} = 0$$

$$2x - 4y - 4x \frac{dy}{dx} - 5 \frac{dy}{dx} = 0$$

$$-(4x + 5) \frac{dy}{dx} = -(2x - 4y)$$

$$\frac{dy}{dx} = \frac{2x - 4y}{4x + 5}$$

Which is required.

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution:

Given $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiate w.r.t. "x"

$$\frac{d}{dx}(4x^2 + 2hxy + by^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(4x^2) + 2h \frac{d}{dx}(xy) + b \frac{d}{dx}(y^2) + 2g \frac{d}{dx}(x) + 2f \frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

$$4(2x) + 2h \left(y \cdot (1) + x \cdot \frac{dy}{dx} \right) + 2by \frac{dy}{dx} + 2g(1) + 2f \frac{dy}{dx} + 0 = 0$$

$$8x + 2hy + 2hx \cdot \frac{dy}{dx} + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$2(hx + by + f) \frac{dy}{dx} = -2(4x + hy + g)$$

$$\frac{dy}{dx} = -\frac{4x + hy + g}{(hx + by + f)}$$

Which is required.

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

Solution:

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

Differentiate w.r.t. "x"

$$\frac{d}{dx}(x\sqrt{1+y} + y\sqrt{1+x}) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x\sqrt{1+y}) + \frac{d}{dx}(y\sqrt{1+x}) = 0$$

$$\sqrt{1+y} \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sqrt{1+y}) + y \cdot \frac{d}{dx}(\sqrt{1+x}) + \sqrt{1+x} \cdot \frac{d}{dx}(y) = 0$$

$$\sqrt{1+y} \cdot (1) + \frac{1}{2}x(1+y)^{-\frac{1}{2}} \frac{dy}{dx} + y \cdot \frac{1}{2}(1+x)^{-\frac{1}{2}}(1) + \sqrt{1+x} \frac{dy}{dx} = 0$$

$$\sqrt{1+y} + \frac{1}{2} \frac{x}{\sqrt{1+y}} \frac{dy}{dx} + \frac{y}{2\sqrt{1+x}} + \sqrt{1+x} \frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x} \right) \frac{dy}{dx} = - \left(\frac{y}{2\sqrt{1+x}} + \sqrt{1+y} \right)$$

$$\left(\frac{x + 2\sqrt{1+y}\sqrt{1+x}}{2\sqrt{1+y}} \right) \frac{dy}{dx} = - \left(\frac{y + 2\sqrt{1+y}\sqrt{1+x}}{2\sqrt{1+x}} \right)$$

$$\frac{dy}{dx} = - \left(\frac{\sqrt{1+y}(y + 2\sqrt{1+y}\sqrt{1+x})}{\sqrt{1+x}(x + 2\sqrt{1+y}\sqrt{1+x})} \right)$$

Which is required.

(vi) $y(x^2 - 1) = x\sqrt{x^2 + 4} = 0$

Solution:

Given $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Differentiate w.r.t. "x"

$$\frac{d}{dx}y(x^2 - 1) = \frac{d}{dx}(x\sqrt{x^2 + 4})$$

$$(x^2 - 1) \frac{d}{dx}(y) + y \frac{d}{dx}(x^2 - 1) = x \frac{d}{dx}(\sqrt{x^2 + 4}) + \sqrt{x^2 + 4} \frac{d}{dx}(x)$$

$$(x^2 - 1) \frac{dy}{dx} + y(2x) = x \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x) + \sqrt{x^2 + 4}(1)$$

$$(x^2 - 1) \frac{dy}{dx} + 2xy = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4}$$

$$(x^2 - 1) \frac{dy}{dx} = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4} - 2xy$$

$$(x^2 - 1) \frac{dy}{dx} = \frac{x^2 + x^2 + 4 - 2xy\sqrt{x^2 + 4}}{\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2xy\sqrt{x^2 + 4}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

Which is required.

Question # 3 Find $\frac{dy}{dx}$ of the following parametric functions:

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$

Solution.

Since $x = \theta + \frac{1}{\theta}$

$$x = \theta + \theta^{-1}$$

Differentiate "x" w.r.t. θ , We have

$$\frac{dx}{d\theta} = 1 + (-1 \cdot \theta^{-2})$$

$$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2 - 1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2 - 1}{\theta^2}$$

Now $y = \theta + 1$

Differentiate "y" w.r.t. θ , We have

$$\frac{dy}{d\theta} = 1 + 0$$

$$\frac{dy}{d\theta} = 1$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}$$

Which is required.

(ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Solution.

Since $x = \frac{a(1-t^2)}{1+t^2}$

Differentiate "x" w.r.t. t , We have

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{a(1-t^2)}{1+t^2} \right)$$

$$\frac{dx}{dt} = a \frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = a \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = -2at \frac{1+t^2+1-t^2}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

$$\frac{dt}{dx} = \frac{-(1+t^2)^2}{4at}$$

Now $y = \frac{2bt}{1+t^2}$

Differentiate "y" w.r.t. t , We have

$$\frac{dy}{dt} = \frac{d}{dt} \left(\frac{2bt}{1+t^2} \right)$$

$$\frac{dy}{dt} = 2b \frac{(1+t^2) \frac{d}{dt}(t) - (t) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b \frac{(1+t^2)(1) - (t)(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b \frac{(1+t^2-2t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b \frac{(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2b \frac{(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{4at}$$

$$\frac{dy}{dx} = -\frac{b(1-t^2)}{2at}$$

Which is required.

Question # 4. Prove that $y \frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$.

Solution

Since $x = \frac{(1-t^2)}{1+t^2}$

Differentiate "x" w.r.t. t , We have

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{(1-t^2)}{1+t^2} \right)$$

$$\frac{dx}{dt} = \frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = -2t \frac{1+t^2+1-t^2}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$\frac{dt}{dx} = \frac{-(1+t^2)^2}{4t}$$

Now $y = \frac{2t}{1+t^2}$

Differentiate "y" w.r.t. t , We have

$$\frac{dy}{dt} = \frac{d}{dt} \left(\frac{2t}{1+t^2} \right)$$

$$\frac{dy}{dt} = 2 \frac{(1+t^2) \frac{d}{dt}(t) - (t) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2 \frac{(1+t^2)(1) - (t)(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2 \frac{(1+t^2-2t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2 \frac{(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2 \frac{(1-t^2)}{(1+t^2)^2} \cdot \frac{-(1+t^2)}{4t}$$

$$\frac{dy}{dx} = -\frac{(1-t^2)}{2t}$$

$$L.H.S = y \frac{dy}{dx} + x$$

$$L.H.S = \frac{2t}{1+t^2} \cdot \frac{(1-t^2)}{2t} + \frac{(1-t^2)}{1+t^2}$$

$$L.H.S = -\frac{(1-t^2)}{1+t^2} + \frac{(1-t^2)}{1+t^2}$$

$$L.H.S = 0$$

$$L.H.S = R.H.S$$

Hence Proved.

Question # 5. Differentiate

(i) $x^2 - \frac{1}{x^2}$ w.r.t. x^4

Solution.

Suppose $y = x^2 - \frac{1}{x^2}$ and $u = x^4$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^2 - \frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{d}{dx} (x^2 - x^{-2})$$

$$\frac{dy}{dx} = 2x + 2x^{-3}$$

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

$$\frac{dy}{dx} = \frac{2(x^4 + 1)}{x^3}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} (x^4)$$

$$\frac{du}{dx} = 4x^3$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{dx}{du}}$$

$$\frac{dy}{du} = \frac{2(x^4 + 1)}{x^3} \cdot \frac{1}{4x^3}$$

$$\frac{dy}{du} = \frac{(x^4 + 1)}{2x^6}$$

Which is required.

(ii) $(1 + x^2)^n$ w.r.t. x^2

Solution.

Suppose $y = (1 + x^2)^n$ and $u = x^2$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx} ((1 + x^2)^n)$$

$$\frac{dy}{dx} = n(1 + x^2)^{n-1} (2x)$$

$$\frac{dy}{dx} = 2nx(1 + x^2)^{n-1}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} (x^2)$$

$$\frac{du}{dx} = 2x$$

Now by chain rule

$$\frac{dy}{du} = 2nx(1 + x^2)^{n-1} \cdot \frac{1}{2x}$$

$$\frac{dy}{du} = n(1 + x^2)^{n-1}$$

Which is required.

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

Solution.

Suppose $y = \frac{x^2+1}{x^2-1}$ and $u = \frac{x-1}{x+1}$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$\frac{dy}{dx} = \frac{(x^2-1) \frac{d}{dx} (x^2+1) - (x^2+1) \frac{d}{dx} (x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$\frac{d}{dx} = \frac{2x(-2)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{x-1}{x+1} \right)$$

$$\frac{du}{dx} = \frac{(x+1) \frac{d}{dx} (x-1) - (x-1) \frac{d}{dx} (x+1)}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{(x+1) - (x-1)}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{x+1-x+1}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{2}{(x+1)^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{dx}{du}}$$

$$\frac{dy}{du} = \frac{-4x}{(x^2-1)^2} \cdot \frac{1}{\frac{2}{(x+1)^2}}$$

$$\frac{dy}{du} = \frac{-2x(x+1)^2}{(x^2-1)^2}$$

$$\frac{dy}{du} = \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2}$$

$$\frac{dy}{du} = \frac{-2x}{(x-1)^2}$$

Which is required.

$$(iv) \frac{ax+b}{cx+d} \text{ w.r.t. } \frac{ax^2+b}{ax^2+d}$$

Solution.

$$\text{Suppose } y = \frac{ax+b}{cx+d} \text{ and } u = \frac{ax^2+b}{ax^2+d}$$

Differentiate "y" w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{ax+b}{cx+d} \right)$$

$$\frac{dy}{dx} = \frac{(cx+d) \frac{d}{dx}(ax+b) - (ax+b) \frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{acx + ad - acx - bc}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{ad - bc}{(cx+d)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{ax^2+b}{ax^2+d} \right)$$

$$\frac{du}{dx}$$

$$= \frac{(ax^2+d) \frac{d}{dx}(ax^2+b) - (ax^2+b) \frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$

$$\frac{du}{dx} = \frac{(ax^2+d)(2ax) - (ax^2+b)(2ax)}{(ax^2+d)^2}$$

$$\frac{du}{dx} = \frac{2ax(ax^2+d - ax^2 - b)}{(ax^2+d)^2}$$

$$\frac{du}{dx} = \frac{2ax(d-b)}{(ax^2+d)^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{du}{dx}}$$

$$\frac{dy}{du} = \frac{ad - bc}{(cx+d)^2} \cdot \frac{1}{\frac{2ax(d-b)}{(ax^2+d)^2}}$$

$$\frac{dy}{du} = \frac{(ad - bc)(ax^2+d)^2}{2ax(d-b)(cx+d)^2}$$

Which is required.

$$(v) \frac{x^2+1}{x^2-1} \text{ w.r.t. } x^3$$

Solution.

$$\text{Suppose } y = \frac{x^2+1}{x^2-1} \text{ and } u = x^3$$

Differentiate "y" w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$\frac{dy}{dx} = \frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = 2x \frac{x^2-1 - x^2-1}{(x^2-1)^2}$$

$$\frac{dy}{dx} = 2x \frac{-2}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}(x^3)$$

$$\frac{du}{dx} = 3x^2$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{du}{dx}}$$

$$\frac{dy}{du} = \frac{-4x}{(x^2-1)^2} \cdot \frac{1}{3x^2}$$

$$\frac{dy}{du} = \frac{-4}{3x(x^2-1)^2}$$

Which is required.

Derivative of Trigonometric function:

$$1. \frac{d}{dx}(\sin x) = \cos x$$

Proof: let $y = \sin x \rightarrow (i)$

$$\Rightarrow y + \delta y = \sin(x + \delta x) \rightarrow (ii)$$

$$\text{by } (ii) - (i) \Rightarrow y + \delta y - y = \sin(x + \delta x) - \sin x$$

$$(\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$\Rightarrow \delta y = 2 \cos \left(\frac{x+\delta x+x}{2} \right) \sin \frac{(x+\delta x-x)}{2}$$

$$\Rightarrow \delta y = 2 \cos \left(\frac{2x+\delta x}{2} \right) \sin \left(\frac{\delta x}{2} \right)$$

dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[2 \cos \left(x + \frac{\delta x}{2} \right) \cdot \frac{\left(\sin \left(\frac{\delta x}{2} \right) \right)}{\delta x} \right]$$

$$\Rightarrow \frac{dy}{dx} = 2 \lim_{\delta x \rightarrow 0} \left[\cos \left(x + \frac{\delta x}{2} \right) \cdot \frac{\left(\sin \left(\frac{\delta x}{2} \right) \right)}{2 \times \frac{\delta x}{2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos x \quad (\because) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Thus $\frac{d}{dx} \sin x = \cos x$

$$2. \frac{d}{dx}(\cos x) = -\sin x$$

Proof: let $y = \cos x \rightarrow (i)$

$$\Rightarrow y + \delta y = \cos(x + \delta x) \rightarrow (ii)$$

$$\text{by } (ii) - (i) \Rightarrow y + \delta y - y = \cos(x + \delta x) - \cos x$$

$$\Rightarrow \delta y = \cos(x + \delta x) - \cos x$$

$$(\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$\Rightarrow \delta y = -2 \sin \left(\frac{x+\delta x+x}{2} \right) \sin \frac{(x+\delta x-x)}{2}$$

$$\Rightarrow \delta y = -2 \sin \left(\frac{2x+\delta x}{2} \right) \sin \left(\frac{\delta x}{2} \right)$$

dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[-2 \sin \left(x + \frac{\delta x}{2} \right) \cdot \frac{\left(\sin \left(\frac{\delta x}{2} \right) \right)}{\delta x} \right]$$

$$\Rightarrow \frac{dy}{dx} = -2 \lim_{\delta x \rightarrow 0} \left[-2 \left(x + \frac{\delta x}{2} \right) \cdot \frac{\left(\sin \left(\frac{\delta x}{2} \right) \right)}{2 \times \frac{\delta x}{2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = -\sin x \quad (\because) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Thus $\frac{d}{dx} \cos x = -\sin x$

3. $\frac{d}{dx} (\tan x) = \sec^2 x$

Proof: let $y = \tan x \rightarrow (i)$

$$\Rightarrow y + \delta y = \tan(x + \delta x) \rightarrow (ii)$$

by (ii) - (i) $\Rightarrow y + \delta y - y = \tan(x + \delta x) - \tan x$

$$\Rightarrow \delta y = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$\Rightarrow \frac{\sin(x+\delta x-x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}$$

$$(\because \sin a \cos b - \cos a \sin b = \sin(a - b))$$

$$\Rightarrow \delta y = \frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x} = \frac{\sin \delta x}{\cos(x+\delta x)\cos x}$$

dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\cos(x + \delta x) \cdot \cos x} \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\sin \delta x}{\delta x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x \cdot \cos x} (1) = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x$$

Thus $\frac{d}{dx} \tan x = \sec^2 x$

4. $\frac{d}{dx} (\sec x) = \sec x \tan x$

Proof:

let $y = \sec x \rightarrow (i)$

$y + \delta x = \sec(x + \delta x) \rightarrow (ii)$

by (ii) - (i) $\Rightarrow y + \delta y - y = \sec(x + \delta x) - \sec x$

$$\Rightarrow \delta y = \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x} = \frac{\cos x - \cos(x + \delta x)}{\cos(x + \delta x) \cdot \cos x}$$

$$\left(\because \cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right)$$

$$= \frac{-2 \sin \left(\frac{x+x+\delta x}{2} \right) \sin \left(\frac{x-x-\delta x}{2} \right)}{\cos(x + \delta x) \cdot \cos x}$$

$$= \frac{-2}{\cos(x + \delta x) \cdot \cos x} \left[\sin \left(x + \frac{\delta x}{2} \right) \sin \left(-\frac{\delta x}{2} \right) \right]$$

$$= \frac{2}{\cos(x + \delta x) \cdot \cos x} \left[\sin \left(x + \frac{\delta x}{2} \right) \sin \left(\frac{\delta x}{2} \right) \right]$$

$$\because \sin(-\theta) = -\sin \theta$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{2}{\cos(x + \delta x) \cdot \cos x} \left[\lim_{\delta x \rightarrow 0} \sin \left(x + \frac{\delta x}{2} \right) \right]$$

$$\left(\frac{\delta x}{2} \right) \lim_{\delta x \rightarrow 0} \sin \left(\frac{\delta x}{2} \right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\cos x \cdot \cos x} (\sin x \cdot 1) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \sec x \tan x \quad \text{hence } \frac{d}{dx} (\sec x) = \sec x \tan x$$

5. $\frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Proof:

let $y = \operatorname{cosec} x \rightarrow (i)$

$y + \delta x = \operatorname{cosec}(x + \delta x) \rightarrow (ii)$

by (ii) - (i) $\Rightarrow y + \delta y - y = \operatorname{cosec}(x + \delta x) - \operatorname{cosec} x$

$$\Rightarrow \delta y = \frac{1}{\sin(x + \delta x)} - \frac{1}{\sin x} = \frac{\sin x - \sin(x + \delta x)}{\sin(x + \delta x) \cdot \sin x}$$

$$\left(\because \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right)$$

$$= \frac{2 \cos \left(\frac{x+x+\delta x}{2} \right) \sin \left(\frac{x-x-\delta x}{2} \right)}{\sin(x + \delta x) \cdot \sin x}$$

$$= \frac{1}{\sin(x + \delta x) \cdot \sin x} \left[2 \cos \left(x + \frac{\delta x}{2} \right) \sin \left(-\frac{\delta x}{2} \right) \right]$$

$$= \frac{1}{\cos(x + \delta x) \cdot \cos x} \left[\cos \left(x + \frac{\delta x}{2} \right) \sin \left(-\frac{\delta x}{2} \right) \right]$$

$$\because \sin(-\theta) = -\sin \theta$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{2}{\sin(x + \delta x) \cdot \sin x}$$

$$\left[\lim_{\delta x \rightarrow 0} 2 \cos \left(x + \frac{\delta x}{2} \right) \lim_{\delta x \rightarrow 0} \sin \left(\frac{\delta x}{2} \right) \right]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin x \cdot \sin x} (\cos x \cdot 1) = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

$$\Rightarrow \text{hence } \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

6. $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$

Proof: let $y = \cot x \rightarrow (i)$

$$\Rightarrow y + \delta y = \cot(x + \delta x) \rightarrow (ii)$$

by (ii) - (i) $\Rightarrow y + \delta y - y = \cot(x + \delta x) - \cot x$

$$\Rightarrow \delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{\cos(x+\delta x-x)\sin x - \sin(x+\delta x)\cos x}{\sin(x+\delta x)\sin x}$$

$$(\because \sin a \cos b - \cos a \sin b = \sin(a - b))$$

$$\Rightarrow \delta y = \frac{\sin(x-\delta x-x)}{\sin(x+\delta x)\sin x} = \frac{\sin(-\delta x)}{\sin(x+\delta x)\sin x}$$

dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-1}{\sin(x + \delta x) \cdot \sin x} \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\sin \delta x}{\delta x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x \cdot \sin x} (1) = \frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2 x$$

$$\Rightarrow \text{Thus } \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

Derivative of inverse trigonometric function:

1. $\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \quad x \in (-1, 1) \text{ or } -1 < x < 1$

Proof: let $y = \sin^{-1} x$

$$\Rightarrow \sin y = x \Rightarrow \frac{d}{dx} (\sin y) = \frac{d}{dx} (x)$$

$$\Rightarrow \cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \quad \because \cos^2 \theta = 1 - \sin^2 \theta$$

$$\Rightarrow \cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \because \sin y = x$$

Hence $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

2. $\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad x \in (-1, 1)$

Proof: let $y = \cos^{-1}x$

$$\Rightarrow \cos y = x \Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}} \quad \because \sin^2 \theta = 1 - \cos^2 \theta$$

$$\Rightarrow \sin \theta = \sqrt{1 - \cos^2 \theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad \because \cos y = x$$

$$\text{Hence } \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

Proof: let $y = \tan^{-1}x$

$$\tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} \quad \because 1 + \tan^2 \theta = \sec^2 \theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} \quad \because \tan y = x$$

$$\Rightarrow \text{Hence } \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$4. \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}, \quad x \in [-1, 1]',$$

$$[-1, 1] = (-\infty, -1) \cup$$

$$(1, +\infty)$$

Proof: let $y = \cot^{-1}x$

$$\cot y = x \Rightarrow \frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosec}^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{1+\cot^2 y} \quad \because \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{1+x^2} \quad \because \cot y = x$$

$$\Rightarrow \text{Hence } \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \quad x \in [-1, 1]$$

Proof: let $y = \sec^{-1}x \Rightarrow \sec y = x$

$$\Rightarrow \frac{d}{dx}(\sec y) = \frac{d}{dx}(x) \Rightarrow \sec y \tan y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} \quad \because 1 + \tan^2 \theta = \sec^2 \theta$$

$$\Rightarrow \tan^2 \theta = \sec^2 - 1$$

$$\Rightarrow \tan \theta = \sqrt{\sec^2 \theta - 1}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \quad \because \sec y = x$$

$$\text{hence } \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$6. \frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}, \quad x \in \mathbb{R}$$

Proof: let $y = \operatorname{cosec}^{-1}x \Rightarrow \operatorname{cosec} y = x$

$$\Rightarrow \frac{d}{dx}(\operatorname{cosec} y) = \frac{d}{dx}(x) \Rightarrow \operatorname{cosec} y \cot y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y} = \frac{-1}{\operatorname{cosec} y \sqrt{\operatorname{cosec}^2 y - 1}}$$

$$\because 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\Rightarrow \cot^2 \theta = \operatorname{cosec}^2 - 1$$

$$\Rightarrow \cot \theta = \sqrt{\operatorname{cosec}^2 \theta - 1}$$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \quad \because \operatorname{cosec} y = x$$

$$\text{hence } \frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

Exercise 2.5

QUESTION NO.1:

Differentiate the following trigonometric functions from the first principals.

- $\sin 2x$
- $\tan 3x$
- $\sin 2x + \cos 2x$
- $\cos x^2$
- $\tan^2 x$
- $\sqrt{\tan x}$
- $\cos \sqrt{x}$

Solution:

(i)

$$\text{Let } y = \sin 2x \rightarrow (1)$$

$$y + \delta y = \sin 2(x + \delta x) \rightarrow (2)$$

$$\text{Eq 2} - \text{eq 1}$$

$$y + \delta y - y = \sin 2(x + \delta x) - \sin 2x$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\text{using } \sin P - \sin Q = 2 \cos \left(\frac{P+Q}{2} \right) \sin \left(\frac{P-Q}{2} \right)$$

$$\delta y = 2 \cos \left(\frac{2x + 2\delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\delta x - 2x}{2} \right)$$

$$= 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \sin \left(\frac{2\delta x}{2} \right)$$

$$\delta y = 2 \cos(2x + \delta x) \sin \delta x$$

Dividing both side by δx

$$\frac{\delta y}{\delta x} = \frac{2 \cos(2x + \delta x) \sin \delta x}{\delta x}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = 2 \cos(2x + 0) \cdot 1$$

$$\frac{dy}{dx} = 2 \cos 2x$$

(ii) **let** $y = \tan 3x \rightarrow (1)$

$$y + \delta y = \tan 3(x + \delta x) \rightarrow (2)$$

$$\text{Eq 2} - \text{eq 1}$$

$$y + \delta y - y = \tan(3x + 3\delta x) - \tan 3x$$

$$\delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$= \frac{\sin(3x + 3\delta x) \cdot \cos 3x - \cos(3x + 3\delta x) \cdot \sin 3x}{\cos(3x + 3\delta x) \cdot \cos 3x}$$

$$\text{using } \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta)$$

$$= \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x) \cdot \cos 3x}$$

dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\cos(3x + 3\delta x) \cdot \cos 3x} \cdot \frac{\sin 3\delta x}{\delta x}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{3}{\cos(3x + 3\delta x) \cdot \cos 3x} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin 3\delta x}{\delta x}$$

$$\frac{dy}{dx} = \frac{3}{\cos(3x + 0)\cos 3x} \cdot 1$$

$$= \frac{3}{\cos 3x \cdot \cos 3x}$$

$$= \frac{3}{\cos^2 3x}$$

$$\frac{dy}{dx} = 3\sec^2 3x$$

(iii) Let $y = \sin 2x + \cos 2x \rightarrow (1)$

$$y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x) \rightarrow (2)$$

Eq2 - eq1

$$y + \delta y - y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - \sin 2x - \cos 2x$$

$$\delta y = [\sin(2x + 2\delta x) - \sin 2x] + [\cos(2x + 2\delta x) - \cos 2x]$$

$$= 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \cdot \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) - 2\sin\left(\frac{2x + 2\delta x + 2x}{2}\right) \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$= 2\cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin \delta x - 2\sin\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin \delta x$$

$$\delta y = [2\cos(2x + 2\delta x) - 2\sin(2x + 2\delta x)] \cdot \sin \delta x$$

dividing both sides by δx

$$\frac{\delta y}{\delta x} = [2\cos(2x + 2\delta x) - 2\sin(2x + 2\delta x)] \cdot \frac{\sin \delta x}{\delta x}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} [2\cos(2x + 2\delta x) - 2\sin(2x + 2\delta x)] \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$= \{2\cos 2x - 2\sin 2x\} \cdot 1$$

$$\frac{dy}{dx} = 2\cos 2x - 2\sin 2x$$

(iv) $\cos x^2$

Solution:

let $y = \cos$

Let $y = \cos x^2 \rightarrow (1)$

$$y + \delta y = \cos(x + \delta x)^2 \rightarrow (2)$$

Eq(2) - eq(1)

$$y + \delta y - y = \cos(x + \delta x)^2 - \cos x^2$$

$$\delta y = -2\sin\left(\frac{x^2 + \delta x^2 + 2x\delta x + x^2}{2}\right) \cdot \sin\left(\frac{x^2 + \delta x^2 + 2x\delta x - x^2}{2}\right)$$

$$\delta y = \frac{-2\sin(2x^2 + 2x\delta x + \delta x^2) \cdot \sin\left(\frac{\delta x(\delta x + 2x)}{2}\right)}{\delta x}$$

$$\frac{\delta y}{\delta x} = -2\sin(2x^2 + 2x\delta x$$

$$+ \delta x^2) \cdot \frac{\sin(\delta x(\delta x + 2x))}{\delta x \cdot \frac{\delta x + 2x}{2}} \cdot \frac{\delta x + 2x}{2}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} -2\sin(2x^2 + 2x\delta x + \delta x^2) \cdot \frac{\sin(\delta x(\delta x + 2x))}{\delta x \cdot \frac{\delta x + 2x}{2}} \cdot \frac{\delta x + 2x}{2}$$

$$\frac{dy}{dx} = -2\sin\left(\frac{2x^2}{2}\right) \cdot 1 \cdot \frac{2x}{2}$$

$$\frac{dy}{dx} = -2x\sin x^2$$

(v)

let $y = \tan^2 x \rightarrow (1)$

$$y + \delta y = \tan^2(x + \delta x) \rightarrow (2)$$

eq2 - eq1

$$y + \delta y - y = \tan^2(x + \delta x) - \tan^2 x$$

$$\delta y = \{\tan(x + \delta x) - \tan x\} \cdot \{\tan(x + \delta x) + \tan x\}$$

$$\delta y = \{\tan(x + \delta x) - \tan x\} \cdot \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}\right]$$

δy

$$= [\tan(x + \delta x)$$

$$- \tan x] \cdot \left[\frac{\sin(x + \delta x) \cdot \cos x - \cos(x + \delta x) \cdot \sin x}{\cos(x + \delta x) \cdot \cos x}\right]$$

$$\delta y = [\tan(x + \delta x) - \tan x] \cdot \left[\frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cdot \cos x}\right]$$

$$\delta y = \frac{[\tan(x + \delta x) - \tan x] \cdot \sin \delta x}{\cos(x + \delta x) \cdot \cos x}$$

dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{[\tan(x + \delta x) - \tan x] \cdot \sin \delta x}{\cos(x + \delta x) \cdot \cos x \cdot \delta x}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{[\tan(x + \delta x) - \tan x]}{\cos(x + \delta x) \cdot \cos x} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = \frac{\tan x + \tan x}{\cos x \cdot \cos x} \cdot 1$$

$$\frac{dy}{dx} = \frac{2\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} = 2\tan x \sec^2$$

(vi)

let $y = \sqrt{\tan x} \dots \dots \dots 1$

$$y + \delta y = \sqrt{\tan(x + \delta x)} \dots \dots \dots 2$$

eq2 - eq1

$$\delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$$

$$\delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$$

$$\times \frac{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{(\sqrt{\tan(x + \delta x)})^2 - (\sqrt{\tan x})^2}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{\tan(x + \delta x) - \tan x}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right]$$

$$\delta y = \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x)\cos x - \cos(x + \delta x) \cdot \sin x}{\cos(x + \delta x)\cos x} \right]$$

$$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x - x)}{\cos(x + \delta x)\cos x} \right]$$

$$\delta y = \frac{\sin(x + \delta x - x)}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x} \cdot \cos(x + \delta x)\cos x}$$

δy

$$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x} \cdot \cos(x + \delta x)\cos x} \cdot \sin \delta x$$

dividing both sides by δx and take limit on both sides

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x} \cdot \cos(x + \delta x)\cos x} \cdot \lim_{\delta x \rightarrow 0} \sin \delta x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan(x)} + \sqrt{\tan x \cdot \cos(x) \cos x}} \cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x \cos^2 x}}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

(vii)

$$\cos\sqrt{x}$$

Solution:

$$\text{Let } y = \cos\sqrt{x} \rightarrow (i)$$

$$\Rightarrow y + \delta y = \cos\sqrt{x + \delta x} \rightarrow (ii)$$

$$\text{by } (ii) - (i) \Rightarrow y + \delta y - y = \cos\sqrt{x + \delta x} - \cos\sqrt{x}$$

$$\Rightarrow \delta y = \cos\sqrt{x + \delta x} - \cos\sqrt{x}$$

$$\because \cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$= -2 \sin\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \sin\left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)$$

dividing by δx

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{-2 \sin\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \sin\left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{\delta x}$$

$$\because (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x}) = \delta x$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{-2 \sin\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \sin\left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{(\sqrt{x + \delta x} + \sqrt{x}) \times 2 \times \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}$$

Take limit $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-\sin\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \sin\left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{(\sqrt{x + \delta x} + \sqrt{x}) \times \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin\left(\frac{\sqrt{x} + \sqrt{x}}{2}\right)}{\sqrt{x} + \sqrt{x}} \quad (1)$$

$$= \frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question # 2.

Differentiate the following w.r.t the variable involved.

(i) $x^2 \sec 4x$

Solution.

$$\text{Let } y = x^2 \sec 4x$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (x^2 \sec 4x)$$

$$\frac{dy}{dx} = (x^2) \frac{d}{dx} (\sec 4x) + (\sec 4x) \frac{d}{dx} (x^2)$$

$$\frac{dy}{dx} = (x^2)(\sec 4x \tan 4x (4x)) + (\sec 4x)(2x)$$

$$\frac{dy}{dx} = 4x^2 (\sec 4x \tan 4x) + 2x \sec 4x$$

Which is required.

(ii) $y = \tan^3 \theta \cdot \sec^2 \theta$

Solution.

$$\text{Let } y = \tan^3 \theta \cdot \sec^2 \theta$$

Differentiate w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (\tan^3 \theta \cdot \sec^2 \theta)$$

$$\frac{dy}{d\theta} = (\sec^2 \theta) \frac{d}{d\theta} (\tan^3 \theta) + (\tan^3 \theta) \frac{d}{d\theta} (\sec^2 \theta)$$

$$\frac{dy}{d\theta} = (\sec^2 \theta)(3 \tan^2 \theta (\sec^2 \theta))$$

$$+ (\tan^3 \theta)(2 \sec \theta (\sec \theta \tan \theta))$$

$$\frac{dy}{d\theta} = (3 \sec^4 \theta \tan^2 \theta) + (2 \tan^4 \theta \sec^2 \theta)$$

$$\frac{dy}{d\theta} = \sec^2 \theta \tan^2 \theta (3 \sec^2 \theta \tan^2 \theta + 2 \tan^2 \theta \sec^2 \theta)$$

(iii) $y = (\sin 2\theta - \cos 3\theta)^2$

Solution.

$$\text{Let } y = (\sin 2\theta - \cos 3\theta)^2$$

Differentiate w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)^2$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta) \left(\frac{d}{d\theta}\right) (\sin 2\theta - \cos 3\theta)$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta) \left[\cos \theta \frac{d}{d\theta} (2\theta) - (-\sin 3\theta) \frac{d}{d\theta} (3\theta)\right]$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)(2 \cos 2\theta + 3 \sin 3\theta)$$

Which is required.

(iv) $y = \cos\sqrt{x} + \sqrt{\sin x}$

Solution.

$$\text{Let } y = \cos\sqrt{x} + \sqrt{\sin x}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (\cos\sqrt{x} + \sqrt{\sin x})$$

$$\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{\sin x}} (\cos x)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{-\sin\sqrt{x}}{\sqrt{x}} \frac{\cos x}{\sqrt{\sin x}} \right)$$

Which is required.

Question # 3. Find $\frac{dy}{dx}$ if:(i) $y = x \cos y$

Solution.

$$\text{Since } y = x \cos y$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (x \cos y)$$

$$\frac{dy}{dx} = (\cos y) \frac{d}{dx} (x) + (x) \frac{d}{dx} (\cos y)$$

$$\frac{dy}{dx} = \cos y (1) + x (-\sin y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \cos y - x \sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} (1 + x \sin y) = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}$$

Which is required.

(ii) $x = y \sin y$

Solution.

Since $x = y \sin y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx} (y \sin y)$$

$$1 = (\sin y) \frac{d}{dx} (y) + (y) \frac{d}{dx} (\sin y)$$

$$1 = (\sin y) \left(\frac{dx}{dx}\right) + (y)(\cos y) \left(\frac{dx}{dx}\right)$$

$$1 = \frac{dx}{dx} (\sin y + y \cos y)$$

$$\frac{dy}{dx} = \frac{1}{\sin y + y \cos y}$$

Question # 4. Find derivative w.r.t x

(i) $y = \cos \sqrt{\frac{1+x}{1+2x}}$

Solution.

Let $y = \cos \sqrt{\frac{1+x}{1+2x}}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\cos \sqrt{\frac{1+x}{1+2x}} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{d}{dx} \left(\sqrt{\frac{1+x}{1+2x}} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+x}{1+2x} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1+2x) \frac{d}{dx} (1+x) - (1+x) \frac{d}{dx} (1+2x)}{(1+2x)^2} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1+2x)(1) - (1+x)(2)}{(1+2x)^2} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{1+2x - 2 - 2x}{(1+2x)^2} \right)$$

$$\frac{dy}{dx} = \left(-\sin \sqrt{\frac{1+x}{1+2x}} \right) \frac{1}{2} \left(\frac{1}{1+x} \right)^{\frac{1}{2}} \left(\frac{-1}{(1+2x)^{2-\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \left(\frac{\sin \sqrt{\frac{1+x}{1+2x}}}{2(1+2x)^{\frac{3}{2}} \sqrt{1+x}} \right)$$

Which is required.

(i) $y = \sin \sqrt{\frac{1+2x}{1+x}}$

Solution.

Let $y = \sin \sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin \sqrt{\frac{1+2x}{1+x}} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{d}{dx} \left(\sqrt{\frac{1+2x}{1+x}} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+2x}{1+x} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}}$$

$$\left(\frac{(1+x) \frac{d}{dx} (1+2x) - (1+2x) \frac{d}{dx} (1+x)}{(1+x)^2} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1+x)(2) - (1+2x)(1)}{(1+x)^2} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{2+2x-1-2x}{(1+2x)^2} \right)$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+2x}{1+x}} \right) \frac{1}{2} \left(\frac{1}{1+2x} \right)^{\frac{1}{2}} \left(\frac{1}{(1+x)^{2-\frac{1}{2}}} \right)$$

$$\frac{dy}{dx} = \left(\frac{\cos \sqrt{\frac{1+2x}{1+x}}}{2(1+x)^{\frac{3}{2}} \sqrt{1+2x}} \right)$$

Which is required.

Question # 5 Differentiate

(i) $\sin x$ w.r.t. $\cot x$

Solution.

Let $y = \sin x$ and $u = \cot x$

we find $\frac{dy}{du}$

Now $y = \sin x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (\sin x)$$

$$\frac{dy}{dx} = \cos x$$

Now $u = \cot x$

Differentiate w.r.t x

$$\frac{du}{dx} = -\operatorname{Cosec}^2 x$$

$$\frac{dx}{du} = -\frac{1}{\operatorname{Cosec}^2 x}$$

$$\frac{dx}{du} = -\sin^2 x$$

Now using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = (\cos x)(-\sin^2 x) = -\sin^2 x \cos x.$$

(ii). $\sin^2 x$ w.r.t $\cos^4 x$

Solution.

Let $y = \sin^2 x$ and $u = \cos^4 x$

we find $\frac{dy}{du}$

Now $y = \sin^2 x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^2 x)$$

$$\frac{dy}{dx} = 2\sin x \cos x$$

Now $u = \cos^4 x$

Differentiate w.r.t x

$$\frac{du}{dx} = 4\cos^3 x(-\sin x) = -4\sin x \cos^3 x$$

$$\frac{dx}{du} = -\frac{1}{4\sin x \cos^3 x}$$

Now using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = (2\sin x \cos x) \left(-\frac{1}{4\sin x \cos^3 x} \right)$$

$$\frac{dy}{du} = -\frac{1}{2\cos^2 x}$$

$$\frac{dy}{du} = -\frac{1}{2} \sec^2 x$$

Question.6.

If $\tan y (1 + \tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$.

Solution.

Since

$$\tan y (1 + \tan x) = 1 - \tan x$$

$$\tan y = \frac{1 - \tan x}{1 + \tan x}$$

$$\tan y = \frac{\tan \frac{\pi}{4} - \tan x}{\tan \frac{\pi}{4} + \tan x}$$

$$\tan y = \tan \left(\frac{\pi}{4} - x \right)$$

$$y = \frac{\pi}{4} - x$$

Differentiate w.r.t " x ", we have

$$\frac{dy}{dx} = 0 - 1$$

$$\frac{dy}{dx} = -1$$

Hence proved.

Question.7.

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$, Prove that $(2y - 1) \frac{dy}{dx} = \sec^2 x$

Solution.

Since

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}} \dots (i)$$

Squaring on both sides, we have

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}$$

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$$

$$y^2 = \tan x + y$$

Differentiate w.r.t " x ", we have

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y - 1) \frac{dy}{dx} = \sec^2 x$$

Hence Proved.

Question.8

If $x = a \cos^3 \theta$, $y = b \sin^3 \theta$, Show that $a \frac{dy}{dx} + b \tan x = 0$

Solution

$$x = a \cos^3 \theta, y = b \sin^3 \theta$$

Diff. " x " w.r.t. " θ ", we have

$$\frac{dx}{d\theta} = \frac{d}{d\theta} (a \cos^3 \theta)$$

$$\frac{dx}{d\theta} = a \cdot 3\cos^2 \frac{d}{d\theta} (\cos \theta)$$

$$\frac{dx}{d\theta} = a \cdot 3\cos^2 (-\sin \theta)$$

$$\frac{dx}{d\theta} = -3a \sin \theta \cos^2$$

$$\frac{dx}{d\theta} = \frac{-1}{3a \sin \theta \cos^2}$$

Diff. " y " w.r.t. " θ ", we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (b \sin^3 \theta)$$

$$\frac{dy}{d\theta} = b \cdot 3\sin^2 \frac{d}{d\theta} (\sin \theta)$$

$$\frac{dy}{d\theta} = b \cdot 3\sin^2 \cos \theta$$

$$\frac{dy}{d\theta} = 3b \sin^2 \cos \theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = 3b \sin^2 \cos \theta \cdot \frac{-1}{3a \sin \theta \cos^2}$$

$$\frac{dy}{dx} = -\frac{b}{a} \tan \theta$$

$$a \frac{dy}{dx} = -b \tan \theta$$

$$a \frac{dy}{dx} + b \tan \theta = 0$$

Hence proved.

Question.10. Differentiate "x", we have

(i) $\text{Cos}^{-1} \frac{x}{a}$

Solution.

Let $y = \text{Cos}^{-1} \frac{x}{a}$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\text{Cos}^{-1} \frac{x}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\frac{\sqrt{a^2 - x^2}}{a}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{a}{\sqrt{a^2 - x^2}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$$

Which is required.

(ii) $\text{Cot}^{-1} \frac{x}{a}$

Solution.

Let $y = \text{Cot}^{-1} \frac{x}{a}$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\text{Cot}^{-1} \frac{x}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{1 + \frac{x^2}{a^2}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\frac{a^2 + x^2}{a^2}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\frac{a^2 + x^2}{a^2}} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{a^2}{a^2 + x^2} \cdot \left(\frac{1}{a} \right)$$

$$\frac{dy}{dx} = -\frac{a}{a^2 + x^2}$$

Which is required.

(iii) $\frac{1}{a} \text{Sin}^{-1} \frac{a}{x}$

Solution.

Let $y = \frac{1}{a} \text{Sin}^{-1} \frac{a}{x}$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{a} \text{Sin}^{-1} \frac{a}{x} \right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx} \left(\frac{a}{x} \right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \frac{a^2}{x^2}}} \cdot \left(\frac{-a}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{\frac{x^2 - a^2}{x^2}}} \cdot \left(\frac{-a}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\frac{\sqrt{x^2 - a^2}}{x}} \cdot \left(\frac{-a}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - a^2}} \cdot \left(\frac{-1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{-1}{x \sqrt{x^2 - a^2}}$$

Which is required.

(iv) $\text{Sin}^{-1} \sqrt{1 - x^2}$

Solution.

Let $y = \text{Sin}^{-1} \sqrt{1 - x^2}$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\text{Sin}^{-1} \sqrt{1 - x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\sqrt{1 - x^2}\right)^2}} \cdot \frac{d}{dx} \left(\sqrt{1 - x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (1 - x^2)}} \cdot \left(\frac{1}{2\sqrt{1 - x^2}} \cdot \frac{d}{dx} (1 - x^2) \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - 1 + x^2}} \cdot \left(\frac{1}{2\sqrt{1 - x^2}} (-2x) \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2}} \cdot \left(\frac{-x}{\sqrt{1 - x^2}} \right)$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \left(\frac{-x}{\sqrt{1 - x^2}} \right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

Which is required.

(v) $\text{Sec}^{-1} \left(\frac{x^2 - 1}{x^2 + 1} \right)$

Solution.

Let $y = \text{Sec}^{-1} \left(\frac{x^2 + 1}{x^2 - 1} \right)$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\text{Sec}^{-1} \left(\frac{x^2 + 1}{x^2 - 1} \right) \right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right) \sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2 - 1}} \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right) \frac{\sqrt{(x^2+1)^2 - (x^2-1)^2}}{(x^2-1)}}$$

$$\left(\frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1) \frac{\sqrt{4x^2 \cdot 1}}{(x^2-1)^2}} \cdot \left(\frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1) \cdot 2x} \cdot (2x(x^2-1) - x^2(2x))$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)} \cdot (-2)$$

$$\frac{dy}{dx} = \frac{-2}{x^2+1}$$

Which is required.

(vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution.

Let $y = Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(Cot^{-1}\left(\frac{2x}{1-x^2}\right) \right)$$

$$\frac{dy}{dx} = - \frac{1}{1 + \left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{d}{dx} \left(\frac{2x}{1-x^2}\right)$$

$$\frac{dy}{dx} = - \frac{1}{1 + \frac{4x^2}{(1-x^2)^2}} \cdot \left(2 \cdot \frac{(1-x^2) \frac{d}{dx}(x) - x \frac{d}{dx}(1-x^2)}{(1-x^2)^2} \right)$$

$$\frac{dy}{dx} = - \frac{1}{\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}} \cdot \left(2 \cdot \frac{(1-x^2) \cdot 1 - x(-2x)}{(1-x^2)^2} \right)$$

$$\frac{dy}{dx} = - \frac{1}{1+x^4-2x^2+4x^2} \cdot (2(1-x^2+2x^2))$$

$$\frac{dy}{dx} = - \frac{1}{1+x^4+2x^2} \cdot (2(1+x^2))$$

$$\frac{dy}{dx} = - \frac{1}{(1+x^2)^2} \cdot (2(1+x^2))$$

$$\frac{dy}{dx} = \frac{-2}{1+x^2}$$

Which is required.

(vii) $Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution.

Let $y = Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff.w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \right)$$

$$\frac{dy}{dx} = - \frac{x^2-1}{x^2-1 \sqrt{1 - \left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx} \left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx} = - \frac{x^2-1}{x^2-1 \sqrt{1 - \frac{(1-x^2)^2}{(1+x^2)^2}}} \cdot \left(\frac{(1+x^2) \frac{d}{dx}(1-x^2) - (1-x^2) \frac{d}{dx}(1+x^2)}{(x^2+1)^2} \right)$$

$$\frac{dy}{dx} = - \frac{x^2-1}{x^2-1 \sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}}} \cdot \left(\frac{(x^2+1)(-2x) - (1-x^2)(2x)}{(x^2+1)^2} \right)$$

$$= \frac{(x^2-1) \sqrt{x^4+2x^2+1 - (x^4+1-2x)}}{(x^2-1)^2} \cdot \frac{2x(x^2-1-x^2-1)}{(x^2+1)^2}$$

$$= \frac{x^2-1}{(x^2-1)^2} \cdot \frac{2x(-2)}{(x^2+1) \sqrt{2x^2+2x^2}} \cdot \frac{2x(-2)}{(x^2+1)^2}$$

$$\frac{-4x}{(x^2+1) \sqrt{4x^2}} = \frac{-4x}{(x^2+1) \cdot 2x} = \frac{-2x}{(x^2+1)}$$

$$\Rightarrow \frac{dy}{dx} = - \frac{2}{x^2+1}$$

Which is required.

Question.11.

Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1} \frac{x}{y}$

Solution.

Since

$$\frac{y}{x} = \tan^{-1} \frac{x}{y}$$

$$y = x \tan^{-1} \frac{x}{y}$$

Diff. w.r.t "x", we have

$$\frac{dy}{dx} = x \frac{d}{dx} \tan^{-1} \frac{x}{y} + \tan^{-1} \frac{x}{y} \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{d}{dx} \left(\frac{x}{y}\right) + \tan^{-1} \frac{x}{y} \cdot 1$$

$$\frac{dy}{dx} = x \frac{1}{1 + \frac{x^2}{y^2}} \left(\frac{y \frac{d}{dx} x - x \frac{dy}{dx}}{y^2} \right) + \frac{y}{x}$$

$$\frac{dy}{dx} = x \frac{1}{\frac{y^2+x^2}{y^2}} \left(\frac{y - x \frac{dy}{dx}}{y^2} \right) + \frac{y}{x}$$

$$\frac{dy}{dx} = x \frac{1}{y^2+x^2} \left(y - x \frac{dy}{dx} \right) + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{x^2+y^2} \left(y - x \frac{dy}{dx} \right) + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2} \frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2+y^2} \frac{dy}{dx} = \frac{xy}{x^2+y^2} + \frac{y}{x}$$

$$\left(1 + \frac{x^2}{x^2+y^2} \right) \frac{dy}{dx} = \frac{y}{x} \left(\frac{x^2}{x^2+y^2} + 1 \right)$$

$$\left(1 + \frac{x^2}{x^2 + y^2}\right) \frac{dy}{dx} = \frac{y}{x} \left(1 + \frac{x^2}{x^2 + y^2}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Hence Proved.

Question.12.

If $y = \tan(p \tan^{-1} x)$, Show that $(1 + x^2)y_1 - p(1 + y^2)$

Solution.

Since

$$y = \tan(p \tan^{-1} x)$$

$$\tan^{-1} y = p \tan^{-1} x$$

Differentiate with respect to "x", we have

$$\frac{1}{1 + y^2} \frac{dy}{dx} = p \frac{1}{1 + x^2}$$

Since $\frac{dy}{dx} = y_1$

$$\frac{1}{1 + y^2} y_1 = p \frac{1}{1 + x^2}$$

$$(1 + x^2)y_1 = p(1 + y^2)$$

$$(1 + x^2)y_1 - p(1 + y^2) = 0$$

Hence Proved.

Derivative of exponential Functions:

if $f(x) = a^x$ where a

> 0 then $f(x)$ is called

exponential function. here the base a is constant and x is variable.

if $f(x) = e^x$ where e

≈ 2.71 then $f(x)$ is called

exponential function here base e is constant and x is variable.

1. $\frac{d}{dx}(e^x) = e^x$

Proof:

Let $y = e^x \rightarrow (i)$

$\Rightarrow y + \delta y = e^{x+\delta x} \rightarrow (ii)$

by $(ii) - (i) \Rightarrow y + \delta y - y = e^{x+\delta x} - e^x$

$\Rightarrow \delta y = e^x \cdot e^{\delta x} - e^x = e^x(e^{\delta x} - 1)$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[e^x \left(\frac{e^{\delta x} - 1}{\delta x} \right) \right]$$

$$\frac{dy}{dx} = e^x \cdot \lim_{\delta x \rightarrow 0} \frac{e^{\delta x} - 1}{\delta x}$$

$$\frac{dy}{dx} = e^x \cdot lne \quad \because \lim_{\delta x \rightarrow 0} \frac{a^x - 1}{x} = lna$$

$$\frac{dy}{dx} = e^x \quad \text{and} \quad \lim_{\delta x \rightarrow 0} \frac{e^x - 1}{x} = lne = 1$$

$$\frac{d}{dx}(e^x) = e^x$$

2. $\frac{d}{dx}(a^x) = a^x lna$

Proof:

Let $y = a^x \rightarrow (i)$

$\Rightarrow y + \delta y = a^{x+\delta x} \rightarrow (ii)$

by $(ii) - (i) \Rightarrow y + \delta y - y = a^{x+\delta x} - a^x$

$$\Rightarrow \delta y = a^x \cdot a^{\delta x} - a^x = a^x(a^{\delta x} - 1)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[a^x \frac{a^{\delta x} - 1}{\delta x} \right]$$

$$\frac{dy}{dx} = a^x \lim_{\delta x \rightarrow 0} \frac{a^{\delta x} - 1}{\delta x} = a^x lna$$

Hence $\frac{d}{dx}(a^x) = a^x lna \quad (\because \lim_{\delta x \rightarrow 0} \frac{a^x - 1}{x} = lna)$

Note: as $a^x = a^x \Rightarrow lna^x = lna^x$

$$\Rightarrow lna^x = x lna$$

$$= x lna \cdot lne \quad \because lna^b = b lna$$

$$= x lna \quad \because lne = 1$$

$$ln^x = lne^{x lna}$$

Derivative of the logarithmic function:

logarithmic function: if $a > 0, a \neq 1$ and $x = a^y$ the function defined by $y = \log_a x$. ($x > 0$) is called the logarithmic of x to the base a .

The logarithmic functions

$\log_e x$ and $\log_{10} x$ are called

natural and common logarithmic resp. $y = \log_e x$

is written as $y = \ln x$

1. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
Proof:

Let $y = \ln x \rightarrow (i)$
 $\Rightarrow y + \delta y = \ln(x + \delta x) \rightarrow (ii)$

$\Rightarrow (ii) - (i) \Rightarrow y + \delta y - y = \ln(x + \delta x) - \ln x$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{1}{\delta x} \ln \left(1 + \frac{\delta x}{x} \right) \right]$$

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \ln \left(1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} \quad \left(\because \lim_{\delta x \rightarrow 0} \left(1 + \frac{1}{x} \right)^n = e \right)$$

$$\frac{dy}{dx} = \frac{1}{x} lne = \frac{1}{x}$$

Hence $\frac{d}{dx}(\ln x) = \frac{1}{x}$

2. $\frac{d}{dx}[\log_a x] = \frac{1}{x} \frac{1}{lna}$

Proof:

Let $u = \log_a x \rightarrow (i)$

$\Rightarrow y + \delta y = \log_a(x + \delta x) \rightarrow (ii)$

$\Rightarrow (ii) - (i) \Rightarrow y + \delta y - y = \log_a(x + \delta x) - \log_a x$

$$\delta y = \log_a \frac{(x + \delta x)}{x} \quad (\because \log_a x - \ln y = \log_a \frac{x}{y})$$

$$\delta y = \log_a \left(1 + \frac{\delta x}{x} \right) = \log_a \left(1 + \frac{\delta x}{x} \right)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{1}{\delta x} \log_a \left(1 + \frac{\delta x}{x} \right) \right]$$

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \log_a \left(1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} \quad \left(\because \lim_{\delta x \rightarrow 0} \left(1 + \frac{1}{x} \right)^x = e \right)$$

$$\frac{dy}{dx} = \frac{1}{x} \log_a e = \frac{1}{x} \cdot \frac{1}{\log_e a} = \frac{1}{x} \cdot \frac{1}{\ln a}$$

$$\text{Hence } \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

Logarithmic differentiation:

Derivative of hyperbolic function:

$$1. \quad \frac{d}{dx} (\sinh x) = \cosh x$$

Proof:

$$\text{we know thta } \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{so } \frac{d}{dx} (\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right)$$

$$\frac{1}{2} \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{1}{2} \left\{ \frac{d}{dx} (e^x) - \frac{d}{dx} (e^{-x}) \right\}$$

$$\frac{1}{2} (e^x - e^{-x}) \frac{d}{dx} (-x) = \frac{1}{2} (e^x - e^{-x}(-1))$$

$$\frac{1}{2} (e^x + e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx} (\sinh x) = \cosh x \quad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

$$2. \quad \frac{d}{dx} (\cosh x) = \sinh x$$

Proof:

$$\text{we know thta } \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{so } \frac{d}{dx} (\cosh x) = \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right)$$

$$\frac{1}{2} \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) = \frac{1}{2} \left\{ \frac{d}{dx} (e^x) + \frac{d}{dx} (e^{-x}) \right\}$$

$$\frac{1}{2} (e^x + e^{-x}) \frac{d}{dx} (-x) = \frac{1}{2} (e^x + e^{-x}(-1))$$

$$\frac{1}{2} (e^x - e^{-x}) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx} (\cosh x) = \sinh x \quad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$3. \quad \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

Proof:

$$\text{We know that } \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{So } \frac{d}{dx} (\tanh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

$$= \frac{(e^x + e^{-x}) \frac{d}{dx} (e^x - e^{-x}) - (e^x - e^{-x}) \frac{d}{dx} (e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x - e^{-x}(-1)) - (e^x - e^{-x})(e^x + e^{-x}(-1))}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

$$\frac{d}{dx} (\tanh x) = \left(\frac{2}{e^x + e^{-x}} \right)^2 = (\operatorname{sech} x)^2 = \operatorname{sech}^2 x$$

$$\Rightarrow \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x \quad \because \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

4.

$$\frac{d}{dx} (\operatorname{cosech} x) = -\operatorname{cosech} x \operatorname{coth} x$$

Proof:

$$\text{we know that } \operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\text{so } \frac{d}{dx} (\operatorname{cosech} x) = \frac{d}{dx} \left(\frac{1}{\sinh x} \right)$$

$$= \frac{\sinh x \frac{d}{dx} (1) - (1) \frac{d}{dx} (\sinh x)}{\sinh^2 x}$$

$$= \frac{\sinh x(0) - \cosh x}{\sinh^2 x}$$

$$\frac{d}{dx} (\operatorname{cosech} x) = \frac{-\cosh x}{\sinh x \cdot \sinh x} = -\operatorname{coth} x \operatorname{cosech} x$$

$$\frac{d}{dx} (\operatorname{cosech} x) = -\operatorname{cosech} x \operatorname{coth} x$$

5.

$$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

Proof:

$$\text{we know that } \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\Rightarrow \frac{d}{dx} (\operatorname{sech} x) = \frac{d}{dx} \left(\frac{1}{\cosh x} \right)$$

$$\Rightarrow = \frac{\cosh x \frac{d}{dx} (1) - (1) \frac{d}{dx} (\cosh x)}{\cosh^2 x}$$

$$= \frac{\cosh x(0) - (1)(\sinh x)}{\cosh^2 x}$$

$$= \frac{-\sinh x}{\cosh x \cdot \cosh x} = -\tanh x \cdot \operatorname{sech} x$$

$$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

6.

$$\frac{d}{dx} (\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

Proof:

$$\text{we know that } \operatorname{coth} x = \frac{1}{\tanh x}$$

$$\Rightarrow \frac{d}{dx} (\operatorname{coth} x) = \frac{d}{dx} \left(\frac{1}{\tanh x} \right)$$

$$= \frac{\tanh x \frac{d}{dx} (1) - (1) \frac{d}{dx} (\tanh x)}{\tanh^2 x}$$

$$= \frac{\tanh x(0) - (1)(\operatorname{sech}^2 x)}{\tanh^2 x}$$

$$= \frac{-\operatorname{sech}^2 x}{\frac{\sinh^2 x}{\cosh^2 x}} = \frac{-\operatorname{sech}^2 x}{\sinh^2 x} \cosh^2 x$$

$$= \frac{-\operatorname{sech}^2 x}{\sinh^2 x} \times \frac{1}{\operatorname{sech}^2 x} = -\frac{1}{\sinh^2 x}$$

Hence $\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$

Derivative of inverse hyperbolic function:

The inverse hyperbolic functions are

1. $y = \sinh^{-1} x$ iff $x = \sinh y$; $x, y \in R$
2. $y = \cosh^{-1} x$ iff $x = \cosh y$; $x \in [1, \infty), y \in [0, \infty)$
3. $y = \tanh^{-1} x$ iff $x = \tanh y$; $x \in (-1, 1), y \in R$
4. $y = \operatorname{coth}^{-1} x$ iff $x = \operatorname{coth} y$; $x \in [-1, 1]', y \in R - \{0\}$
5. $y = \operatorname{sech}^{-1} x$ iff $x = \operatorname{sech} y$; $x \in (0, 1], y \in [0, \infty)$
6. $y = \operatorname{cosech}^{-1} x$ iff $x = \operatorname{cosech} y$; $x \in R - \{0\}, y \in R - \{0\}$

Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

Proof:

Let $y = \sinh^{-1} x \Rightarrow x = \sinh y$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2} \quad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow 2x = e^y - e^{-y} = e^y - \frac{1}{e^y} = \frac{(e^y)^2 - 1}{e^y}$$

$$\Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} \quad \text{using } x$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2 \pm \sqrt{4(x^2 + 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = 2 \left(\frac{x \pm \sqrt{x^2 + 1}}{2} \right)$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \quad , e^y = x - \sqrt{x^2 + 1}$$

As e^y is +ve for $y \in R$, so (rejected being -ve for $x \in R$)

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \quad ,$$

$$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$$

Hence $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

Note:

$$\begin{aligned} \because e^y &= x \\ \ln e^y &= \ln x \\ y \ln e &= \ln x \\ y(1) &= \ln x \\ y &= \ln x \end{aligned}$$

Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

Proof:

Let $y = \cosh^{-1} x \Rightarrow x = \cosh y$

$$\Rightarrow x = \frac{e^y + e^{-y}}{2} \quad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\Rightarrow 2x = e^y + e^{-y} = e^y + \frac{1}{e^y} = \frac{(e^y)^2 + 1}{e^y}$$

$$\Rightarrow 2xe^y = e^{2y} + 1 \Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow (e^y)^2 - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} \quad \text{using } x$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2 \pm \sqrt{4(x^2 - 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = 2 \left(\frac{x \pm \sqrt{x^2 - 1}}{2} \right)$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow e^y = x + \sqrt{x^2 - 1} \quad , e^y = x - \sqrt{x^2 - 1}$$

As e^y is +ve for $y \in R$, so (rejected being -ve for $x > 1$)

$$\Rightarrow e^y = x + \sqrt{x^2 - 1} \quad ,$$

$$\Rightarrow y = \ln(x + \sqrt{x^2 - 1})$$

Hence $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

Note:

$$\begin{aligned} \because e^y &= x \\ \ln e^y &= \ln x \\ y \ln e &= \ln x \\ y(1) &= \ln x \\ y &= \ln x \end{aligned}$$

1. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad x, y \in R$

Proof:

Let $y = \sinh^{-1} x$

$$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(\sinh y) = \cosh y \frac{dy}{dx}$$

$$\Rightarrow \cosh \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

$$\because \cosh^2 x - \sinh^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

($\because \sinh y = x$)

2. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \quad x \in [1, \infty), y \in [0, \infty)$

Proof:

Let $y = \cosh^{-1} x$

$$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(\cosh y) = \sinh y \frac{dy}{dx}$$

$$\Rightarrow \sinh \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2 y}} = \frac{1}{\sqrt{\cosh^2 y - 1}}$$

$$\because \cosh^2 x - \sinh^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

3.

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

Proof:

Let $y = \tanh^{-1} x ; x \in (-1, 1), y \in \mathbb{R}$

$$\Rightarrow \tanh y = x$$

$$\Rightarrow \frac{d}{dx}(\tanh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 - x^2} \quad (\because \operatorname{sech}^2 x + \tanh^2 x = 1)$$

4.

$$\frac{d}{dx}(\operatorname{coth}^{-1} x) = \frac{1}{1 - x^2}$$

Proof:

Let $y = \operatorname{coth}^{-1} x ; |x| > 1$

$$\Rightarrow \operatorname{coth} y = x$$

$$\Rightarrow \frac{d}{dx}(\operatorname{coth} y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{-\operatorname{cosech}^2 y} = \frac{1}{1 - \operatorname{coth}^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 - x^2} \quad (\because -\operatorname{cosech}^2 x + \operatorname{coth}^2 x = 1)$$

$$5. \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

Proof:

Let $y = \operatorname{sech}^{-1} x$

$$\Rightarrow \operatorname{sech} y = x \Rightarrow \frac{d}{dx}(\operatorname{sech} y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{sech} y \tanh y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{sech} y \tanh y} = \frac{-1}{\operatorname{sech} y \sqrt{\tanh^2 y}}$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} \quad (\because \operatorname{sech}^2 y + \tanh^2 y = 1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$5. \frac{d}{dx}(\operatorname{cosech}^{-1} x) = -\frac{1}{x\sqrt{1+x^2}}, \quad x \in \mathbb{R} - \{0\}$$

Proof:

Let $y = \operatorname{cosech}^{-1} x$

$$\Rightarrow \operatorname{cosech} y = x \Rightarrow \frac{d}{dx}(\operatorname{cosech} y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosech} y \operatorname{coth} y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosech} y \operatorname{coth} y} = \frac{-1}{\operatorname{cosech} y \sqrt{\operatorname{coth}^2 y}}$$

$$(\because \operatorname{coth}^2 x - \operatorname{cosec}^2 x = 1)$$

$$(\operatorname{coth}^2 x = 1 + \operatorname{cosec}^2 x)$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosech} y \sqrt{1 + \operatorname{cosec}^2 y}} = \frac{-1}{x\sqrt{1+x^2}}$$

Exercise 2.6

Question # 1) Find $f'(x)$ if

i) $f(x) = e^{\sqrt{x}-1}$

Solution:

Let $f(x) = e^{\sqrt{x}-1}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(e^{\sqrt{x}-1})$$

$$f'(x) = e^{\sqrt{x}-1} \frac{d}{dx}(\sqrt{x}-1)$$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}}$$

Which is required.

ii) $f(x) = x^3 e^{\frac{1}{x}}$

Solution:

Let $f(x) = x^3 e^{\frac{1}{x}}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(x^3 e^{\frac{1}{x}})$$

$$f'(x) = e^{\frac{1}{x}} \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(e^{\frac{1}{x}})$$

$$f'(x) = e^{\frac{1}{x}}(3x^2) + x^3 \left(e^{\frac{1}{x}}\right) \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$f'(x) = e^{\frac{1}{x}}(3x^2) + x^3 \left(e^{\frac{1}{x}}\right) \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$f'(x) = e^{\frac{1}{x}}(3x^2) + x^3 \left(e^{\frac{1}{x}}\right) \left(\frac{-1}{x^2}\right)$$

$$f'(x) = x e^{\frac{1}{x}}(3x - 1)$$

Which is required.

iii) $f(x) = e^x(1 + \ln x)$

Solution:

Let $f(x) = e^x(1 + \ln x)$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(e^x(1 + \ln x))$$

$$f'(x) = e^x \frac{d}{dx}(1 + \ln x) + (1 + \ln x) \frac{d}{dx}(e^x)$$

$$f'(x) = e^x \left(0 + \frac{1}{x}\right) + (1 + \ln x)(e^x)$$

$$f'(x) = \frac{e^x + x(1 + \ln x)(e^x)}{x}$$

$$f'(x) = \frac{(1 + x(1 + \ln x))e^x}{x}$$

Which is required.

iv) $f(x) = \frac{e^x}{e^{-x}+1}$

Solution:

Let $f(x) = \frac{e^x}{e^{-x}+1}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$

$$f'(x) = \frac{(e^{-x}+1) \frac{d}{dx}(e^x) - (e^x) \frac{d}{dx}(e^{-x}+1)}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{(e^{-x} + 1)(e^x) - (e^x)(e^{-x}(-1))}{(e^{-x} + 1)^2}$$

$$f'(x) = \frac{(1 + e^x) - (-1)}{(e^{-x} + 1)^2}$$

$$f'(x) = \frac{2 + e^x}{(e^{-x} + 1)^2}$$

Which is required.

v) $f(x) = \ln(e^x + e^{-x})$

Solution:

Let $f(x) = \ln(e^x + e^{-x})$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(\ln(e^x + e^{-x}))$$

$$f'(x) = \frac{1}{e^x + e^{-x}} \frac{d}{dx}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{e^x + e^{-x}}(e^x + e^{-x}(-1))$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$$

$$f'(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Which is required.

vi) $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Solution.

Let $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx} \left(\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}} \right)$$

$$f'(x) = \frac{(e^{ax} + e^{-ax}) \frac{d}{dx}(e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax}) \frac{d}{dx}(e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{(e^{ax} + e^{-ax})(e^{ax}(a) - e^{-ax}(-a)) - (e^{ax} - e^{-ax})(e^{ax}(a) + e^{-ax}(-))}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[(e^{ax} + e^{-ax})^2 - (e^{ax} - e^{-ax})^2]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[(e^{ax})^2 + (e^{-ax})^2 + 2(e^{ax})(e^{-ax}) - (e^{ax})^2 - (e^{-ax})^2 - 2(e^{ax})(e^{-ax})]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[4e^0]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{4a}{(e^{ax} + e^{-ax})^2}$$

Which is required.

vii) $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$

Solution:

Let $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(\sqrt{\ln(e^{2x} + e^{-2x})})$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x} + e^{-2x})}} \frac{d}{dx}(\ln(e^{2x} + e^{-2x}))$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x} + e^{-2x})}} \left[\frac{1}{e^{2x} + e^{-2x}} \frac{d}{dx}(e^{2x} + e^{-2x}) \right]$$

$$f'(x) = \frac{2(e^{2x} - e^{-2x})}{2\sqrt{\ln(e^x + e^{-x})(e^{2x} + e^{-2x})}}$$

$$f'(x) = \frac{(e^{2x} - e^{-2x})}{\sqrt{\ln(e^x + e^{-x})(e^{2x} + e^{-2x})}}$$

Which is required.

viii) $f(x) = \ln\sqrt{(e^{2x} + e^{-2x})}$

Solution:

Let $f(x) = \ln\sqrt{(e^{2x} + e^{-2x})} = \frac{1}{2}(\ln(e^{2x} + e^{-2x}))$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx} = \frac{1}{2} \frac{d}{dx}(\ln(e^{2x} + e^{-2x}))$$

$$f'(x) = \frac{2e^{2x} - 2e^{-2x}}{2(e^{2x} + e^{-2x})}$$

$$f'(x) = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

$$f'(x) = \tanh 2x$$

Question # 2. Find $\frac{dy}{dx}$ if

Solution.

i) $y = x^2 \ln\sqrt{x}$

Let $y = x^2 \ln\sqrt{x} = \frac{1}{2}x^2 \ln x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2}x^2 \ln x \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left[(x^2) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x^2) \right]$$

$$\frac{dy}{dx} = \frac{1}{2} \left[x^2 \left(\frac{1}{x} \right) + \ln x (2x) \right]$$

ii) $y = x\sqrt{\ln x}$

Solution.

Let $y = x\sqrt{\ln x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (x\sqrt{\ln x})$$

$$\frac{dy}{dx} = (x) \frac{d}{dx}(\sqrt{\ln x}) + (\sqrt{\ln x}) \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \frac{1}{2\sqrt{\ln x}} \frac{d}{dx}(\ln x) + \sqrt{\ln x} (1)$$

$$\frac{dy}{dx} = x \frac{1}{2\sqrt{\ln x}} \left(\frac{1}{x} \right) + \sqrt{\ln x} (1)$$

$$\frac{dy}{dx} = \frac{1 + 2(\sqrt{\ln x})^2}{2\sqrt{\ln x}}$$

$$\frac{dy}{dx} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

Which is required.

iii) $y = \frac{x}{\ln x}$

Solution.

Let $y = \frac{x}{\ln x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{\ln x} \right)$$

$$\frac{dy}{dx} = \left(\frac{(\ln x) \frac{d}{dx}(x) - (x) \frac{d}{dx}(\ln x)}{(\ln x)^2} \right)$$

$$\frac{dy}{dx} = \left(\frac{(\ln x)(1) - (x) \left(\frac{1}{x} \right)}{(\ln x)^2} \right)$$

$$\frac{dy}{dx} = \frac{\ln x - 1}{(\ln x)^2}$$

iv) $y = x^2 \ln \left(\frac{1}{x} \right)$

Solution.

$$\text{Let } y = x^2 \ln \left(\frac{1}{x} \right) = x^2 \ln(x^{-1}) = -x^2 \ln(x)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (-x^2 \ln(x))$$

$$\frac{dy}{dx} = - \left[(x^2) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x^2) \right]$$

$$\frac{dy}{dx} = - \left[(x^2) \left(\frac{1}{x} \right) + (\ln x)(2x) \right]$$

$$\frac{dy}{dx} = -[x + 2x \ln x]$$

$$\frac{dy}{dx} = -x(1 + 2 \ln x)$$

Which is required.

v) $y = \ln \sqrt{\frac{x^2-1}{x^2+1}}$

Solution.

$$\text{Let } y = \ln \sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{2} \ln \left(\frac{x^2-1}{x^2+1} \right)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \ln \left(\frac{x^2-1}{x^2+1} \right) \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{\frac{x^2-1}{x^2+1}} \frac{d}{dx} \left(\frac{x^2-1}{x^2+1} \right) \right]$$

$$\frac{dy}{dx} = \frac{1(x^2+1)}{2(x^2-1)} \left(\frac{(x^2+1) \frac{d}{dx}(x^2-1) - (x^2-1) \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \right)$$

$$\frac{dy}{dx} = \left(\frac{(x^2+1)(2x) - (x^2-1)(2x)}{2(x^2-1)(x^2+1)} \right)$$

$$\frac{dy}{dx} = \left(\frac{2x(x^2+1 - x^2+1)}{2(x^2-1)(x^2+1)} \right)$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^4-1} \right)$$

Which is required.

vi) $y = \ln(x + \sqrt{x^2+1})$

Solution.

$$\text{Let } y = \ln(x + \sqrt{x^2+1})$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Which is required.

vii) $y = \ln(9 - x^2)$

Solution.

$$\text{Let } y = \ln(9 - x^2)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (\ln(9 - x^2))$$

$$\frac{dy}{dx} = \frac{1}{9 - x^2} \frac{d}{dx} (9 - x^2)$$

$$\frac{dy}{dx} = \frac{1}{9 - x^2} (-2x)$$

$$\frac{dy}{dx} = \frac{-2x}{9 - x^2}$$

Which is required.

viii) $y = e^{-2x} \cdot \sin 2x$

Solution.

$$\text{Let } y = e^{-2x} \cdot \sin 2x$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (e^{-2x} \cdot \sin 2x)$$

$$\frac{dy}{dx} = (e^{-2x})(\cos 2x (2)) + (\sin 2x)(e^{-2x}(-2))$$

$$\frac{dy}{dx} = 2e^{-2x}(\cos 2x - \sin 2x)$$

Which is required.

ix) $y = e^{-x}(x^3 + 2x^2 + 1)$

Solution.

$$\text{Since } y = e^{-x}(x^3 + 2x^2 + 1)$$

Differentiating w.r.t "x"

$$\frac{dy}{dx} = (x^3 + 2x^2 + 1) \frac{d}{dx}(e^{-x}) + e^{-x} \frac{d}{dx}(x^3 + 2x^2 + 1)$$

$$\frac{dy}{dx} = (x^3 + 2x^2 + 1)(e^{-x}(-1)) + e^{-x}(3x^2 + 4x)$$

$$\frac{dy}{dx} = e^{-x}(-x^3 - 2x^2 - 1 + 3x^2 + 4x)$$

$$\frac{dy}{dx} = -e^{-x}(x^3 - x^2 - 4x + 1)$$

x) $y = xe^{\sin x}$

Solution.

$$\text{Since } y = xe^{\sin x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(xe^{\sin x})$$

$$\frac{dy}{dx} = (e^{\sin x}) \frac{d}{dx}(x) + (x) \frac{d}{dx}(e^{\sin x})$$

$$\frac{dy}{dx} = (e^{\sin x})(1) + (x)(e^{\sin x}(\cos x))$$

$$\frac{dy}{dx} = e^{\sin x}(1 + x \cos x)$$

Which is required.

xi) $y = 5e^{3x-4}$

Solution.

$$\text{Let } y = 5e^{3x-4}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (5e^{3x-4})$$

$$\frac{dy}{dx} = 5e^{3x-4} \frac{d}{dx}(3x-4)$$

$$\frac{dy}{dx} = 5e^{3x-4} (3)$$

$$\frac{dy}{dx} = 15e^{3x-4}$$

Which is required.

$$\text{(ii)} \quad y = (x+1)^x$$

Solution.

$$\text{Let } y = (x+1)^x$$

Differentiate w.r.t x

Taking ln on both sides

$$\ln y = \ln(x+1)^x$$

$$\ln y = x \ln(x+1)$$

$$\frac{1}{y} \frac{dy}{dx} = (\ln(x+1)) \frac{d}{dx}(x) + x \frac{d}{dx}(\ln(x+1))$$

$$\frac{1}{y} \frac{dy}{dx} = (\ln(x+1))(1) + (x) \frac{1}{(x+1)} \left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y \left(\frac{\ln(x+1) + 1}{x+1} \right)$$

$$\frac{dy}{dx} = (x+1)^x \left(\frac{\ln(x+1) + 1}{x+1} \right)$$

$$\frac{dy}{dx} = (x+1)^{x-1} (\ln(x+1) + 1)$$

Which is required.

$$\text{(iii)} \quad y = (\ln x)^{\ln x}$$

Solution.

$$\text{Let } y = (\ln x)^{\ln x}$$

Differentiate w.r.t x

Taking ln on both sides

$$\ln y = \ln(\ln x)^{\ln x}$$

$$\ln y = \ln x \cdot \ln(\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = (\ln(\ln x)) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(\ln(\ln x))$$

$$\frac{1}{y} \frac{dy}{dx} = (\ln(\ln x)) \left(\frac{1}{x}\right) + (\ln x) \frac{1}{(\ln x)} \left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y \left(\frac{\ln(\ln x) + 1}{x} \right)$$

$$\frac{dy}{dx} = (\ln x)^x \left(\frac{\ln(\ln x) + 1}{x} \right)$$

$$\frac{dy}{dx} = \frac{(x+1)^x}{x} (\ln(\ln x) + 1)$$

Q#3) Find $\frac{dy}{dx}$ if:

$$\text{(i)} \quad y = \cosh 2x$$

Solution.

$$\text{Let } y = \cosh 2x$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\cosh 2x)$$

$$\frac{dy}{dx} = \sinh 2x \frac{d}{dx}(2x)$$

$$\frac{dy}{dx} = \sinh 2x (2)$$

$$\frac{dy}{dx} = 2 \sinh 2x$$

Which is required.

$$\text{(ii)} \quad y = \sinh 3x$$

$$\text{Let } y = \cosh 2x$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh 3x)$$

$$\frac{dy}{dx} = \cosh 2x \frac{d}{dx}(3x)$$

$$\frac{dy}{dx} = \cosh 2x (3)$$

$$\frac{dy}{dx} = 3 \cosh 3x$$

$$\text{(iii)} \quad y = \tanh^{-1}(\sin x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Sol:

$$\text{Let } y = \tanh^{-1}(\sin x)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\tanh^{-1}(\sin x))$$

$$\frac{dy}{dx} = \frac{1}{1 - (\sin x)^2} \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{1 - \sin^2 x} (\cos x)$$

$$\frac{dy}{dx} = \frac{\cos x}{\cos^2 x} = \sec x$$

$$\text{(iv)} \quad y = \sinh^{-1}(x^3)$$

$$\text{Let } y = \sinh^{-1}(x^3)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(x^3))$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^3)^2}} \frac{d}{dx}(x^3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^6}} (3x^2)$$

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1 - x^6}}$$

$$\text{(v)} \quad y = \ln(\tanh x)$$

$$\text{Let } y = \ln(\tanh x)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(\tanh x))$$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \frac{d}{dx}(\tanh x)$$

$$\frac{dy}{dx} = \frac{\cosh x}{\sinh x} (\text{sech}^2 x)$$

$$\frac{dy}{dx} = \frac{\cosh x}{\sinh x \cosh^2 x}$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x}$$

$$\text{(vi)} \quad y = \sinh^{-1}\left(\frac{x}{2}\right)$$

Sol:

$$\text{Let } y = \sinh^{-1}\left(\frac{x}{2}\right)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}\left(\frac{x}{2}\right))$$

$$\frac{dy}{dx} = \frac{1}{1 - (\sin x)^2} \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \frac{x^2}{4}}} \left(\frac{1}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4 + x^2}{4}}} \left(\frac{1}{2}\right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4 + x^2}} \left(\frac{1}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4 + x^2}}$$

Successive differentiation (or higher derivative)

Let y

$= f(x)$ be a function, then its successive or higher are given below.

1st derivative 2nd derivative 3rd derivative

y'	y''	y'''
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
y_1	y_2	y_3
D_y	D_y^2	D_y^3
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{df}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$

Exercise 2.7

Question # 1

Find y_2 if:

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Solution.

Let $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

Again differentiate w.r.t x

$$y_2 = \frac{d}{dx}(10x^4 - 12x^3 + 12x^2 + 1)$$

$$y_2 = 40x^3 - 36x^2 + 24x$$

Which is required.

(ii) $y = (2x + 5)^{\frac{3}{2}}$

Solution.

$$\frac{dy}{dx} = \frac{d}{dx}(2x + 5)^{\frac{3}{2}}$$

$$y_1 = \frac{3}{2}(2x + 5)^{\frac{1}{2}}(2)$$

$$y_1 = 3(2x + 5)^{\frac{1}{2}} = 3\sqrt{2x + 5}$$

Again differentiate w.r.t x

$$y_2 = 3 \frac{d}{dx}(\sqrt{2x + 5})$$

$$y_2 = \frac{3}{2\sqrt{2x + 5}}(2)$$

$$y_2 = \frac{3}{\sqrt{2x + 5}}$$

Which is required.

(iii) $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

Sol: Since $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

$$y = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

Differentiating w.r.t " x "

$$y_1 = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

Again differentiate w.r.t x

$$y_2 = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$y_2 = \frac{-1}{4}x^{-\frac{3}{2}} + \frac{-3}{4}x^{-\frac{5}{2}}$$

$$y_2 = \frac{-1}{4}x^{-\frac{3}{2}} + \frac{-3}{4}x^{-\frac{5}{2}}$$

Which is required.

Question # 2 Find y_2 if:

(i) $y = x^2 \cdot e^{-x}$

Solution.

Let $y = x^2 \cdot e^{-x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cdot e^{-x})$$

$$y_1 = (x^2)(e^{-x}(-1)) + (e^{-x})(2x)$$

$$y_1 = e^{-x}(2x - x^2)$$

Again differentiate w.r.t x

$$y_2 = \frac{d}{dx}(e^{-x}(2x - x^2))$$

$$y_2 = ((2x - x^2))(e^{-x}(-1)) + (e^{-x})(2 - 2x)$$

$$y_2 = e^{-x}(-2x + x^2 + 2 - 2x)$$

$$y_2 = e^{-x}(x^2 - 4x + 2)$$

Which is required.

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{d}{dx} \cdot \ln\left(\frac{2x+3}{3x+2}\right)$$

$$= \frac{d}{dx} [\ln(2x + 3) - \ln(3x + 2)]$$

$$y_1 = 2(2x + 3)^{-1} - 3(3x + 2)^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y_2 = 2 \frac{d}{dx}(2x + 3)^{-1} - 3 \frac{d}{dx}(3x + 2)^{-1}$$

$$= 2(-1)(2x + 3)^{-2} \frac{d}{dx}(2x + 3) - 3(-1)(3x + 2)^{-2} \frac{d}{dx}(3x + 2)$$

$$= -2(2x + 3)^{-2}(2) + 3(3x + 2)^{-2}(3)$$

$$y_2 = -\frac{4}{(2x + 3)^2} + \frac{9}{(3x + 2)^2}$$

$$= \frac{-4(3x + 2)^2 + 9(2x + 3)^2}{(2x + 3)^2(3x + 2)^2}$$

$$= \frac{-4(9x^2 + 4 + 12x) + 9(4x^2 + 3 + 12x)}{(2x + 3)^2(3x + 2)^2}$$

$$= \frac{-36x^2 - 16 - 48x + 36x^2 + 81 + 108x}{(2x + 5)^2(3x + 2)^2}$$

$$y_2 = \frac{60x + 65}{(2x + 5)^2(3x + 2)^2}$$

Q#3) Find y_2 if:

(i) $x^2 + y^2 = a^2$

Sol:

Differentiate w.r.t x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$$

$$2x + 2yy_1 = 0$$

$$2yy_1 = -2x$$

$$y_1 = -\frac{x}{y}$$

Differentiating again w.r.t "x"

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{dy_1}{dx} = -\frac{(y)\frac{d}{dx}(x) - (x)\frac{d}{dx}(y)}{(y)^2}$$

$$y_2 = -\left(\frac{(y)(1) - (x)(y_1)}{(y)^2}\right)$$

Put value of " y_1 "

$$y_2 = -\left(\frac{(y) - (x)\left(-\frac{x}{y}\right)}{(y)^2}\right)$$

$$y_2 = -\left(\frac{\left(\frac{y^2 + x^2}{y}\right)}{(y)^2}\right)$$

$$y_2 = -\left(\frac{y^2 + x^2}{y^3}\right)$$

$$y_2 = \frac{-a^2}{y^3}$$

(ii) $x^3 - y^3 = a^3$

Sol:

Differentiate w.r.t x

$$\frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}(a^3)$$

$$3x^2 - 3y^2y_1 = 0$$

$$3y^2y_1 = 3x^2$$

$$y_1 = \frac{x^2}{y^2}$$

Differentiating again w.r.t "x"

$$\frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\frac{dy_1}{dx} = \frac{(y^2)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2 = \frac{(y^2)(2x) - (x^2)(2yy_1)}{y^4}$$

Put value of " y_1 "

$$y_2 = \frac{(y^2)(2x) - (x^2)\left(2y\left(\frac{x^2}{y^2}\right)\right)}{y^4}$$

$$y_2 = \frac{(2xy^2) - \left(\frac{2x^4}{y}\right)}{y^4}$$

$$y_2 = \frac{\left(\frac{2y^2x^2 - 2x^4}{y}\right)}{y^4}$$

$$y_2 = \frac{2x(y^3 - x^3)}{y^5}$$

$$= \frac{-2x}{y^5}(x^3 - y^3)$$

$$\frac{d^2y}{dx^2} = -\frac{2x}{y^5}(a^3) \quad \because x^3 - y^3 = a^3$$

$$y_2 = -\frac{2xa^3}{y^5}$$

(iii) $x = a\cos\theta, y = a\sin\theta$

Sol:

$$x = a\cos\theta, y = a\sin\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin\theta) = a\cos\theta$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{d}{d\theta} \cdot \frac{d\theta}{dx} = (a\cos\theta)\left(\frac{1}{-a\sin\theta}\right)$$

$$\frac{dy}{dx} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta \Rightarrow y_1 = -\cot\theta$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\theta) = -(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta\left(\frac{1}{-a\sin\theta}\right) = \frac{1}{\sin^2\theta}\left(\frac{1}{-a\sin\theta}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{a\sin^3\theta}$$

$$y_2 = \frac{-1}{a\sin^3\theta}$$

Which is required

(iv) $x = at^2, y = bt^4$

Sol:

$$x = at^2 \Rightarrow t^2 = \frac{x}{a}$$

$$y = bt^4 \Rightarrow y = b\left(\frac{x}{a}\right)^2$$

$$y = \frac{bx^2}{a^2}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{b}{a^2} \frac{d}{dx}(x^2)$$

$$y_1 = \frac{b}{a^2}(2x)$$

Again differentiate w.r.t x

$$y_2 = \frac{2b}{a^2} \frac{d}{dx}(x)$$

$$y_2 = \frac{2b}{a^2}$$

$$(v) \quad x^2 + y^2 + 2gx + 2fy + c = 0$$

Sol: differentiate w.r.t "x"

$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$2x + 2yy_1 + 2g + 2fy_1 = 0$$

$$yy_1 + fy_1 = -x - g$$

$$y_1(y + f) = -(x + g)$$

$$y_1 = -\frac{x + g}{y + f}$$

Differentiating again w.r.t "x"

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x + g}{y + f}\right)$$

$$\frac{dy_1}{dx} = -\frac{(y + f)\frac{d}{dx}(x + g) - (x + g)\frac{d}{dx}(y + f)}{(y + f)^2}$$

$$y_2 = -\left(\frac{(y + f)(1) - (x + g)(y_1)}{(y + f)^2}\right)$$

Put value of "y₁"

$$y_2 = -\left(\frac{(y + f)(1) - (x + g)\left(-\frac{x + g}{y + f}\right)}{(y + f)^2}\right)$$

$$y_2 = -\left(\frac{(y + f)^2 + (x + g)^2}{(y + f)^3}\right)$$

$$y_2 = -\left(\frac{y^2 + f^2 + 2fy + x^2 + g^2 + 2gx}{(y + f)^3}\right)$$

$$y_2 = -\left(\frac{(x^2 + y^2 + 2gx + 2fy + c) + f^2 + g^2 - c}{(y + f)^3}\right)$$

$$y_2 = -\left(\frac{(0) + f^2 + g^2 - c}{(y + f)^3}\right)$$

$$y_2 = \frac{-f^2 - g^2 + c}{(y + f)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-f^2 - g^2 + c}{(y + f)^3}$$

$$\frac{d^2y}{dx^2} = \frac{c - f^2 - g^2}{(y + f)^3}$$

Q#4 Find y₄ if:

$$(i) \quad y = \sin 3x$$

Sol: Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x) = \cos 3x \frac{d}{dx}(3x)$$

$$y_1 = \cos 3x (3)$$

$$y_1 = 3\cos 3x$$

Again differentiate w.r.t x

$$\frac{d^2y}{dx^2} = y_2 = \frac{d}{dx}(3\cos 3x) = 3(-\sin 3x) \frac{d}{dx}(3x)$$

$$3(-\sin 3x (3))$$

$$y_2 = -9\sin 3x$$

Again differentiate w.r.t x

$$\frac{d^3y}{dx^3} = y_3 = \frac{d}{dx}(-9\sin 3x) = -9(\cos 3x) \frac{d}{dx}(3x)$$

$$= -9 \cos 3x (3)$$

$$y_3 = -27\cos 3x$$

Again differentiate w.r.t x

$$\frac{d^4y}{dx^4} = y_4 = \frac{d}{dx}(-27\cos 3x)$$

$$= -27(-\sin 3x) \frac{d}{dx}(3x)$$

$$= -27(-\sin 3x (3))$$

$$y_4 = 81\sin 3x$$

Which is required

$$(ii) \quad y = \cos^3 x$$

$$\text{Sol: } \begin{cases} \because \cos 3x = 4\cos^3 x - 3\cos x \\ \Rightarrow \cos 3x + 3\cos x = 4\cos^3 x \\ \Rightarrow \frac{1}{4}(\cos 3x + 3\cos x) = \cos^2 x \end{cases}$$

Thus

$$y = \frac{1}{4}[\cos 3x + 3\cos x]$$

$$\frac{dy}{dx} = y_1 = \frac{1}{4}\left[\frac{d}{dx}(\cos 3x) + \frac{d}{dx}(3\cos x)\right]$$

$$= \frac{1}{4}[(-\sin 3x(3) + 3(-\sin x))]$$

$$y_1 = -\frac{3}{4}(\sin 3x + \sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{3}{4}\left(\frac{d}{dx}(\sin 3x) + \frac{d}{dx}(\sin x)\right)$$

$$= -\frac{3}{4}(\cos 3x (3) + \cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = y_2 = -\frac{3}{4}[(3\cos 3x + \cos x)]$$

$$\Rightarrow \frac{d^3y}{dx^3} = y_3 = -\frac{3}{4}\left[3\left(\frac{d}{dx}(\cos 3x) + \frac{d}{dx}(\cos x)\right)\right]$$

$$= -\frac{3}{4}[3(-\sin 3x)(3) + (-\sin x)]$$

$$= -\frac{3}{4}(-9\sin 3x - \sin x)$$

$$y_3 = \frac{3}{4}(9\sin 3x + \sin x)$$

$$\Rightarrow \frac{d^4y}{dx^4} = y_4 = \frac{3}{4}\left[9\frac{d}{dx}(\sin 3x) + \frac{d}{dx}(\sin x)\right]$$

$$= \frac{3}{4}[9\cos 3x(3) + \cos x]$$

$$= \frac{3}{4}[27\cos 3x + \cos x]$$

$$= \frac{3}{4}(27(4\cos^3 x - 3\cos x) + \cos x)$$

$$= \frac{3 \times 27 \times 4}{4}\cos^3 x - \frac{3 \times 27 \times 3}{4}\cos x + \frac{3}{4}\cos x$$

$$81\cos^3 x - \frac{243}{4}\cos x + \frac{3}{4}\cos x$$

$$y_4 = 81\cos^3 x + \left(\frac{3 - 243}{4}\right)\cos x$$

$$\Rightarrow y_4 = 81\cos^3 x - \frac{240}{4}\cos x$$

$$y_4 = -60\cos x + 81\cos^3 x$$

$$y_4 = 81\cos^3 x - 60\cos x$$

$$\begin{aligned}
 \text{(iii)} \quad y &= \ln(x^2 - 9) = \ln(x - 3)(x + 3) \\
 y &= \ln(x - 3) + \ln(x + 3) \\
 \Rightarrow \frac{dy}{dx} &= y_1 = \frac{d}{dx} [\ln(x - 3) + \ln(x + 3)] \\
 y_1 &= \frac{1}{x - 3} + \frac{1}{x + 3} = (x - 3)^{-1} + (x + 3)^{-1} \\
 \frac{d^2y}{dx^2} &= y_2 = \frac{d}{dx} (x - 3)^{-1} + \frac{d}{dx} (x + 3)^{-1} \\
 y_2 &= (-1)(x - 3)^{-2} + (-1)(x + 3)^{-2} \\
 \frac{d^3y}{dx^3} &= y_3 = \frac{d}{dx} (-1)(x - 3)^{-2} \\
 &\quad + (-1)(x + 3)^{-2} \\
 y_3 &= (-1)(-2)(x - 3)^{-3} \\
 &\quad + (-1)(-2)(x + 3)^{-3} \\
 \frac{d^4y}{dx^4} &= y_4 = \frac{d}{dx} ((-1)(-2)(x - 3)^{-3} + \frac{d}{dx} (-1)(-2)(x + 3)^{-3}) \\
 &= (-1)(-2)(-3)(x - 3)^{-4} \\
 &\quad + (-1)(-2)(-3)(x + 3)^{-4} \\
 y_4 &= -\frac{6}{(x - 3)^4} - \frac{6}{(x + 3)^4} \\
 &= -6 \left[\frac{1}{(x - 3)^4} + \frac{1}{(x + 3)^4} \right]
 \end{aligned}$$

Q#5)

if $x = \sin m\theta$, $y = \cos \theta$, show that $(1 - x^2)y_2 - xy_1 + m^2y = 0$

Sol: Since $x = \sin m\theta \Rightarrow \theta = \frac{1}{m} \sin^{-1}(x)$

And $y = \cos \theta \Rightarrow \theta = \cos^{-1}(y)$

$\Rightarrow \frac{1}{m} \sin^{-1}(x) = \cos^{-1}(y)$

Differentiate w.r.t x

$\frac{1}{m} \frac{d}{dx} (\sin^{-1}(x)) = \frac{d}{dx} (\cos^{-1}(y))$

$\frac{1}{m} \frac{1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-y^2}} (y_1)$

$\Rightarrow m \sqrt{1-y^2} = -\sqrt{1-x^2} y_1$

On squaring

$(1 - x^2)y_1^2 = m^2(1 - y^2)$

Differentiating again w.r.t x

$(1 - x^2)(2y_1y_2) + (-2x)(y_1^2) = m^2(-2yy_1)$

Taking common $2y_1$

$2y_1[(1 - x^2)y_2 - xy_1] = -2m^2yy_1$

$(1 - x^2)y_2 - xy_1 = -m^2y$

$(1 - x^2)y_2 - xy_1 + m^2y = 0$

Hence proved.

Q#6) $f y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + 2y = 0$

Sol: $y = e^x \sin x$

Differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx} (e^x \sin x)$

$\frac{dy}{dx} = (e^x)(\cos x) + (\sin x)(e^x)$

$\frac{dy}{dx} = e^x(\cos x + \sin x)$

Differentiating again w.r.t x

$\frac{d^2y}{dx^2} = (e^x)(-\sin x + \cos x) + (\cos x + \sin x)(e^x)$

$\frac{d^2y}{dx^2} = e^x(-\sin x + \cos x + \cos x + \sin x)$

$\frac{d^2y}{dx^2} = e^x(2\cos x)$

Now consider

$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + 2y$

$= (e^x(2\cos x))$

$- 2(e^x(\cos x + \sin x)) + 2(e^x \sin x)$

$= e^x(2\cos x - 2\sin x - 2\cos x + \sin x)$

$= e^x(0)$

$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + 2y = 0$

Hence proved.

Q#7) If $y = e^{ax} \sin bx$, show that $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$

Sol: $y = e^{ax} \sin bx$

differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx} (e^{ax} \sin bx)$

$\frac{dy}{dx} = (e^{ax})(\cos bx (b)) + (\sin bx)(e^{ax}(a))$

$\frac{dy}{dx} = e^{ax}(b \cos bx + a \sin bx)$

Differentiating again w.r.t x

$\frac{d^2y}{dx^2} = (e^{ax})(-b \sin bx (b) + a \cos bx (a))$

$+ (b \cos bx + a \sin bx)(e^{ax}(a))$

$\frac{d^2y}{dx^2} = e^{ax}(-b^2 \sin bx + a b \cos bx + a b \cos bx$

$+ a^2 \sin bx)$

$\frac{d^2y}{dx^2} = e^{ax}(-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx)$

Now consider

$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y$

$= (e^{ax}(-b^2 \sin bx + 2ab \cos bx$

$+ a^2 \sin bx))$

$- 2a(e^{ax}(b \cos bx + a \sin bx)) + a^2$

$+ b^2(e^{ax} \sin bx)$

$= e^{ax}(-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx$

$- 2ab \cos x - 2a^2 \sin bx + a^2 \sin bx$

$+ b^2 \sin bx)$

$= 0$

$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$

Hence proved.

Q#8) If $y = (\cos^{-1} x)^2$,

prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$

Sol:

Let $y = (\cos^{-1} x)^2$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} ((\cos^{-1} x)^2)$$

$$y_1 = 2 \cos^{-1} x \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

On squaring

$$y_1^2 = 4 (\cos^{-1} x)^2 \left(\frac{1}{1-x^2} \right)$$

$$(1-x^2)y_1^2 = 4 (\cos^{-1} x)^2$$

$$(1-x^2)y_1^2 = 4y_1$$

Differentiating again w.r.t x

$$(1-x^2)(2y_1y_2) + (-2x)(y_1^2) = 4y_1$$

Taking common $2y_1$

$$2y_1[(1-x^2)y_2 - xy_1] = 4y_1$$

$$(1-x^2)y_2 - xy_1 = 2$$

$$(1-x^2)y_2 - xy_1 - 2 = 0$$

Hence proved.

Q#9) If $y = a \cos(\ln x) + b \sin(\ln x)$, prove that

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Solution.

$$\text{Let } y = a \cos(\ln x) + b \sin(\ln x)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} (a \cos(\ln x) + b \sin(\ln x))$$

$$\frac{dy}{dx} = a \left(-\sin(\ln x) \left(\frac{1}{x} \right) \right) + b \left(\cos(\ln x) \left(\frac{1}{x} \right) \right)$$

Multiply by x

$$x \frac{dy}{dx} = -a \sin(\ln x) + b \cos(\ln x)$$

Differentiating again w.r.t x

$$\begin{aligned} (x) \left(\frac{d^2y}{dx^2} \right) + (1) \left(\frac{dy}{dx} \right) &= -a \left(\cos(\ln x) \left(\frac{1}{x} \right) \right) \\ &+ b \left(-\sin(\ln x) \left(\frac{1}{x} \right) \right) \end{aligned}$$

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = - \left(\frac{1}{x} \right) (a \cos(\ln x) + b \sin(\ln x))$$

Multiply by x

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -y$$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Hence proved

McLaurin Series: The series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(x)$$

is known as "McLaurin Series expansion."

Proof:

We know that power series of $f(x)$ in x

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n \\ &\rightarrow (i) \\ &\Rightarrow f(0) = a_0 \Rightarrow f(0) \end{aligned}$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1}$$

$$\Rightarrow f'(0) = a_1 \Rightarrow a_1 = f'(0)$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2}$$

$$f''(x) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

$$f'''(x) = 6a_3 + 24a_4x + \dots + n(n-1)(n-2)a_nx^{n-3}$$

$$\Rightarrow f'''(0) = 6a_3 \Rightarrow a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

Similarly, $\Rightarrow a_n = \frac{f^n(0)}{n!}$

Putting values of $a_1, a_2, a_3, \dots, a_n$ in (1)

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\ &+ \frac{x^n}{n!} f^n(0) \end{aligned}$$

NOTE:

A function f can be expanded in Machianrium series if the function is defined in the interval containing 0 and its derivative exist at $x = 0$ the expansion is only valid if it is converge.

The above expansion is also named as "McLaren's theorem"

Note:

McLar series for $\frac{1}{x+1}$ is the geometric series with first term 1 and common ratio x

$$\because S = \frac{a}{1-r} \text{ we have}$$

$$1 - x + x^2 - x^3 + \dots + \frac{1}{1-(-x)} = \frac{1}{1+x}$$

Taylor Series:

The series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!} h^2 + \frac{f'''(x)}{3!} h^3 + \dots + \frac{f^n(x)}{n!} h^n$$

Is known as "Taylor series expansion"

Proof:

Let the power series of $f(x)$ in $(x-a)$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \dots + a_n(x-a)^n \rightarrow (1)$$

$$\Rightarrow f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + a_3(a-a) + \dots + a_n(a-a)^n$$

$$\Rightarrow f(a) = a_0 \Rightarrow a_0 = f(a)$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \dots + na_n(x-a)^{n-1}$$

$$\Rightarrow f'(a) = a_1 \Rightarrow a_1 = f'(a)$$

$$f''(x) = 2a_2 + 6a_3(x-a) + 2a_4(x-a)^2 + \dots + n(n-1)a_n(x-a)^{n-2}$$

$$\Rightarrow f''(a) = 2a_2 \Rightarrow a_2 = \frac{f''(a)}{2!}$$

Similarly, $a_n = \frac{f^n(a)}{n!}$

Putting values of $a_1, a_2, a_3, \dots, a_n$ in (1)

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(x)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots + \frac{f^n(a)}{n!}(x - a)^n \rightarrow (2)$$

This expansion is the Taylor series for f at $x = a$.

Replace x by $x + h$ and a by x we get

$$f(x + h) = f(x) + f'(x)(x + h - x) + \frac{f''(x)}{2!}(a + h - x)^2 + \frac{f'''(x)}{3!}(x + h - x)^3 + \dots + \frac{f^n(x)}{n!}(x + h - x)^n + \dots$$

$$f(a + h) = f(x) + f'(x)(h) + \frac{f''(x)}{2!}(h)^2 + \frac{f'''(x)}{3!}(h)^3 + \dots + \frac{f^n(x)}{n!}(h)^n + \dots$$

Note:

if we put $a = 0$ in (2) then $f(x)$

$$= f(0) + f'(0)x + \frac{f'''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n$$

Which is McLaurin's series.

The above expansion is also named as Taylor theorem

Exercise 2.8

Question # 1. Apply the Maclaurin series expansion to prove that:

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{4} + \dots$

Solution. Let $f(x) = \ln(x + 1)$

$$f(0) = \ln(0 + 1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x + 1}$$

$$f'(0) = \frac{1}{0 + 1} = 1$$

$$f''(x) = \frac{-1}{(x + 1)^2}$$

$$f''(0) = \frac{-1}{(0 + 1)^2} = -1$$

$$f'''(x) = \frac{2}{(x + 1)^3}$$

$$f'''(0) = \frac{2}{(0 + 1)^3} = 2$$

$$f^{(4)}(x) = \frac{-6}{(x + 1)^4}$$

$$f^{(4)}(0) = \frac{-6}{(0 + 1)^4} = -6$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \dots$$

$$\ln(x + 1) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots$$

$$\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Hence Proved.

(ii) $\text{Cos}x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Solution. Let $f(x) = \text{Cos}x$

$$f(0) = \text{Cos}0 = 0$$

$$f'(x) = -\text{Sin}x$$

$$f'(0) = -\text{Sin}0 = 0$$

$$f''(x) = -\text{Cos}x$$

$$f''(0) = -\text{Cos}0 = -1$$

$$f'''(x) = \text{Sin}x$$

$$f'''(0) = \text{Sin}0 = 0$$

$$f^{(4)}(x) = \text{Cos}x$$

$$f^{(4)}(0) = \text{Cos}0 = 1$$

$$f^{(5)}(x) = -\text{Sin}x$$

$$f^{(5)}(0) = -\text{Sin}0 = 0$$

$$f^{(6)}(x) = -\text{Cos}x$$

$$f^{(6)}(0) = -\text{Cos}0 = -1$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + \frac{x^6}{6!}f^{(6)}(0) + \dots$$

$$\text{Cos}x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \dots$$

$$\text{Cos}x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Hence Proved.

(iii) $\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$

Solution. Let $f(x) = \sqrt{1 + x}$

$$f(0) = \sqrt{1 + 0} = 1$$

$$f'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}}$$

$$f'(0) = \frac{1}{2}(1 + 0)^{-\frac{1}{2}} = \frac{1}{2}$$

$$f''(x) = \frac{1}{2} \cdot \frac{-1}{2}(1 + x)^{-\frac{3}{2}}$$

$$f''(0) = \frac{1}{2} \cdot \frac{-1}{2}(1 + 0)^{-\frac{3}{2}} = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}(1 + x)^{-\frac{5}{2}}$$

$$f'''(0) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}(1 + 0)^{-\frac{5}{2}} = \frac{3}{8}$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$\sqrt{1+x} = 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\left(\frac{1}{8}\right) + \dots$$

$$\cos x = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \dots$$

Hence Proved.

(iv) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Solution. Let $f(x) = e^x$

$$f(0) = e^0 = 1$$

$$f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

$$f''(x) = e^x$$

$$f''(0) = e^0 = 1$$

$$f'''(x) = e^x$$

$$f'''(0) = e^0 = 1$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

Hence Proved.

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$

Solution. Let $f(x) = e^{2x}$

$$f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x}$$

$$f'(0) = 2e^0 = 2$$

$$f''(x) = 4e^{2x}$$

$$f''(0) = 4e^0 = 4$$

$$f'''(x) = 8e^{2x}$$

$$f'''(0) = 8e^0 = 8$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \dots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} \dots$$

Hence Proved.

Question # 2 Show that $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \dots$

And evaluate $\cos 61^\circ$.

Solution.

Let $f(x+h) = \cos(x+h)$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

By Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x)$$

$$+ \frac{h^3}{3!}(\sin x) + \dots$$

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x$$

+ ...
Hence Proved.

Now, Put $x = 60^\circ$ and $h = 1^\circ = \frac{\pi}{180}$ rad

$$= 0.01745 \text{ rad}$$

$$\cos(60^\circ + 1^\circ) = \cos 60^\circ - (0.01745)\sin 60^\circ - \frac{(0.01745)^2}{2!}\cos 60^\circ + \frac{(0.01745)^3}{3!}\sin 60^\circ + \dots$$

$$\cos(61^\circ) = 0.5 - (0.01745)(0.866) - \frac{(0.01745)^2}{2!}(0.5)$$

$$+ \frac{(0.01745)^3}{3!}(0.866) + \dots$$

$$\cos(61^\circ) = 0.5 - 0.0151117 - 0.000076125 + 0.000000072 + \dots$$

$$\cos(61^\circ) = 0.484812247 \approx 0.4848$$

Which is required.

Question # 3 Show that $2^{x+h} = 2^x \left[1 + (\ln 2)h + (\ln 2)^2 \frac{h^2}{2!} + (\ln 2)^3 \frac{h^3}{3!} + \dots \right]$

Solution.

Let $f(x+h) = 2^{x+h}$

$$f(x) = 2^x$$

$$f'(x) = 2^x \cdot \ln 2$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

By Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x)$$

$$+ \frac{h^3}{3!}(\sin x) + \dots$$

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x$$

+ ...
Hence Proved

Geometrical meaning of a derivative:

Let us draw a curve of the function $y = f(x)$ take two neighboring points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the curve. Draw a tangent line to the curve at pt.

$p(x, y)$ Such that it makes an angle ϕ with $x - axis$.

Also Draw a secant line to the curve passing through the pts. $p(x, y)$ and $Q(x + \delta x, y + \delta y)$. the secant line makes angle ϕ with $x - axis$.

Draw $\perp PM$ and QN from P and Q on $x - axis$. Also

draw perpendicular PR on QN . in ΔPQR

$$\tan \phi = \frac{|QR|}{|PR|} \quad |QR| = \delta y \quad |PR| = \delta x$$

$$\Rightarrow \tan \phi = \frac{\delta y}{\delta x}$$

When $Q \rightarrow P$ then $\delta x \rightarrow 0$ $\delta y \rightarrow 0$

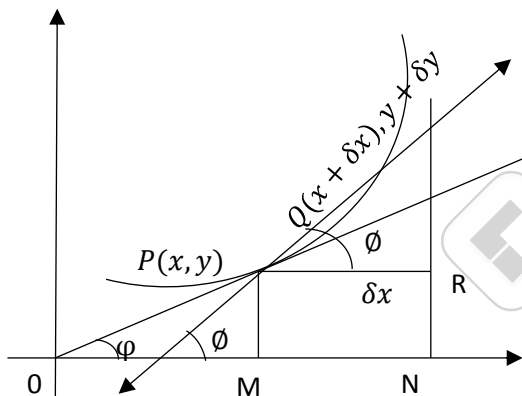
$\phi \rightarrow \phi$ it means secant line becomes tangent line.

Now,

$$\lim_{\phi \rightarrow \phi} \tan \phi = \lim_{\phi \rightarrow \phi} \frac{\delta y}{\delta x} \Rightarrow \tan \phi = \frac{dy}{dx}$$

But $\tan \phi$ is slope of tangent line. Thus geometrically derivative of

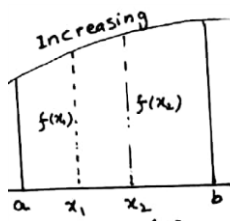
y at x represent the slope of tangent line at $p(x, y)$



Increasing and decreasing function:

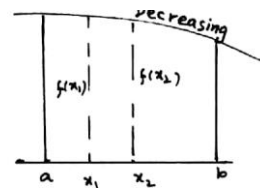
Increasing functions:

A function f is said to be increasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then $f(x_1) < f(x_2)$ whenever $x_1 < x_2$



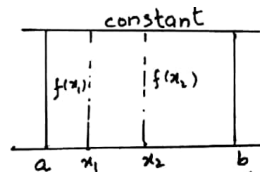
Decreasing function:

A function f is said to be decreasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then $f(x_1) > f(x_2)$ whenever $x_1 < x_2$



Constant function:

A function f is said to be constant function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then $f(x_1) = f(x_2)$ whenever $x_1 < x_2$



Alternative definitions:

Increasing functions:

A differentiable function f is said to be increasing function if $f'(x) > 0 \forall x \in (a, b)$

Decreasing function:

A differentiable function f is said to be decreasing function if $f'(x) < 0 \forall x \in (a, b)$

Constant function:

A differentiable function f is said to be constant function if $f'(x) = 0 \forall x \in (a, b)$

Stationary point:

Any point where f is neither increasing nor decreasing is called stationary point.

Provided that $f'(x) = 0$ at that point.

Relative Maxima:

A function $f(x)$ has relative maxima $f(c)$ at $x = c \in (a, b)$ if

$$(i) f(c - \delta x) > 0 \quad (ii) f'(c) = 0 \quad (iii) f'(c + \delta x) < 0$$

where δx is small + ve is small number.

Relative Minima:

A function $f(x)$ has relative minima $f(c)$ at $x = c \in (a, b)$ if

$$(i) f(c - \delta x) < 0 \quad (ii) f'(c) = 0 \quad (iii) f'(c + \delta x) > 0$$

where δx is small + ve is small + ve number.

Alternative definitions.

Relative maxima:

A function $f(x)$ has relative maxima $f(c)$ if $f'(x)$ changes sign from + ve to - ve as x change through C .

Relative minima:

A function $f(x)$ has relative minima $f(c)$ if $f'(x)$ changes sign from - ve to + ve as x change through C

Relative Extrema:

Both relative maxima and relative minima are called in general "relative extreme"

Critical Valves and critical points:

if $c \in D_f$ and $f'(x) = 0$ or $f'(c)$ does not exist, Then the number c is called a critical value for f while the point $(c, f(x))$ on the graph of f is Named as a critical as a critical point.

Important note:

There are functions which have extrema (maxima or minima) at the points where their derivatives do not exist. For example

$$f(x) = |x| \quad \text{and} \quad \phi(x) = \begin{cases} 2 - x & , x > 0 \\ x + 2, & x \leq 0 \end{cases}$$

do not exist at $(0, 0)$ and $(0, 2)$ resp.

But

f has minima at $(0, 0)$ and ϕ has relative maxima at $(0, 2)$

First derivative:

Let

f be differentiable in neighbourhood of C where $f'(c) = 0$

1. A function $f(x)$ has relative maxima at $x = c \in$

(a, b) if $f'(x)$ has relative maxima at $x = f'(c - \delta x) < 0$

2. A function $f(x)$ has relative minima at $x = c \in (a, b)$ if $f'(c - \delta x) < 0, f'(c) = 0$ and

$$f'(c + \delta x) > 0$$

Second derivative:

Let

f be differentiable in neighbourhood of C where $f'(c) = 0$

Then,

1. f has relative maxima at c if $f''(c) < 0$

2. f has relative minima at c if $f''(c) > 0$

Turning point:

A stationary point is called a turning point or a minimum point.

Point of inflexion:

A point at which the function has neither maximum nor minimum value is called point of inflexion.

Procedure for finding maxima and minima of a function:

Let $y = f(x)$ be given function

Step1. Find $\frac{dy}{dx}$ or $f'(x)$

Step11: put $\frac{dy}{dx} = 0$ $f'(x) = 0$

stepIII find $\frac{d^2y}{dx^2}$ or $f''(x)$ at the roots.

StepIV check the sign of $f''(x)$ at the roots

*if $f''(x) > 0$ then $f(x)$ has minimum value

*if $f''(x) < 0$ then $f(x)$ has maximum value.

*if $f''(x) = 0$

then no information about the function then use first derivative rule"

Exercise 2.9**Question No.1**

Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case

$$f(x) = \text{Sin}x \quad ; x \in [-\pi, \pi]$$

Solution.

Given that

$$f(x) = \text{Sin}x \quad ; x \in [-\pi, \pi]$$

$$f'(x) = \text{Cos}x$$

$$\text{Cos}x = 0$$

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

So we have Sub-intervals

$$\left(-\pi, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$$

$$f'(x) = \text{Cos}x < 0 \text{ whenever } x \in \left(-\pi, -\frac{\pi}{2}\right)$$

So $f(x)$ is decreasing in the interval $\left(-\pi, -\frac{\pi}{2}\right)$

$$f'(x) = \text{Cos}x > 0 \text{ whenever } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

So $f(x)$ is increasing in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f'(x) = \text{Cos}x < 0 \text{ whenever } x \in \left(\frac{\pi}{2}, \pi\right)$$

So $f(x)$ is decreasing in the interval $\left(\frac{\pi}{2}, \pi\right)$

$$\text{(ii)} \quad f(x) = \text{Cos}x \quad ; x \in$$

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Solution.

Given that

$$f(x) = \text{Cos}x \quad ; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$f'(x) = -\text{Sin}x$$

$$\text{Put } f'(x) = 0$$

$$-\text{Sin}x = 0$$

$$x = 0$$

So we have Sub-intervals $\left(-\frac{\pi}{2}, 0\right), \left(0, \frac{\pi}{2}\right)$

$$f'(x) = -\text{Sin}x > 0 \text{ whenever } x \in \left(-\frac{\pi}{2}, 0\right)$$

So $f(x)$ is increasing in the interval $\left(-\frac{\pi}{2}, 0\right)$

$$f'(x) = -\text{Sin}x < 0 \text{ whenever } x \in \left(0, \frac{\pi}{2}\right)$$

So $f(x)$ is decreasing in the interval $\left(0, \frac{\pi}{2}\right)$.

$$\text{(iii)} \quad f(x) = 4 - x^2 \quad ; x \in [-2, 2]$$

Solution.

Given that

$$f(x) = 4 - x^2 \quad ; x \in [-2, 2]$$

$$f'(x) = -2x$$

$$\text{Put } f'(x) = 0$$

$$-2x = 0$$

$$x = 0$$

So we have Sub-intervals $(-2, 0)$, $(0, 2)$

$$f'(x) = -2x > 0 \text{ whenever } x \in (-2, 0)$$

So $f(x)$ is increasing in the interval $(-2, 0)$

$$f'(x) = -2x < 0 \text{ whenever } x \in (0, 2)$$

So $f(x)$ is decreasing in the interval $(0, 2)$.

$$(iv) \quad f(x) = x^2 + 3x + 2 \quad ; x \in [-4, 1]$$

Solution.

$$\Rightarrow f'(x) = 2x + 3$$

$$\text{Put } f'(x) = 0 \Rightarrow 2x + 3 = 0$$

$$\Rightarrow x = -\frac{3}{2}$$

$$\because 2x + 3 \text{ is +ve in } \left(-\frac{3}{2}, 1\right)$$

So $f(x)$ is increasing on the interval $\left(-\frac{3}{2}, 1\right)$

$$\because 2x + 3 \text{ is}$$

$$-ve \text{ in } \left(-4, -\frac{3}{2}\right) \text{ so } f(x) \text{ is decreasing}$$

On the interval $\left(-4, -\frac{3}{2}\right)$

Question. 2.

Find the extreme values of the following functions defined as

$$(i) \quad f(x) = 1 - x^3$$

Solution.

Given that

$$f(x) = 1 - x^3$$

Differentiate w.r.t x

$$f'(x) = -3x^2 \text{ --- (i)}$$

For stationary points. Put $f'(x) = 0$

$$-3x^2 = 0$$

$$x = 0$$

Differentiate (i) w.r.t x , we have

$$f''(x) = -6x \text{ --- (ii)}$$

Now put $x = 0$ in (ii)

$$f''(0) = -6(0) = 0$$

So second derivative test fails to determine the extreme points

Put $x = 0 - \varepsilon$ in (i)

$$f'(x) = -3(-\varepsilon)^2 = -3\varepsilon^2 < 0$$

Put $x = 0 + \varepsilon$ in (i)

$$f'(x) = -3(-\varepsilon)^2 = -3\varepsilon^2 < 0$$

As $f'(x)$ does not change its sign before and after $x = 0$.

Since at $x = 0$, $f(x) = 1$ therefore $(0, 1)$ is the point inflection.

$$(ii) \quad f(x) = x^2 - x - 2$$

Solution.

Given that

$$f(x) = x^2 - x - 2$$

Differentiate w.r.t x

$$f'(x) = 2x - 1 \rightarrow (i)$$

For stationary points, Put $f'(x) = 0$

$$2x - 1 = 0$$

$$x = \frac{1}{2}$$

Differentiate (i) w.r.t x , we have

$$f''(x) = 2 \text{ --- (ii)}$$

Now put $x = \frac{1}{2}$ in (ii)

$$f''\left(\frac{1}{2}\right) = 2$$

So second derivative test $f(x)$ is minimum at $x = \frac{1}{2}$.

$$\text{Now } f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$$

$$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$$

$$(iii) \quad f(x) = 5x^2 - 6x + 2$$

Solution.

Given that

$$f(x) = 5x^2 - 6x + 2$$

Differentiate w.r.t x

$$f'(x) = 10x - 6 \text{ --- (i)}$$

For stationary points. Put $f'(x) = 0$

$$10x - 6 = 0$$

$$x = \frac{3}{5}$$

Differentiate (i) w.r.t x , we have

$$f''(x) = 10 \text{ --- (ii)}$$

Now put $x = \frac{3}{5}$ in (ii)

$$f''\left(\frac{3}{5}\right) = 10$$

So second derivative test $f(x)$ is minimum at $x = \frac{3}{5}$.

Now

$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$

$$f\left(\frac{3}{5}\right) = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$$

$$(iv) \quad f(x) = 3x^2$$

Solution.

Given that

$$f(x) = 3x^2$$

Differentiate w.r.t x

$$f'(x) = 6x \text{ --- (i)}$$

For stationary points. Put $f'(x) = 0$

$$6x = 0$$

$$x = 0$$

Differentiate (i) w.r.t x , we have

$$f''(x) = 6 \text{ --- (ii)}$$

Now put $x = 0$ in (ii)

$$f''(0) = 6$$

So second derivative test $f(x)$ is minimum at $x = 0$.

Now

$$f(0) = 3(0)^2$$

$$f(0) = 0.$$

$$(v) \quad f(x) = 3x^2 - 4x + 5$$

Solution.

$$\Rightarrow f'(x) = 6x - 4 \Rightarrow f''(x) = 6 \rightarrow (i)$$

$$\text{Put } f'(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0 \text{ put in (i)}$$

$$\Rightarrow f''(0) = 6 > 0 \text{ so } f(x) \text{ has minimum value at } x = 0 \text{ and } f(0) = 3(0)^2 = 0$$

$$(vi) \quad f(x) = 2x^3 - 2x^2 - 36x + 3$$

Solution.

Given that

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

Differentiate w.r.t "x"

$$f'(x) = 6x^2 - 4x - 36 \dots (i)$$

For stationary points . Put $f'(x) = 0$

$$6x^2 - 4x - 36 = 0$$

$$3x^2 - 2x - 18 = 0$$

$$x = \frac{2 \pm \sqrt{(2)^2 - 4(3)(-18)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 216}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$x = \frac{2 \pm 2\sqrt{55}}{6}$$

$$x = \frac{1 \pm \sqrt{55}}{3}$$

Differentiate (i) w.r.t "x", we have

$$f''(x) = 12x - 4 \rightarrow (ii)$$

Now put $x = \frac{1+\sqrt{55}}{3}$ in (ii)

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4(1+\sqrt{55}) - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4 + 4\sqrt{55} - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4\sqrt{55} > 0$$

So second derivative test $f(x)$ is minimum at $x =$

$$\frac{1+\sqrt{55}}{3}.$$

Now

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(1+\sqrt{55})^3 - \frac{2}{9}(1+\sqrt{55})^2 - \frac{36}{3}(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(1+3\sqrt{55}+3.55+55\sqrt{55}) - \frac{2}{9}(1+2\sqrt{55}+55) - 12(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(166+58\sqrt{55}) - \frac{2}{9}(56+2\sqrt{55}) - 12(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{332}{27} + \frac{116}{27}\sqrt{55} - \frac{112}{9} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}(247+220\sqrt{55})$$

Now put $x = \frac{1-\sqrt{55}}{3}$ in (ii)

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right) - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 4(1-\sqrt{55}) - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 4 - 4\sqrt{55} - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = -4\sqrt{55} < 0$$

So second derivative test $f(x)$ is maximum at

$$x = \frac{1-\sqrt{55}}{3}.$$

Now

$$f\left(\frac{1-\sqrt{55}}{3}\right) = 2\left(\frac{1-\sqrt{55}}{3}\right)^3 - 2\left(\frac{1-\sqrt{55}}{3}\right)^2 - 36\left(\frac{1-\sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{2}{27}(1-\sqrt{55})^3 - \frac{2}{9}(1-\sqrt{55})^2 - \frac{36}{3}(1-\sqrt{55}) + 3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{2}{27}(1-3\sqrt{55}+3.55-5\sqrt{55}) - \frac{2}{9}(1-2\sqrt{55}+55) - 12(1-\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(166-58\sqrt{55}) - \frac{2}{9}(56-2\sqrt{55}) - 12(1-\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{332}{27} - \frac{116}{27}\sqrt{55} - \frac{112}{9} + \frac{4}{9}\sqrt{55} - 12 + 12\sqrt{55} + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}(247-220\sqrt{55})$$

(vii) $f(x) = x^4 - 4x^2$

Solution:

$\Rightarrow f'(x) = 4x^3 - 8x \Rightarrow f''(x) = 12x^2 - 8$
 $\rightarrow (i)$

Put $f'(x) = 0$ so $4x^3 - 8x = 0$
 $\Rightarrow 4x(x^2 - 2) = 0$

$\Rightarrow 4x(x^2 - 2) =$

$\Rightarrow 4x = 0 \quad x^2 - 2 = 0$

$\Rightarrow x = 0 \quad x = \pm\sqrt{2}$

\Rightarrow thus $x = 0, x = \sqrt{2}, x = -\sqrt{2}$

Put $x = 0$ in (1)

$\Rightarrow f''(0) = 12(0)^2 - 8 = -8 < 0$

\Rightarrow So $f(x)$ has maximum value at $x = 0$ and

$\Rightarrow f(0) = (0)^4 - 4(0)^2 = 0$

Put $x = \sqrt{2}$ in (1)

$\Rightarrow f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 12(2) - 8 = 16 > 0$

\Rightarrow So $f(x)$ has minimum value at $x = \sqrt{2}$ and

$f(\sqrt{2}) = 12(\sqrt{2})^2$

$= [(\sqrt{2})^2]^2 - 4(2) = (2)^2 - 8 = 4 - 8 = -4 < 0$

Put $x = -\sqrt{2}$ in (1)

$f''(-\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 12(2) - 8 = 24 - 8 = 16 > 0$

so $f(x)$ has minimum value at

$x = -\sqrt{2}$ ans $f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2$

$[(\sqrt{2})^2]^2 - 4(2)$

$= (2)^2 - 8 = 4 - 8 = -4$

(viii) $f(x) = x^4 - 4x^2$

Solution.

$\Rightarrow f'(x) = \frac{d}{dx} \{(x-2)^2(x-1)\}$
 $= (x-2) \frac{d}{dx} (x-1) + (x-1) \frac{d}{dx} (x-2)^2$
 $= (x-2)^2 (1(x-1)) + 2(x-2) \frac{d}{dx} (x-2)$

$= (x-2)^2 + 2(x-1)(x-2)(1)$

$f'(x) = (x-2) + 2(x-1)(x-2)$

$= (x-2)\{x-2+2x-2\}$

$f'(x) = (x-2)(3x-4)$

$f''(x) = \frac{d}{dx} \{(x-2)(3x-4)\}$

$= (x-2) \frac{d}{dx} (3x-4)$

$+ (3x-4) \frac{d}{dx} (x-2)$

$(x-2)(3) + (3x-4)(1)$

$= 3x - 6 + 3x - 4$

$= 6x - 10$

Put $f'(x) = 0$, so $(x-2)(3x-4) = 0$

$x-2 = 0 \quad 3x-4 = 0$

$x = 2 \quad x = \frac{4}{3}$

Put $x = 2$ in (1) $f''(2) = 6(2) - 10 = 12 - 10 = 2 > 0$

so $f(x)$ has minimum value at $x = 2$ and

$f(2) = (2-2)^2(2-1) = 0$

put $x = \frac{4}{3}$ in (1) $f''(\frac{4}{3}) = 6(\frac{4}{3}) - 10 = 8 - 10 = -2 < 0$

so $f(x)$ has maximum value at $x = \frac{4}{3}$ and

$f(\frac{4}{3}) = (\frac{4}{3}-2)^2(\frac{4}{3}-1) = (\frac{4-6}{3})^2(\frac{4-3}{3})$

$= (\frac{-2}{3})^2(\frac{1}{3}) = \frac{4}{27}$

(ix) $f(x) = 5 + 3x - x^3$

$\Rightarrow f'(x) = 3 - 3x^2$

$\Rightarrow f''(x) = -6x \rightarrow (1)$

put $f'(x) = 0, 3 - 3x^2 = 0$

$\Rightarrow 3(1 - x^2) = 0 \quad 3 \neq 0$ so $1 - x^2 = 0$

$\Rightarrow 1 = x^2$

$\Rightarrow x = \pm 1$

Put $x = 1$ in (i) has maximum value at x

$= 1$ and $f(1) = 5 + 3(1) - (1)^3$

$= 5 + 3 - 1 = 7$

put $x = -1$ in (i) $f''(-1) = -6(-1) = 6 > 0$

So $f(x)$ has minimum value at $x = 1$

and $f(-1) = 5 + 3(-1) - (-1)^3 = (5 - 3 - (-1))$

$f(-1) = 2 + 1 = 3$

Question.3.

Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$

$f(x) = \sin x + \cos x$

Solution. Given function

$f(x) = \sin x + \cos x \quad ; x \in [0, 2\pi]$

Differentiate with respect to "x".

$f'(x) = \cos x - \sin x \dots (i)$

For Stationary points, Put $f'(x) = 0$

$\cos x - \sin x = 0$

$-\sin x = -\cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \tan^{-1} 1 = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ when $x \in [0, 2\pi]$

Now diff. (i) w.r.t "x"

$f''(x) = -\sin x - \cos x$

Now put $x = \frac{\pi}{4}$ in (ii)

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$$

So second derivative test $f(x)$ is Maximum at $x = \frac{\pi}{4}$.

Now

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Now put $x = \frac{5\pi}{4}$ in (ii)

$$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) = -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)$$

$$f''\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0$$

So second derivative test $f(x)$ is Minimum at $x = \frac{5\pi}{4}$.

Now

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right)$$

$$f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

Question.4.

Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$.

Solution. Given that

$$y = \frac{\ln x}{x}$$

Diff. w.r.t. " x ", we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\ln x}{x}\right) \\ \frac{dy}{dx} &= \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \\ \frac{dy}{dx} &= \frac{1 - \ln x}{x^2} \end{aligned} \quad \dots (i)$$

For critical points. Put $\frac{dy}{dx} = 0$

$$\begin{aligned} \frac{1 - \ln x}{x^2} &= 0 \\ 1 - \ln x &= 0 \\ \ln x &= 1 \\ \ln x &= \ln e \\ x &= e \end{aligned}$$

Now diff. (i) with respect to " x ", we have

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1 - \ln x}{x^2}\right) \\ \frac{d^2y}{dx^2} &= \frac{x^2 \cdot \left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} \\ \frac{d^2y}{dx^2} &= \frac{-x - 2x + 2x \ln x}{x^4} \\ \frac{d^2y}{dx^2} &= \frac{-3x + 2x \ln x}{x^4} \end{aligned}$$

At $x = e$, we have

$$\frac{d^2y}{dx^2}\bigg|_{x=e} = \frac{-3(e) + 2(e)\ln(e)}{(e)^4}$$

$$\frac{d^2y}{dx^2}\bigg|_{x=e} = \frac{-3e + 2e}{e^4}$$

$$\frac{d^2y}{dx^2}\bigg|_{x=e} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

Hence y has a maximum value at $x = e$.

Question.5.

Show that $y = x^x$ has maximum value at $x = \frac{1}{e}$.

Solution. Given that

$$y = x^x$$

Taking Log on both sides

$$\ln y = \ln x^x = x \ln x$$

Diff. w.r.t. " x ", we have

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \left(\frac{1}{x}\right) + \ln x \cdot 1$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\frac{dy}{dx} = x^x(1 + \ln x) \quad \dots (i).$$

For critical points, Put $\frac{dy}{dx} = 0$

$$x^x(1 + \ln x) = 0$$

As $x^x \neq 0$ then, $1 + \ln x = 0$

$$1 + \ln x = 0$$

$$\ln x = -1$$

$$\ln x = -\ln e$$

$$\ln x = \ln e^{-1}$$

$$x = e^{-1}$$

$$x = \frac{1}{e}$$

Now diff. (i) with respect to " x ", we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^x(1 + \ln x))$$

$$\frac{d^2y}{dx^2} = (1 + \ln x) \frac{d}{dx}x^x + x^x \frac{d}{dx}(1 + \ln x)$$

$$\frac{d^2y}{dx^2} = (1 + \ln x)x^x(1 + \ln x) + x^x \left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = x^x \left[(1 + \ln x)^2 + \frac{1}{x} \right]$$

$$\frac{d^2y}{dx^2} = x^x \left[\frac{x(1 + \ln x)^2 + 1}{x} \right]$$

At $x = \frac{1}{e}$, we have

$$\frac{d^2y}{dx^2}\bigg|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[\frac{\frac{1}{e} \left(1 + \ln\left(\frac{1}{e}\right)\right)^2 + 1}{\frac{1}{e}} \right]$$

$$\frac{d^2y}{dx^2} > 0$$

Hence y has a minimum value at $x = \frac{1}{e}$.

Application of maxima and minima:

1. If we first from the relation of the form $y = f(x)$ from the given information.

2. Find the maximum or minimum value of f as required by using 1st or 2nd derivative value.

*mostly we will use 2nd derivative rule.

Exercise 2.10

Question No.1. find two integers whose sum is 20 and their product will be maximum.

Solution:

Let x and y be req. integer's product $f(x) = xy$

By given condition $x + y = 30$

$$\Rightarrow y = 30 - x \rightarrow (i)$$

$$\text{So, } f(x) = x(30 - x) = 30x - x^2$$

$$\Rightarrow f'(x) = 30 - 2x$$

$$\Rightarrow f''(x) = -2 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \quad 30 - 2x = 0$$

$$\Rightarrow 2x = 30 \Rightarrow x = 15 \text{ put in (i)}$$

$$\Rightarrow f''(15) = -2 <$$

0 thus $f(x)$ has maximum value

$$\text{At } x = 15 \text{ and put in (i)} \Rightarrow y = 30 - 15 = 15$$

So req. integer are 15 and 15.

Question No: 2 divide 20 into two parts so that the sum of their squares will be minimum.

Solution:

let x and y be req. two parts of 20. so $x + y = 20$

$$\Rightarrow y = 20 - x \rightarrow (i)$$

by given condition $f(x) = x^2 + y^2$

$$\Rightarrow f(x) = x^2 + (20 - x)^2 = x^2 + 400 + x^2 - 40x$$

$$\Rightarrow f'(x) = 4x - 40 \Rightarrow f''(x) = 4 \rightarrow (ii)$$

$$\Rightarrow \text{Put } f'(x) = 0 \quad 4x - 40 = 0$$

$$x - 10 = 0 \Rightarrow x = 10 \text{ put in (i)}$$

$$f''(10) = 4 > 0 \text{ so } f(x) \text{ has minimum value at } x = 10$$

$$\text{and put in (i)} y = 20 - 10 = 10$$

So req. two parts of 20 are 10 and 10.

Q3. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution:

Let x and y be req. integers by given condition;

$$x + y = 12 \quad y = 12 - x \rightarrow (i)$$

$$\text{Product} = f(x) = x^2y = x^2(12 - x)$$

$$\Rightarrow f(x) = 12x^2 - x^3$$

$$\Rightarrow f'(x) = 24x - 3x^2 \Rightarrow f''(x) = 24 - 6x \rightarrow (ii)$$

$$\text{put } f'(x) = 0 \Rightarrow 24x - 30x^2 = 0 \Rightarrow 2(8 - x) = 0$$

$$\Rightarrow x = 0 \text{ and } x = 8$$

$$\text{Put } x = 8 \text{ in (ii)} \Rightarrow f''(8) = 24 - 48 < 0$$

So $f(x)$ has minimum value at $x = 0$

which is not required

$$\text{So Put } x = 8 \text{ in (i)} f''(8) = 24 - 48 = -24 < 0$$

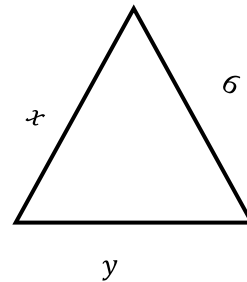
$$f''(8) = -24 < 0$$

So $f(x)$ has maximum value at $x = 8$ and put in (i)

$$y = 12 - 8 = 4 \text{ so required are 8 and 4.}$$

Question No.4 The perimeter of a triangle is 16cm. if one side of the other sides for maximum area of the triangle?

Solution:



Let x and y be other sides.

$$\because \text{perimeter} = x + y + 6 (\text{sum of all sides})$$

$$\Rightarrow 16 = x + y + 6$$

$$\Rightarrow 16 - 6 = x + y$$

$$\Rightarrow x + y = 10$$

$$\Rightarrow \text{And } y = 10 - x \rightarrow (i)$$

$$\because S = \frac{x + y + 6}{2} = \frac{16}{2} \text{ or } S = 8$$

Also area $A^2 = f(x) = s(s - x)(s - y)(s - 6)$

$$\Rightarrow f(x) = 8(8 - x)(8 - y)(8 - 6)$$

$$\Rightarrow f(x) = 16(8 - x)(8 - y)$$

$$= 16(8 - x)(8 - (10 - x))$$

$$f(x) = (8 - x)(x - 2)$$

$$f'(x) = 16 \frac{d}{dx} \{(8 - x)(x - 2)\}$$

$$= 16 \{(8 - x)(1) + (x - 2)(-1)\}$$

$$= 16(8 - x - x + 2) = 16(10 - 2x)$$

$$f'(x) = 160 - 32x = 0$$

$$f''(x) = -32 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 160 - 32x = 0$$

$$\Rightarrow 32x = 160 \Rightarrow x = 5$$

$$\text{put } x = 5 \text{ in (i)} \Rightarrow f''(5) = -32 < 0$$

Thus $f(x)$ has maximum value at $x = 5$

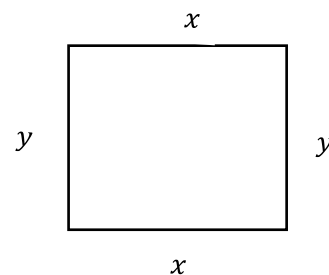
and put in (i)

$$y = 10 - 5 = 5$$

Hence req. sides are 5cm and 5cm.

Question No.5 find the dimension of a rectangular of largest area having perimeter 12cm.

Solution:



Let x and y be required dimensions.

$$\text{Perimeter} = x + y + x + y (\text{sum of all sides})$$

$$\Rightarrow 12 = 2x + 2y$$

$$\Rightarrow x + y = 60$$

$$\text{Andy} = 60 - x \rightarrow (i)$$

\therefore Area of $f(x) = xy$ rectangle

$$\Rightarrow f(x) = x(60 - x)$$

$$f(x) = 60x - x^2$$

$$\Rightarrow f'(x) = 60 - 2x$$

$$\Rightarrow f''(x) = -2 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \text{ so } 60 - 2x = 0$$

$$f''(x) = -2 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \text{ so } 60 - 2x = 0$$

$$2x = 60 \Rightarrow x = \frac{60}{2} = 30 \text{ put in (ii)}$$

$$\Rightarrow f''(30) = -2 < 0$$

so $f(x)$ has maximum (largest) value at $x = 30$ and

$$\text{put in (i)} y = 60 - 30 = 30$$

thus required dimensions are 30cm^2 where its

perimeter is minimum.

Question NO.6 find the length of sides of a variable rectangular having area 36cm^2 where its perimeter is minimum.

Solution:

let x and y be required length

$$\text{Perimeter} = x + y + x + y$$

$$\Rightarrow f(x) = 2x + 2y$$

$$\Rightarrow f(x) = 2(x + y) \rightarrow (i)$$

$$\text{area of rectangle} = xy$$

$$\Rightarrow 36 = xy \Rightarrow y = \frac{36}{x}$$

$$\text{so (i)} \Rightarrow f(x) = 2\left(x + \frac{36}{x}\right)$$

$$f(x) = 2x + 72x^{-1}$$

$$f'(x) = 2 - \frac{72}{x^2}, \quad f''(x) = -72(-2)x^{-3}$$

$$f''(x) = \frac{144}{x^3} \rightarrow (iii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 2 - \frac{72}{x^2} = 0$$

$$2 = \frac{72}{x^2} \Rightarrow x^2 - 36 \Rightarrow x = \pm 6$$

$$x = 6 \quad x = -6 \text{ (rejected length is always positive)}$$

$$\Rightarrow \text{Put } x = 6 \text{ in (iii)} f''(6) = \frac{144}{(6)^3} = \frac{144}{216} > 0$$

So $f(x)$ has minimum value at $x = 6$ and put in (i)

$$\Rightarrow y = \frac{36}{6} = 6$$

So sides of rectangle are 6cm and 6cm .

Question No.7 A box with a square base and open top is to have a volume of 4 cube dm. find the dimensions of the box which will require the least material.

Solution:

Let length, width, and height of box are x, x and y respectively. So

$$\text{Volume} = x \times x \times y = x^2 y \rightarrow (i)$$

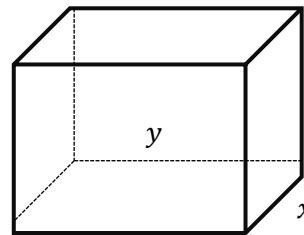
$$\Rightarrow 4 = x^2 y \Rightarrow y = \frac{4}{x^2}$$

\therefore surface area of the box =

(area of the square + area of 4 vertical width)

$$\Rightarrow f(x) = x \times x \times 4(x \times y)$$

$$\Rightarrow f(x) = x^2 = 4xy$$



$$x^2 + 4x\left(\frac{4}{x^2}\right) \quad \therefore y = \frac{4}{x^2}$$

$$f(x) = x^2 + 16x^{-1} \Rightarrow f'(x) = 2x + 16(-1)x^{-2}$$

$$f'(x) = 2x - \frac{16}{x^2}$$

$$\Rightarrow f''(x) = 2 + 16(-1)(-2)x^{-3}$$

$$f''(x) = 2 + \frac{32}{x^3} \rightarrow (ii)$$

$$\text{put } f'(x) = 0 \text{ so } 2x - \frac{16}{x^2} = 0 \Rightarrow \frac{2x^3 - 16}{x^2} = 0$$

$$\Rightarrow 2(x^3 - 8) = 0 \Rightarrow (x^3 - (2)^3) = 0$$

$$\text{Put in (ii)} f''(x) = 2 + \frac{32}{(2)^2} > 0$$

$$\Rightarrow f(x) \text{ has minimum (least) value at } x = 2 \text{ and}$$

$$\text{put in (i)} \Rightarrow y = \frac{4}{4} = 1$$

Hence

$2\text{dm}, 2\text{dm}$ and 1dm is the dimension of box

Question No.8 find the dimensions of a rectangular garden having perimeter 80m . if its area is to be maximum.

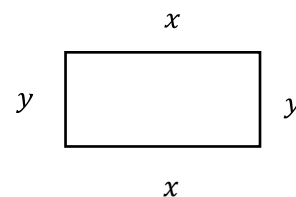
Solution: Let x and y be required dimensions.

$$\text{perimeter} = x + y + x + y$$

$$80 = 2x + 2y \Rightarrow x + y = 40$$

$$\Rightarrow y = 40 - x \rightarrow (i)$$

$$\text{area of rectangular} = f(x) = xy$$



$$f(x) = xy = x(40 - x) = 40x - x^2$$

$$f'(x) = 40 - 2x \Rightarrow f''(x) = -2 \rightarrow (ii)$$

$$\text{put } f'(x) = 0 \text{ so } 40 - 2x = 0 \Rightarrow 2x = 40$$

$$\Rightarrow x = 20 \text{ put in (i)} \Rightarrow f''(20) = -2 < 0$$

$$\Rightarrow f(x) \text{ has maximum value at } x = 20$$

$$\text{and put in (i)} y = 40 - 20 \Rightarrow y = 20$$

So dimensions of rectangular garden are 20m and 20m

Question No.9 An open tank of square base of side x and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of x if the tank with lead will be least.

Solution:

Let length, width and height of tank with square base are $x, x, \text{ and } y$ respectively.

$$\therefore \text{volume} = x \times x \times y = x^2 y$$

$$y = \frac{v}{x^2} \rightarrow (i)$$

$$\therefore \text{surface area} =$$

area of square + area of four verticle walls

$$\Rightarrow f(x) = x^2 + 4xy$$

$$\Rightarrow f(x) = x^2 + 4x\left(\frac{v}{x^2}\right) = x^2 + \frac{4v}{x}$$

$$f(x) = x^2 + 4vx^{-1}$$

$$\Rightarrow f'(x) = 2x + 4v(-1)(x^{-2}) = 2x - \frac{4v}{x^2}$$

$$\Rightarrow f''(x) = 2 + 4v(-1)(-2)x^{-3} = 2 + \frac{8v}{x^3} \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 2 + 4v(-1)(-2)(x^{-3}) = 2 + \frac{8v}{x^3}$$

$\rightarrow (ii)$

$$\text{Put } f'(x) = 0 \Rightarrow 2x - \frac{4v}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 4v}{x^2} = 0 \Rightarrow 2^3 = 4v$$

$$\Rightarrow x^3 = \frac{4v}{2} = 0 \Rightarrow x = (2v)^{\frac{1}{3}} \text{ put in (ii)}$$

$$f''(2v)^{\frac{1}{3}} = 2 + 8v \left[(2v)^{\frac{1}{3}} \right]^{-3} = 2 + 8v(2v)^{-1}$$

$$= 2 + \frac{8v}{2v} = 2 + 4 = 6 > 0$$

$$\Rightarrow f(x) \text{ has minimum value at } x = (2v)^{\frac{1}{3}}$$

$$\text{and put in (i)} y = \frac{x^3}{2} = \frac{x}{2}$$

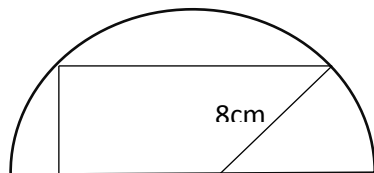
$y = \frac{x}{2}$ which is required height (depth) interms of x .

Question No.10 find the dimensions of the rectangular of maximum area which fits inside the semi-circle of radius 8cm as shown in figure.

Solution:

Let O be centre of semi-circle
let $ABCD$ be rectangular with fits inside the semi-circle

Of radius 8cm as
Shown in figure.



let $|AC| = |BD| = y$

ALSO let $|AC| = |BD| = x$

so length of rectangular

$$= 2x \text{ and width of rectangular} = y$$

\therefore radius 8cm (given) so $|OC| = 8\text{cm}$

in ΔOAC $x^2 + y^2 = (8)^2$ using pathagourm theorem

$$\Rightarrow y^2 = 64 - x^2 \Rightarrow y = \pm\sqrt{64 - x^2}$$

$$y = \sqrt{64 - x^2} \rightarrow (i)$$

$y = -\sqrt{64 - x^2}$ rejected being length

\therefore Area of rectangular $= 2xy \Rightarrow A = f(x) = 2xy$

$$\Rightarrow A^2 = f(x) = 4x^2(64 - x^2) = 256x^2 - 4x^4$$

$$\Rightarrow f(x) = (x - 18)^2 + (y - 1)^2$$

$$f(x) = (x - 18)^2 + (x^2 + 1 - 1) \therefore y = x^2 + 1$$

$$\Rightarrow f'(x) = 519x^2x - 16x^3$$

$$f'' = 512 - 48x^2 \rightarrow (ii)$$

$$\text{put } f'(x) = 0 \Rightarrow 512x - 16x^3 = 0$$

$$\Rightarrow 16x(32 - x^2) = 0$$

$$\Rightarrow 16x = 0 \quad x^2 = 32 \Rightarrow x = \pm\sqrt{32}$$

$$\Rightarrow x = 0, \quad x = \sqrt{32},$$

$x = -\sqrt{32}$ rejected being length

$$\text{So put } x = \text{put } x = \sqrt{32} \text{ in (ii)} \Rightarrow f''(x)(\sqrt{32})$$

$$= 512 - 48(\sqrt{32})^2$$

$$\Rightarrow f''(\sqrt{32}) = 512 - 48(32) = -1024 < 0$$

$\Rightarrow f(x)$ has maximum value at $x = \sqrt{32}$

$$\text{and put in (i)} y = \sqrt{64 - (\sqrt{32})^2} = \sqrt{64 - 32} = \sqrt{32}$$

$$y = \sqrt{32}$$

Thus length of rectangular $= 2x = 2(4\sqrt{2}) = 8\sqrt{2}$

Width of rectangular $= y = 4\sqrt{2}$

Hence dimensions are $8\sqrt{2}$ and $4\sqrt{2}$

Question No.11 find the point on the curve $y = x^2 - 1$ that is closed to the point $(3, -1)$

Solution:

$$\therefore y = x^2 - 1 \rightarrow (i)$$

Let $p(x, y)$ be required point.

Let $d =$ distance b/w (x, y) and $(3, -1)$

$$\Rightarrow d = \sqrt{(x - 3)^2 + (y + 1)^2}$$

$$\Rightarrow d^2 = (x - 3)^2 + (y + 1)^2 \quad \text{take } d^2 = f(x)$$

$$\Rightarrow f(x) = (x - 3)^2 + (y + 1)^2$$

$$f(x) = x^2 + 9 - 6x + (x^2 - 1)^2 + 1 + 2(x^2 - 1)$$

$$\therefore y = x^2 - 1$$

$$= x^2 + 9 - 6x + x^4 + 1 - 2x^2 + 1 + 2x^2 - 2$$

$$f(x) = x^4 + x^2 - 6x + 9$$

$$\Rightarrow f'(x) = 4x^3 + 2x - 6$$

$$\Rightarrow f''(x) = 12x^2 + 2 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 4x^3 + 2x - 6 = 0$$

$$2x^3 + x - 3 = 0 \quad \text{put } x = 1$$

$$\Rightarrow (x - 1)(2x^2 + 2x + 3) = 0$$

$$\Rightarrow x = 1, 2x^2 + 2x + 3 = 0 \quad (\text{rejected being complex})$$

so put $x = 1$ in (ii)

	2	0	1	-3
1	↓	2	2	3
	2	2	3	0

$2x^2 + 2x + 3$ and $x - 1$ are factors of $2x^3 + x - 3 = 0$

$$f''(x) = 12(1)^2 + 2 = 14 > 0$$

$\Rightarrow f(x)$ has minimum value at $x = 1$ and put in (i)

$$\Rightarrow y = (1)^2 - 1 = 0$$

So $(1, 0)$ is the required point closest to $(3, -1)$

Question No.12

Find the point on the curve $y = x^2 - 1$ so

$(1, 0)$ is the required point closet to $(18, -1)$

Solution:

$$\therefore y = x^2 - 1 \rightarrow (i)$$

Let $p(x, y)$ be the required point.

$\therefore d =$ distance b/w $p(x, y)$ and $(18, 1)$

$$\Rightarrow d = \sqrt{(x - 18)^2 + (y - 1)^2}$$

$$\Rightarrow d^2 = \sqrt{(x - 18)^2 + (y - 1)^2} \quad \text{take } d^2 = f(x)$$

$$= x^2 + 324 - 36x + x^4$$

$$f(x) = x^4 + x^2 - 36x + x^4$$

$$f'(x) = 4x^3 + 2x - 36$$

$$f''(x) = 12x^2 + 2 \rightarrow (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 4x^3 + 2x - 36 = 0$$

$$\Rightarrow 2x^3 + x - 18 = 0 \quad \text{for } x = 2$$

$$\Rightarrow (x - 2)(2x^2 + 4x + 9) = 0$$

$$x = 2, \quad 2x^2 + 4x + 9 = 0 \text{ rejected being complex}$$

$$\text{Put } x = 2 \text{ in (ii)}$$

2	2	0	1	-18
	↓	4	8	18
	2	2	9	0

$2x^2 + 4x + 9$ and $x - 2$ are factors of

$$2x^3 + x - 18 = 0$$

put $x = 2$ in (ii)

$$\Rightarrow f''(x) = 12(2)^2 + 2 > 0$$

$\Rightarrow f(x)$ has minimum value at $x = 2$ put in (i)

$$\Rightarrow y = (2)^2 + 1 = 5$$

So $(2,5)$ is the required point which is closest to $(18,1)$

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