MATHEMATICS 12 INTERMEDIATE PART II

Bilal Articles

Chapter 1.

FUNCTION AND LIMITS

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Contents

Function:

If A and B be two non-empty sets then f is said to be a function from set A to set B written as ; $f: A \to B$ and defined as

$i) D_f = A \; ii) \; for \; every \; a \in A \; there exist \; only \; one \; b \in B \; s.that \; (a, b) \in f$

Domain:

The set of all possible inputs of a function is called domain.

*the domain of every function $f(x)$ is defined.

*the valves at which at $f(x)$ becomes undefined or complex valued will be excluded from real numbers.

*domain is also known as pre-images.

Range:

The set of all possible out puts of a function is called range.

*range is also known as images.

Types of functions:

i) **Algebraic function:**

Any function generated by algebraic operations is known as algebraic function. Algebraic functions are classified as below.

ii) **Polynomial function:**

A function P of the form

$$p(x) = a_n x_n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

$for \; all \; x,$

$where \; the \; coefficients \; a_n, a_{n-1}, a_{n-2}. a_2, a_1, a_0 \; are \; real \; numbers \; and \; exponents \; are \; non-negative$

$integers, is \; called \; a \; polynomial \; function.$

iii) **Linear Function:**

If the degree of polynomial function is 1. Then it is called linear function.

iv) **Quadratic Function:**

If the degree of polynomial function is 2. Then it is called a quadratic function.

v) Identity function:

A function for which $f(x) = y \; or \; y = x$ is called identity function. It is denoted by I

vi) **Constants Function:**

A function for which $f(x) = b \; or \; y = b$ is called constant function.

vii) **Rational function:**

The quotient of two polynomials such as $f(x) = \frac{}{Q(x)}$ $where \; Q(x) \neq 0 \; is \; called \; rational \; function$

viii) **Exponential Function:**

A function in which the variable appears as exponent (power) is called exponential function.

$e.g; y = e^{ax}, y = e^x \; e.t.c$

ix) **Logarithmic Functions:**

$$if \; x = a^y \; then \; y = loga^x \; where \; a > o, a \neq 1 \; is \; called \; logarithmic \; functions.$$

*$\log_{10} x$ *is knonw as common logarithm.*

*$\log_e x$ *is known as natural logarithm.*

x) **Hyperbolic Function:**

$$sinhx = \frac{e^x - e^{-x}}{}, \quad coshx = \frac{e^x + e^{-x}}{}$$

$$x \quad -x$$

$$tanhx = \frac{}{e^x + e^{-x}}, \quad sechx = \frac{}{e^x_x + e^{-x}_{-x}}$$

$$cschx = \frac{}{e^x - e^{-x}}, \quad cothx = \frac{}{e^x - e^{-x}}$$

xi) **Inverse Hyperbolic function:**

$$\sinh\_^1 x = \ln(x + \sqrt{x^2 + 1}), \forall x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$$

$$\tanh^{-1} x = \frac{1}{} \ln(\frac{1 + x}{}), |x| < 1$$

$$\operatorname{csch}^{-1} x = \ln(\frac{1}{} + \frac{\sqrt{1 + x^2}}{}), x \neq 0$$

$$\operatorname{sech}\_^1 x = \ln(\frac{1}{} + \frac{\sqrt{1 - x^2}}{}), 0 < x \leq 1$$

$$\coth\_^1 x = \frac{}{2} \ln(\frac{}{}), |x| < 1$$

xii) **Explicit function:**

If y is easily expressed in terms of x, then y is called explicit function.

$symbolically \; y = f(x)$

xiii) **Implicit function:**

If the two variables x and y are so mixed up such that y cannot be expressed in terms of x, then this type of function. Symbolically $f(x, y) = 0$

xiv) **Parametric function:**

If x and y are expressed in terms of third variable (say t) such as $x = f(t), y = g(t) \; then \; these \; equations \; are$

Called parametric equations.

xv) **Even function:**

A function f is said to be an even if $f(-x) = f(x)$ for every $x \; in \; domain \; of \; f.$

xvi) **Odd function**:

A function f is said to be odd if $f(-x) = -f(x)$ for every number x in the domain of f

Exercise 1.1

Q1. Given that

$$a) f(x) = x^2 - x$$

$$b) f(x) = \sqrt{x + 4} \qquad find \; i) f(-2)$$

$ii) \; f(a) \qquad iii) \; f(x - 1) \qquad iv) \; f(x^2 + 4)$

Solution:

$(a) f(x) = x^2 - x$

$(i) f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

$ii) \; f(0) = (0)^2 - 0 = 0$

$$iii) \; f(x - 1) = (x - 1)^2 - (x - 1)$$
$$= x^2 + 1 - 2x - x + 1$$

$= x^2 - 3x + 2$

$iv)\ f(x^2+4) = ((x^2+4)^2 - (x^2+4)$

$= x^4 + 16 + 8x^2 - x^2 - 4$

$= x^4 + 7x^2 + 12$

(b) $f(x) = \sqrt{x+4}(i) f(-2) = \sqrt{-2+4} = \sqrt{2}$

$ii) f(0) = \sqrt{0+4} = \sqrt{4} = 2$

$iii) f(x-1) = \sqrt{x-1+4} = \sqrt{x+3}$

$xiv)\ f(x^2+4) = \sqrt{x^2+4-4} = \sqrt{x^2+8}$

Q2. Find $\frac{f(a+h)-f(a)}{}$ ***and simlify where***

$i)\ f(x) = 6x - 9 \quad ii)\ f(x) = sinx$

$iii) f(x) x^3 + 2x^2 - 1 \quad (iv) f(x) = cosx$

Solution:

$i)\ f(x) = 6x - 9$

$\frac{f(a+h)-f(a)}{h} \quad \frac{\{6(a+h)-9\}-(6a-9)}{h}$

$= \frac{(6a+6h-9-6a+9)}{} = \frac{6h}{} = 6$

$ii)\ f(x) = sinx$

$\frac{f(a+h)-f(a)}{h} \quad \frac{sin(a+h)-sina}{h}$

$= \frac{1}{}\{2cos\left(\frac{a+h+a}{}\right) sin\left(\frac{}{}\right)\}$

$= \frac{1}{}\{2cos\left(\frac{}{}\right) sin\left(\frac{h}{}\right)\}$

$= \frac{1}{}\{2cos\left(a+\frac{h}{}\right) sin\left(\frac{}{}\right)\}$

$iii)\ (f(x) = x^3 + 2x^2 - 1$

$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$

$a^3 + b^3 + 3a^2h + 3ah^2 + 2a^2 + 2h^2 + 4ah - 1$

$f(a) = a^3 + 2a^2 - 1$

$\frac{f(a+h)-f(a)}{h}$

$\frac{a^3+h^3+3a^2+3ah^2+2a^2+2h^2+4ah-1-a^3-2a^2+1}{h}$

$\frac{h(h^2+3a^2+3ah+2h+4a)}{h}$

$= h^2 + 3a^2 + 3ah + 2h + 4a$

$iv)\ f(x) = cosx$

$\frac{f(a+h)-f(a)}{h} \quad \frac{cos(a+h)-cosa}{h}$

$= \frac{1}{}\left(-2sin\left(\frac{}{}\right) sin\left(\frac{}{}\right)\right)$

$\frac{1}{}\left(-2sin\left(\frac{2a+b}{}\right) sin\left(\frac{}{}\right)\right)$

$-\frac{2}{h} sin\left(\frac{a+h}{2}\right) sin\left(\frac{h}{2}\right)$

Q3. Express the following (a) the perimeter P of square as a function of its area A.

Solution:

Let each side of square be "x" then

Perimeter:

$p = 4x \to (i)$

Area:

$a = x \times x = x^2 \Rightarrow x = \sqrt{A}$

put valve of x in (i)

⇨ $P = 4\sqrt{A}$

b) The area A of a circle as a function of its circumference C.

Solution:

let r be the radius of circle then

Then

Area $= \pi r^2 \to (i)$

Circumference:

$C = 2\pi r \Rightarrow r = \frac{}{2\pi}$ *put in(i)*

⇨ $A = \pi\left(\frac{}{2\pi}\right)^2 = \pi.\left(\frac{}{2\pi}\right)^2 = \frac{2}{2\pi}$

⇨ $A = \frac{2}{2\pi}$

(C) the volume V of a cube as a function of the area A of its base.

solution:

let each side of cube be x then

volume:

$V = x \times x \times x$

$V = x^3 \to (i)$

Area of base:

$A = x^2 \Rightarrow x = \sqrt{A}$ *put in (i)*

⇨ $V = (\sqrt{A})^3 \Rightarrow V = A^2$

Q4. Find the domain and range of the functions g defined below.

$(i) g(x) = 2x - 5$

$D_y = (-\infty, +\infty), R_y = (-\infty, +\infty)$

$ii) g(x) = \sqrt{x^2 - 4}$

g(x) becomes complex valued when $x^2 - 4 < 0$

or $x^2 < 4$ *or* $-2 < x < 2$

$D_y = R - (-2,2), R_y = [0, +\infty)$

$(iii) g(x) = \sqrt{x+1}$

g(x) becomes complex valued when $x + 1 < 0$ *or*

$x < -1$ *so* $D_g = [-1, +\infty)$

$iv)\ g(x) = |x - 3|$

$D_y = (-\infty, +\infty), R_y = [0, \infty)$

$6x + 7$ *if* $x \le -2$

$x \quad 4x - 3$ *if* $x > -2$

$D_y = (-\infty, -2] \cup (-2, +\infty)$

$R_y = (-\infty, -5] \cup (-11, +\infty)$

$vi)\ g(x) = \frac{x^2+3x+2}{}, x \ne -1$

$D_y = R - \{-1\} \quad \because = \frac{x^2+3x+2}{}$

$R_y = R - \{1\} \quad = \frac{(x+1)(x+2)}{x+1}$

$g(x) = x + 2$

$viii)\ g(x) = \frac{x^2-16}{x-4}, x \ne 4 \quad g(-1) = -1 + 2$

$= 1$

$D_y = R - \{4\}$ $\quad \because \frac{{}^2 - 16}{xx - 4}$

R

${}_y = R - \{8\}$ $\quad = \frac{(x-4)(x+4)}{x-4}$

$$g(x) = x + 4$$
$$g(x) = 4 + 4 = 8$$

Q5. Given $f(x) = x^3 - ax^2 + bx + 1$ if $f(2) = -3$, $f(-1) = 0$ find the value of a and b

Solution:

$f(x) = x^3 - ax^2 + bx + 1$

- ⇨ $f(2) = (2)^3 - a(2)^2 + b(2) + 1$
- ⇨ $-3 = 8 - 4a + 2b + 1$
- ⇨ -4a+2b+12=0
- ⇨ $-2a + b + 6 = 0 \rightarrow (i)$
- ⇨ $also\ f(-1) = (-1)^3 + b(-1) + 1$
- ⇨ $0 = -1 - a - b + 1$
- ⇨ $-a - b = 0 \rightarrow (ii)$
- ⇨ $(i) + (ii) \quad -2a + b + 6 = 0$

$$-a - b = 0$$
$$-3 + 6 = 0 \Rightarrow -3a = -6 \Rightarrow a = 2$$

Put in $ii)$ $\quad -2 - b = 0 \Rightarrow b = -2$

Q6. A stone falls from a height of h after x second is approximately given by $h(x) = 40 - 10x^2$

i) when is the height of the stone when

(a) $x = 1\ sec$?

b) $x = 1.5\ sec$ (c) $x = 1.7\ sec$

d) when does the stone strike the ground.

Solution:

$h(x) = 40 - 10x^2$

(a) $h(1) = 40 - 10(1)^2 = 40 - 10 = 30$

b) $h(1.5) = 40 - 10(1.5)^2 = 40 - 22.5 = 17.5m$

c) $h(1.7) = 40 - 10(1.7)^2 = 40 - 28.9 = 11.1m$

ii) when does stone strikes the ground then $h(x) = 0$

$$h(x) = 40 - 10x^2$$

- ⇨ $0 = 40 - 10x^2$
- ⇨ $10x^2 = 40$
- ⇨ $x^2 = 4$
- ⇨ $x = \pm 2$
- ⇨ $x = 2$, (neglect − 2)

Q7. Show that the parametric equation:

i) $x = at^2, y = 2at$ represented the equation: of parabola $y^2 = 4ax$

Solution:

$$x = at^2 \rightarrow (1)$$
$$y = 2at \Rightarrow t = \frac{}{2a}\ put\ in\ (i)$$

- ⇨ $x = a\left(\frac{}{2a}\right)^2 = a.\frac{}{4a^2}$
- ⇨ $x = \frac{y^2}{4a} \Rightarrow y^2 = 4ax$

(ii) $x = acos\theta, y = bsin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

$$x = acos\theta \rightarrow (1)$$
$$y = bsin\theta \rightarrow (2)$$

From (1)

$$\frac{x}{a} = cos\theta \rightarrow (3)$$

From (2)

$$\frac{y}{b} = sin\theta \rightarrow (4)$$

Squaring and adding (3) and (4)

$$\left(\frac{}{}\right)^2 + \left(\frac{}{}\right)^2 = (cos\theta)^2 + (sin\theta)^2$$
$$= \cos^2\theta + \sin^2\theta$$
$$\frac{{}^2}{a^2} + \frac{{}^2}{b^2} = 1$$

$iii) x = asec\theta \rightarrow (1)$

$y = btan\theta \rightarrow (2)$

From 1)

$$\frac{}{a} = sec\theta \Rightarrow \frac{{}^2}{a^2} = \sec^2\theta \rightarrow (3)$$

From 2)

$$\frac{y}{} = tan\theta \Rightarrow \frac{y^2}{} = \tan^2\theta \rightarrow (4)$$

(3) − (4)

- ⇨ $\frac{}{a^2_2} - \frac{}{b^2_2} = \sec^2\theta - \tan^2\theta$
- ⇨ $\frac{}{a^2} - \frac{}{b^2} = 1$

Q8.prove the identities

i) $sinh2x = 2sinhxcoshx$

Solution:

$R.H.S = 2sinhxcoshx$

- ⇨ $2.\frac{e^x - e^{-x}}{}.\frac{e^x + e^{-x}}{} = \frac{e^{2x} - e^{-2x}}{}$
- ⇨ $sinh2x = L.H.S$

Hence $sinh2x = 2sinhxcoshx$

$iii)$ $\operatorname{sech}^2 x = 1 - \tanh^2 x$

Solution:

$R.H.S = 1 - tanh^2 x$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}}^{\ 2} \qquad \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$
$$\frac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2}$$
$$\frac{(e^{2x} + e^{-2x} + 2) - e^{2x} - e^{-2x} + 2)}{(e^x + e^{-x})^2}$$
$$= \frac{}{(e^x + e^{-x})^2} = \left(\frac{}{e^x + e^{-x}}\right)^2$$
$$= \left(\frac{}{e^x + e^{-x}}\right)^2 = \frac{}{cosh^2} = sech^2 x = L.H.S$$

(iii) $\operatorname{csch}^2 x = \coth^2 x - 1$

Solution:

$R.H.S = \coth^2 x - 1$

$$= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) - 1 = \frac{(e^x + e^{-x})}{(e^x - e^{-x})} - 1$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x - e^{-x})^2}$$

$$= \frac{4}{(e^x - e^{-x})^2} = \left(\frac{2}{e^x - e^{-x}}\right)^2$$

$$= \left(\frac{1}{\frac{e^x - e^{-x}}{2}}\right) = \frac{1}{\sinh^2 x} = cseh^2 x = L.H.S$$

Hence $csch^2 x = \coth^2 x - 1$

Q9. Determine whether the given function f is even or odd.

Solution:

$i) f(x) = x^3 + x$

- ⇨ $f(-x) = (-x)^3 + (-x) = -x^3 - x$
- ⇨ $= -(x^3 + x) = -f(x)$
- ⇨ *thus $f(x)$ is odd.*
- ⇨ $ii)\ f(x) = (x+2)^2$
- ⇨ $f(-x) = (-x+2)^2 \neq \pm f(x)$

thus $f(x)$ is neither even nor odd.

$iii)\ \ f(x) = x\sqrt{x^2} + 5$

- ⇨ $f(-x) = x\sqrt{(-x)^2} + 5$
- ⇨ $= -x\sqrt{x^2} + 5 = -f(x)$

thus $f(x)$ is neither even nor odd.

v) $f(x) = x^{\frac{1}{3}} + 6$

$\Rightarrow f(x) = x^{\frac{1}{3}} + 6$

$[(-x)^2] + 6$

$$= (x\ )^{\frac{1}{3}} + 6$$

$$= (x^2)^{\left(\frac{1}{3}\right)} + 6 = x^{\frac{2}{3}} + 6 = f(x)$$

thus $f(x)$ is even.

$\boldsymbol{vi)\ f(x) = \frac{x^3}{x+1}}$

$$\frac{(-x)^2 - (-x)}{(-x)^2 + 1} \qquad \frac{-x^3 + x}{x^2 + 1}$$

⇨ $= -\frac{(x^3 - x)}{} = -f(x)$

thus $f(x)$ is oddd.

Composition of function:

If f is a function from set A to set B and g is a function from set B to set C then composition of f and g is denoted by

$$(fog)(x) = f(g(x)) \forall\ x \in A$$

Inverse of a function:

Let f be a bijective $(1 - 1$ *and onto* $)$ *function from set* A *ro set* b *i.e* $f: A \rightarrow B$ *then its inverse is* $f\_^1$ *which is surjective (onto) function from* B *to* A *i.e* $f\_^1: B \rightarrow A$ *in this case* $D_f: R_f$ *on to* $R_f = D_{f^{-1}}$

Exercise 1.2

Q1. The real valued functions f and g are defined below. find

$\boldsymbol{(a) fog(x)\ (b) gof(x)\ \ (c) fof(x)\ \ (d) gog(x)}$

i) $\quad f(x) = 2x + 1;\ \ g = \frac{3}{x-1}, x \neq 1$

S0lution:

$$(a) fog(x) = f(g(x)) = f\left(\frac{3}{x-1}\right)$$

$$= 2\left(\frac{3}{x-1}\right) + 1 = \frac{6}{x-1} + 1 = \frac{6 + x - 1}{x-1}$$

$$\frac{5+x}{x-1}$$

b) $gof(x) \ = \ g(f(x)) \ = \ g(2x+1)$

$$= \frac{}{2x+1-1} = \frac{}{2x}$$

c) $fof(x) = f(f(x)) = f(2x+1)$

$$= 2(2x+1) + 1 = 4x + 3$$

d) $gog(x) = g(g(x)) = g\left(\frac{}{x-1}\right) = \frac{}{\frac{}{x-1} - 1}$

$$\frac{3}{\frac{3-(x-1)}{x-1}} \qquad \frac{3(x-1)}{3-x+1} \qquad \frac{3(x-1)}{4-x}$$

$\boldsymbol{ii) f(x) = \sqrt{x} + 1, g(x) = \frac{1}{}}$

Solution:

$$a)\ fog(x) = f(g(x))$$

$$= f\left(\frac{1}{}\right) = \sqrt{\frac{1}{}} + 1 = \sqrt{\frac{\ + x^2}{1}} = \sqrt{1 + x^2}$$

$$b) gof(x) = g(f(x)) = g(\sqrt{x} + 1) = \frac{1}{(\sqrt{x} + 1)}$$

$$\frac{1}{x+1}$$

$$c)\ fof(x) = f(f(x)) = f(\sqrt{x} + 1) = \sqrt{\sqrt{x} + 1} + 1$$

$$d) gog(x) = g(g(x)) = g\left(\frac{}{}\right) = \frac{1}{\left(\frac{}{}\right)^2} = \frac{1}{x^4} = x^4$$

$\boldsymbol{(iii)\ f(x) = \frac{}{\sqrt{x-1}}, g(x) = (x^2 + 1)^2}$

Solution:

a)$fog(x) = f(g(x))$

$$f((x^2+1)^2) = \frac{1}{\sqrt{(x^2+1)^2 - 1}}$$

$$\frac{1}{\sqrt{x^4 + 1 + 2x^2 - 1}} \qquad \frac{1}{\sqrt{x^4 + 2x^2}} \qquad \frac{1}{\sqrt{x^2(x^2+2)}}$$

$$= \boldsymbol{\frac{1}{x\sqrt{x^2 + 2}}}$$

$$\boldsymbol{b)\ gof(x) = g(f(x)) = g\left(\frac{}{\sqrt{}}\right)}$$

$$= \boldsymbol{\left[\left(\frac{}{}\right)^2 + 1\right]^2 = \left(\frac{}{x-1} + 1\right)^2}$$

$$= \boldsymbol{\left(\frac{}{x-1}\right)^2 = \left(\frac{}{x-1}\right)^2}$$

$\boldsymbol{c)}$

$$\boldsymbol{fof(x) = f(f(x)) = f\left(\frac{1}{\sqrt{x-1}}\right)}$$

$$=\frac{1}{\sqrt{\frac{1}{\sqrt{x-1}}}-1}=\frac{1}{\left(\frac{1-\sqrt{x-1}}{\sqrt{x-1}}\right)^{\frac{1}{2}}}$$

$$=\left(\frac{1-\sqrt{x-1}}{\sqrt{x-1}}\right)^{-1\backslash 2}$$

$$\frac{\sqrt{x-1}}{1-\sqrt{x-1}}^{\frac{1}{2}} \quad \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x}-16}}$$

d)

$$gog(x)=g(g(x))$$
$$=g(x^2+1)$$
$$=((x^2+1)^2+1)^2$$

(iv)

$f(x)=3x^4-2x^2, g(x)=\sqrt{x}$

Solution:

$$a)\ fog(x)=f(g(x))$$
$$=f(\ )=3(\sqrt{x})^4-2(\ )^2$$
$$=3(16/x^2)-\frac{8}{}=\frac{48}{}-\frac{8}{}=\frac{48-8x}{}$$

b)

$$gof(x)=g(f(x))=g(3x^2-2x^2)$$
$$=g(3x^4-2x^2)$$
$$=\sqrt{3x^4-2x^2}=\sqrt{x^2(3x^2-2)}=x\sqrt{3x^2-2}$$

c)

$$fof(x)=f(f(x))=f(3x^4-2x^2)$$
$$=3(3x^4-2x^2)^4-2(3x^4-2x^2)^2$$

d)

$$gog(x)=g(g(x))=g(\sqrt{\ })^{1}$$
$$=\frac{2}{\sqrt{\ }}=\frac{2}{(\ )^{\frac{1}{2}}}=2(\ )^{\frac{1}{2}}$$
$$=2\left(\frac{x}{}\right)^{\frac{1}{2}}=2\sqrt{\sqrt{x}}=\sqrt{2}.\sqrt{2}\ \sqrt{\sqrt{2}}$$
$$=\sqrt{2}\sqrt{x}$$

Q2.

For the real valued function f defined below, find

$(a) f^{-1}(x)\quad (b) f^{-1}(-1)$ **and verify**

$$f(f^{-1}(x))=f^{-1}(f(x))=x$$

$i)\ f(x)=-2x+8$

Solution:

$f(x)=-2x+8$

let $y=f(x)$ then

$$y=-2x+8 \Rightarrow \frac{y-8}{-2}=x$$

⇨ $x=\frac{y-8}{-2}$

⇨ $f^{-1}(y)=\frac{y-8}{-2} \Rightarrow \because y=f(x)$

⇨ $f^{-1}(y)=x$

Replace y by x we have

⇨ $f^{-1}(x)=\frac{x-8}{}$

⇨ put $x=-1, f^{-1}(-1)=\frac{-1-8}{}=\frac{9}{}$

ii) $f(x)=3x^2 2+7$

Solution:

$$f(x)=3x^2+7$$
$$let\ y=f(x) then\ y=3x^2+7$$

⇨ $\frac{y-7}{}=x^3$

⇨ $x=\left(\frac{y-7}{}\right)^{\frac{1}{3}}$

⇨ $\because y=f(x)\Rightarrow f^{-1}(y)=x$

⇨ $f^{-1}(y)=\left(\frac{y-7}{}\right)^{\frac{1}{3}}$

Replace y by x we have

⇨ $f^{-1}(x)=\left(\frac{x-7}{}\right)^{\frac{1}{3}}$

Put $x=-1\ f^{-1}(-1)=\left(\frac{-8}{}\right)^{\frac{1}{3}}$

Verification:

$$f(f^{-1}(x))=f\left[(\ )^{\frac{1}{3}}\right]=3\left[(\ )^{\frac{1}{2}}\right]^2+7$$
$$=3(\ )+7=x-7+7=x$$
$$f^{-1}(f(x))=f^{-1}(3x^3+7)=\left(\frac{3x^2+7-7}{}\right)^{\frac{1}{3}}$$
$$=\left(\frac{3x^3}{}\right)^{\frac{1}{3}}=x$$

$hence\ f(f^{-1}(x))=f^{-1}(f(x))=x$

iii) $f(x)=(-x+9)^3$

⇨ $y=f(x)=(-x+9)^3$

$let\ y=f(x) then\ y=(-x+9)^3$

$$y^{\frac{1}{3}}=-x+9$$

⇨ $y^5-9=-x$

⇨ $x=9-y^{\frac{1}{5}}$

⇨ $f^{-1}(y)=9-y^{\frac{1}{3}}$

$(\because y=f(x)\Rightarrow f^{-1}(y)=x$

$replace\ y\ by\ x\ we\ have$

$$f^{-1}(x)=9-x^{\frac{1}{3}}$$

Put $x=-1, f^{-1}(-1)=9-(-1)^{\frac{1}{3}}=9-(-1)=0$

Verification:

$$f(f^{-1}(x))=f(9-x^{\frac{1}{3}})=[-(9-x^{\frac{1}{3}})+9]^3$$

$= (-9 + x^{\frac{1}{3}} + 9)^{3} = x$

$f^{-1}(f(x)) = f^{-1}((-x+9)^3)$

$= 9 - ((- - x + 9)^3)^{\frac{1}{3}} = 9 - (-x + 9)$

$= 9 + x - 9 = x$

Hence $f(f\_^1(x)) = f\_^1(f(x)) = x$

iv) $f(x) = {}^{2x+1}$

Let $y = f(x) then\ y = {}^{2x+1}$

⇨ $(x-1)y = 2x + 1$
⇨ $xy - y = 2x + 1$
⇨ $xy - 2x = y + 1$
⇨ $x(y-2) = 1 + y$
⇨ $x = {}^{1+y}$
⇨ $replace\ y\ by\ x\ we\ have$
⇨ $f^{-1}(x) = {}^{1+x}$
⇨ $put\ x = -1, f^{-1}(-1) = {}^{1+(-1)} = 0$

Verification:

$f(f^{-1}(x)) = f\left(\frac{1+x}{x-2}\right) = \frac{2\left(\frac{1+x}{x-2}\right)+1}{\frac{1+x}{x-2}}$

$\frac{2\ 1+x\ +x+2}{1+x-(x-2)} = \frac{3x}{2x+1-2x+2} = \frac{3x}{3}$

$x - 2$

Hence $f(f\_^1(x)) = f\_^1(f(x)) = x$

Q3.

Without finding the inverse, state the domain and range of f^{-1} $)\ i)\ f(x) = \sqrt{x+2}\quad ii)\ f(x) = {}_{x-1}, x \neq 4$

$iii)\ f(x) = {}_{x+3}, x \neq -3$

$iv)\ f(x) = (x-5)^2, x \geq 5$

Solution:

i) $f(x) = \sqrt{x+2}$

$\because f(x) becomes\ complex\ valued\ when\ x + 2 < 0$

$or\ x < -2$

$D_f = [-2, +\infty), R_f = [0, +\infty)$

By definition of inverse function,

$D_{f^{-1}} = R_f = [0, +\infty)$

By definition of inverse function,

$D_{f^{-1}} = R_f = [0, +\infty), R_{f^{-1}} = D_f = [-2, +\infty)$

ii)

$f(x) = \frac{x-1}{x-4}, x \neq 4$

$D_f = R - \{4\}, \because f(x) = \frac{x-1}{x-4}, x \neq 4$

$R_f = R - \{1\} \quad y = \frac{x-1}{x-4}$

⇨ $yx - 4y = x - 1$

$xy - x = 4y - 1$

⇨ $x(y-1) = 4y - 1$
⇨ $x = \frac{4y-1}{y-1}$

$f^{-1}(x) = \frac{4x-1}{x-1}, x \neq 1$

By def. of inverse function.

$D_{f^{-1}} = R_f = R - \{1\}$

$R_{f^{-1}} = D = R - \{4\}$

iii)

$f(x) = {}_{x+3}, x \neq -3$

$D = R - \{-3\} \because f(x) = {}_{x+3}, x \neq -3$

$R_f = R - \{0\} \quad y = {}_{x+3}$

By def. of inverse $\quad x + 3 = {}_y$

$D_{f^{-1}} = R_f = R - \{0\} \quad x = {}_y - 3$

$R_{f^{-1}} = D = R - \{-3\} \quad f\_^1(x) = {}^1 - 3, x \neq 0$

$R_{f^{-1}} = D = R - \{-3\}$

iv)

$f(x) = (x-2)^2, \quad x \geq 5$

$D = [5, +\infty) \quad , R_f = [0, +\infty)$

By definition of inverse function.

$D_{f^{-1}} = R_f = [0, +\infty), \quad R_{f^{-1}} = D = [5, +\infty)$

Limits of functions:

Let $f(x)$ be a function then a number L is said to be limit of $f(x)$ when x approaches to a from both left and right hand side of a , symbolically it is written as;
$\lim_{x \to a} f(x) = L$

And read as "limit of $f\ of\ x$ as approaches to a is equal to L"

Theorems on limits of functions:

i) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$= L + M$

ii) $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

$= L - M$

iii) $\lim_{x\to a}[k, f(x)] = k \lim_{x\to a} f(x) = kL$

iv) $\lim_{x\to a} f(x)g(x) = k \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = LM$

v) $\lim \left(_{g(x)}\right) = \frac{}{\lim_{x\to a} g(x)} = {}_M$

vi) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$

Theorem:

Prove that $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$ $where\ n\ is\ an\ integer$

And $a > 0$

Proof:

Case 1:

Suppose n is a +ve integer.

$$L.H.S = \lim_{x\to a} \frac{x^n - a^n}{x - a} \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{(n-3)}a^2 + \cdots + xa^{n-2} + a^{n-1}}{x-a}$$

$$= \lim_{x\to a} \; x^{n-1} + x^{n-2}a + x^{(n-3)}a^2 + \cdots + xa^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-2}.a + a^{n-3}.a^2 + \cdots + xa^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1} + a^{n-1}$$

$$= na^{n-1}$$

$$\text{thus } \lim \; x \; n \; x - a = na^{n-1}$$

Case 11:

Suppose n is +ve.

$$\text{let } n \text{ is } -ve$$

$(\textit{where m is} + \textit{ve integer})$

$$\text{then } \lim_{x\to a} \; n \; n = \lim_{x\to a} \; -m \; -m$$

$$= \lim(x^{-m} - a^{-m}).\; x - a = \lim (\;\text{—}\;.\; a^n) = \; x - a$$

$$= \lim \left(ax^m a \right) x^m \;.\; x - a$$

$$= \lim \left(x - a^m \right)\left(x \; -1 \; a \right) x^m$$

$$= \lim_{x\to a} \left(x - a^m \right) \lim_{x\to a} \left(-1 \right)$$

$$ma^{m-1} \left(a^{2m} \right)$$

$$-ma^{m-1-2m} = -ma^{(-m-1)} = na^{(n-1)}$$

Thus $\lim_{x-a} \; x \; n = na^{n-1} \because n = -m$

Theorem:

Prove that $\lim_{x\to a} \sqrt{x+a} - \sqrt{a} = 1$

***proof*:**

$$L.H.S = \lim_{x\to 0} \sqrt{x+a} - \sqrt{a} \; (-) \text{ form}$$

$$x\to 0 \quad \frac{\sqrt{x+a} - \sqrt{a}}{x} \quad \frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} + \sqrt{a}}$$

$$\frac{x + a - a}{x\to 0 \; x(\sqrt{x+a} + \sqrt{a})}$$

$$= \lim \; (\sqrt{x+a} + \sqrt{a})$$

$$= \lim_{x\to 0} \frac{1}{(\sqrt{a} + \sqrt{a})}$$

$$= \lim_{x\to 0} \frac{1}{(2\sqrt{a})}$$

$$\text{thus } \lim_{x\to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = 1/2\sqrt{a}$$

Theorem:

Prove that $\lim_{x\to +\infty} \left(1 + \frac{1}{n}\right)^n = e$

Using Binomial theorem we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots$$

$$= 1 + 1 + {}^1 \left({}^{n-1}\right) + {}^1 \left({}^{n-1}\right)\left({}^{n-2}\right) + \ldots.$$

$$2 + {}_{2!}\left(1 - {}_n\right) + {}_{3!}\left(1 - {}_n\right)\left(1 - {}_n\right) + \cdots$$

$$\text{when } n \to \infty, {}_n, {}_n, {}_n \text{ .. all tend to term}$$

Thus,

$$\lim \left(1 + {}_n\right)^n = 2 + {}_{2!} + {}_{3!} + {}_{4!} + \cdots$$

$$= 2 + 0.5 + 0.16667 + ..$$

$$= 2.718281$$

Thus,

$$\lim_{x\to +\infty} \left(1 + 1\right)^n = e$$

Deduction:

$$\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$$

We know that

$$\lim_{n\to\infty} \left(1 + 1\right)^n = e \to (1)$$

$$\text{put } n = {}_x \Rightarrow x = {}_n \text{ in(i)}$$

$$\text{when } n \to \infty, x \to 0$$

So (i)

$$\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$$

Theorem:

Prove that

$$\lim_{x\to 0} \; a^x - 1 = log \; a$$

Proof:

$$L.H.S = \lim_{x\to a} \; a^x - 1$$

$$\text{put } a^x - 1 = y \Rightarrow a^x = 1 + y$$

So $x = log_a(1 + y)$

As $x \to 0 \;, y \to 0 \; so$

$L.H.S = \lim \; log_a(1+y)$

$$= \lim \; log_a(1 + y)$$

$$= \lim_{y\to 0} \frac{1}{y \, log_a(1+y)} \qquad = \lim_{y\to 0} \frac{1}{log_a(1+y)^{\frac{1}{y}}}$$

$$= \frac{}{log_a e} = log_e a \qquad \because \lim(1 + y)^{\frac{1}{y}} = e$$

$$R.H.S$$

Thus $\lim_{x\to 0} \frac{a^x - 1}{x} = log_e a$

Deduction:

$$\lim_{x\to 0} \left(\frac{e^x - 1}{x}\right) = log_e e = 1$$

Since we know that

$$\lim_{x\to0}\left(\frac{a^x-1}{x}\right)=\log_e a \to (i)$$

put $a=e$ in (1) we get

$$\lim_{x\to0}\frac{e^x-1}{x}=\log_e e=1$$

Important results to remember:

$$i)\ \lim_{x\to+\infty}(e^x)=\infty$$

$$ii)\ \lim_{x\to-\infty}(e^x)=\lim_{x\to-\infty}\left(\frac{}{}\right)=0$$

$$iii)\ \lim_{x\to\pm\infty}\left(\frac{}{}\right)=0\ \text{ehere } a \text{ is any real numbers.}$$

The Sandwich theorem:

let f, g and h be functiions s. that

$f(x)\le g(x)\le h(x)$ For all numbers x in some open interval containing "c" itself .if $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} h(x)=L$ then $g(x)$ is sandwhich b/w $f(x)$ and $h(x)$ so that $\lim_{x\to c} g(x)=L$

Theorem:

If θ is measured in radian, then $\lim_{\theta\to0}\frac{\sin\theta}{}=1$

Proof:

draw a unit circle (radius 1) in which

A

C B D

Area of $\triangle OAB=\frac{1}{}(base)(perpendicual)$

$=\frac{}{2}|OB||AC|$ where $\frac{}{|OA|}=\sin\theta$

$=\frac{1}{}(1)(\sin\theta)$ $|AC|=|OA|\sin\theta$

$=\frac{1}{}\sin\theta$ $|AC|=\sin\theta$

$\because$ radius $=|OA|=|OB|=1$

Area of sector $OAB=\frac{}{2}r^2\theta$

$=\frac{1}{}(1)^2\theta=\frac{1}{}\theta$

Area of $\triangle OAD=\frac{}{2}(base)(perpendicual)$

$=\frac{1}{}|OA||AD|$ where $\frac{AD}{|\ \ |}=\tan\theta$

$=\frac{}{2}\tan\theta$ $|AD|=\tan\theta$

Now by (1)

$$\frac{1}{2}\sin\theta<\frac{1}{2}\theta<\frac{1}{2}\tan\theta$$

$$or\quad \sin\theta<\theta<\tan\theta$$

$$or\quad \frac{\sin\theta}{\sin\theta}<\frac{\theta}{\sin\theta}<\frac{\sin\theta}{\cos\theta}\times\frac{1}{\sin\theta}\quad(\div\ y\ in\theta)\ b\ s$$

$$or\quad 1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}$$

take reciprocal and limit $\theta\to0$

$$\lim_{\theta\to0}(1)>\lim_{\theta\to0}\frac{\sin\theta}{\theta}>\lim_{\theta\to0}\cos\theta$$

$$1>\lim_{\theta\to0}\frac{\sin\theta}{\theta}>1$$

applying sandwhich therom

$$\lim_{\theta\to0}\frac{\sin\theta}{}=1$$

Exercise 1.3

Q1. Evaluate each limit by using theorems of limits.

$i)\ \lim_{x\to3}(2x+4)$

Solution:

$$\lim_{x\to3}(2x+4)$$

$$=\lim_{x\to3}2x+\lim_{x\to3}4=2(3)+4=10$$

$ii)\ \lim_{x\to1}(3x^2-2x+4)$

Solution:

$=3(1)^2-2(1)+4=3-2+4=5$

$iii)\ \lim_{x\to3}\ x^2+x+4$

solution: $\sqrt{(3)^2+3+4}=\sqrt{16}=4$

$iv)\ \lim_{x\to2} x\sqrt{x^2-4}$

solution: $(2)\sqrt{(2)^2-4}=0$

$v)\ \lim(\sqrt{x^3+1}-\sqrt{x^2+5})$

$\sqrt{(2)^3+1}-\sqrt{(2)^3+5}=3-3=0$

$(vi)\ \lim_{x\to-2}\frac{2x^3+5x}{}$

$\frac{2(-2)^3+5(-2)}{3(-2)-2}$ $\frac{-16-10}{-8}$ $\frac{26}{-8}$ $\frac{13}{4}$

Q2. Evaluate each limit by using algebra techniques.

$i)\ \lim_{x\to-1}\frac{x^3-x}{}$

Solution:

$\lim_{x\to-1}\frac{x^3-x}{}$ $\left(\frac{0}{}\right)$ *form*

$$=\lim_{x\to1}\frac{x(x^2-1)}{}=\lim_{x\to1}\frac{x(x-1)(x+1)}{}$$

$$=\lim_{x\to-1}x(x-1)=(-1)(-1-1)=2$$

$ii)$

$$\lim_{x\to0}\left(\frac{3x^3+4x}{}\right)$$

Solution:

$\lim_{x\to0}\frac{3x^3+4x}{}$ $\left(\frac{}{}\right)$ *form*

$$=\lim_{x\to0}\frac{x\ 3x^2+4)}{x(x+1)}=\lim_{x\to0}\frac{3x^2+4}{x+1}$$

$$=\frac{3(0)^2+4}{0+1}=\frac{4}{1}=4$$

$iii)$

$\lim_{x\to2}\frac{x^3-8}{x^2+x-6}$ $\left(\frac{0}{0}\right)$ *form*

$$= \lim_{x\to 2} \frac{(x)^3-(2)^3}{x^2+3x-2x-6}$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2+4+2x)}{x(x+3)-2(x+3)}$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2+4+2x)}{(x+3)(x-2)}$$
$$\lim_{x\to 2} \frac{x^2+4+2x}{x+3} \quad \frac{(2)^2+4+2(2)}{2+3} \quad \frac{12}{5}$$

iv)

$$\lim_{x\to 1} x^3-3x^2+3x-1 \qquad (\frac{0}{}) \; fo\; r\; m$$
$$= \lim \frac{x-}{x(\quad -1)} \qquad \because (x-1)^3$$
$$= x^3-3x^2+3x-1$$
3 2

$$\text{l m}_{\text{i}}\; x(x-1)(x+1) = \text{l m}_{\text{i}}\; x(x+1)$$
2
$$= \frac{}{1(1+1)} = 0$$

v)

$$\lim_{x\to -1} x^3+x$$

Solution:

$$\lim\; (\frac{3}{x^2-1}) \qquad (\frac{}{0})\; form$$
$$\lim_{x\to -1} \frac{x^2(x+1)}{(x-1)(x+1)} \quad \lim_{x\to -1}\frac{x^2}{x-1} \quad \frac{(-1)^2}{-1-1} \quad \frac{1}{-2}$$

vi)

$$\lim_{x\to 4} \frac{2x^2-32}{x^3-4x^2}$$

Solution:

$$\lim_{x-4} \frac{2x^2-32}{} \qquad (\frac{}{})\; form$$
$$\lim_{x\to 4} \frac{2(x^2-16)}{x^2(x-4)} \quad \lim_{x\to 4} \frac{2(x-4)(x+4)}{x^2(x-4)}$$
$$\lim_{x\to 4} \frac{2(x+4)}{} = \frac{2(4+4)}{} = 1$$

vii)

$$\lim \frac{\sqrt{x}-\quad 2}{2} \times \sqrt{x}+\sqrt{2}$$
$$= \lim \frac{-\qquad)}{(x-2 \quad x+\quad 2}$$
$$\lim \frac{}{(x-2)(\sqrt{x}-\sqrt{2}} = \lim \frac{}{\sqrt{x}+\sqrt{2}} = \frac{}{\sqrt{2}+\sqrt{2}}$$
$$= \frac{}{2\sqrt{2}}$$

viii)

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Solution:

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \qquad (\frac{0}{0}) \quad form$$
$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

ix)

$$\lim_{\theta\to a} \frac{x^n-a^n}{x^m-a^m}$$

Solution:

$$\lim \frac{n \quad n}{x^m-a^m} \qquad (\frac{}{0})\; form$$

dividing up and down by $x-a$

$$= \lim\; (\frac{\frac{x^n-a^n}{x-a}}{x^m-a^m}) = \lim_{x\to a} \frac{\frac{x^n-a^n}{x-a}}{x^m-a^m}$$
$$= \frac{n-1}{ma} \qquad (\because \lim \frac{x \quad n}{x-a} = na^{n-1})$$
$$= \frac{}{m} a^{n-1} - {}^{m+1} = \frac{}{m} a^{n-m}$$

Q3. Evaluate the following limits.

i) $\lim_{x\to 0} sin7x$

Solution:

$$\lim_{x\to 0} sin7x \qquad (\frac{}{})\; form$$
$$= 7\; (\lim_{x\to 0} sin7x) = 7(1) = 7$$
$$\because \lim_{\theta\to 0} sin\theta = 1$$

ii)

$$\lim_{x\to 0} sinx^0$$

Solution:

$$\lim_{x\to 0} sinx^0 \qquad (\frac{0}{})\; form$$
$$= {}_{x\to 0}\; \frac{sin\pi x \; 80}{x} \qquad \because 1^0 = \frac{}{180} rad$$
$$so\; x^0 = \frac{}{180} rad$$
$$\lim_{x\to 0} \frac{sin\pi x}{180} \qquad \frac{\pi}{180}$$
$$\frac{}{180}$$
$$1 \times \frac{}{180} = \frac{}{180}$$

Iii)

$$\lim_{\theta\to 0} \frac{1-cos\theta}{sin\theta}$$

Solution:

$$\lim_{\theta\to 0} \frac{1-cos\theta}{sin\theta} \qquad (\frac{0}{0})\; form$$
$$= \lim_{\theta\to 0} \frac{1-cos\theta}{sin\theta} \times \frac{1+cos\theta}{1+cos\theta}$$
$$= \lim_{0\to 0} \frac{1-\cos^2\theta}{sin\theta(1+cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{0}{1+1} = 0$$

iv)

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x} \qquad \left(\frac{0}{0}\right) form$$

$put\ \pi - x = t$

⇨ $x = \pi - t$

$when\ x\to\pi\ \ then\ t\to o$

So

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x} = \lim_{t\to 0}\frac{\sin(\pi-t)}{t}$$

$$= \lim_{t\to 0}\frac{\sin t}{t} \qquad \because \sin(\pi-\theta) = \sin\theta$$

$= 1$

$v)$

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}$$

Solution:

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} \qquad \left(\frac{0}{0}\right) form$$

$$\lim\left(\frac{\frac{\sin ax}{ax}\times ax}{\frac{\sin bx}{bx}\times bx}\right)$$

$$= \left(\frac{\lim\frac{\sin ax}{ax}\times ax}{\lim_{x\to 0}\frac{\sin bx}{bx}\times bx}\right) = \frac{1\times ax}{1\times bx} = \frac{a}{b}$$

vi)

$$\lim_{x\to 0}\frac{x}{\tan x}$$

Solution:

$$\lim\frac{x}{\tan x} \qquad \left(\frac{0}{0}\right) form$$

$= \lim x.\cot x \qquad \because \cot x = \frac{1}{\tan x}$

$$= \lim x.\frac{\cos x}{\sin x}$$

$$= \lim_{x\to 0}\frac{x}{\sin x}.\lim_{x\to 0}\cos x$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^{-1}.\lim_{x\to 0}\cos x$$

$$= (1)^{-1}.\cos x = 1.1 = 1$$

vii)

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

Solution:

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} \qquad \frac{0}{0} form$$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2} \qquad \because 1-\cos\theta = 2\sin\left(\frac{\theta}{2}\right)$$

⇨ $1-\cos 2\theta = 2\sin^2\theta$

⇨ $= 2\left(\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\right)^2 = 2(1)^2 = 2$

(viii)

$$\lim_{x\to 0}\left(\frac{1-\cos x}{\sin^2\theta}\right)$$

Solution:

$$\lim_{x\to 0}\left(\frac{1-\cos x}{\sin^2\theta}\right) \qquad \left(\frac{0}{0}\right) form$$

$$\lim_{x\to 0}\frac{1-\cos x}{1-\cos^2 x} \qquad \because \sin^2\theta+\cos^2\theta = 1$$

⇨ $\sin^2\theta = 1-\cos^2\theta$

⇨ $\lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)} \qquad = (1-\cos\theta)(1+\cos\theta)$

⇨ $= \lim\frac{1}{1+\cos x} = \frac{1}{1+\cos(0)} = \frac{1}{1+1} = \frac{1}{2}$

ix)

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$$

Solution:

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} \qquad \left(\frac{0}{0}\right) form$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\ \lim_{\theta\to 0}\sin\theta = 1.\sin\theta = 1.0 = 0$$

$x)$

$$\lim_{x\to 0}\frac{\sec x-\cos x}{x}$$

Solution:

$$\lim_{x\to 0}\frac{\sec x-\cos c}{x} \qquad \left(\frac{0}{0}\right) form$$

$$= \lim\frac{1}{x}\left(\frac{1}{\cos x}-\cos x\right) = \lim\left(\frac{1-\cos^2 x}{x\cos x}\right)$$

$$= \lim\left(\frac{1}{\cos x}\right)\left(\frac{\sin^2 x}{x}\right) = \lim\frac{\sin x}{x}.\lim\frac{\sin x}{\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\tan x = 1.\tan 0 = 0$$

xi)

$$\lim_{\theta\to 0}\frac{1-\cos p\theta}{1+\cos q\theta}$$

Solution:

$$\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta} \qquad \left(\frac{0}{0}\right) form$$

$$= \lim_{\theta\to 0}\frac{2\sin^2\left(\frac{p\theta}{2}\right)}{2\sin^2\left(\frac{q\theta}{2}\right)} = \frac{\left(\lim\sin\left(\frac{p\theta}{2}\right)\right)^2}{\left(\lim_{\theta\to 0}\sin\left(\frac{q\theta}{2}\right)\right)^2}$$

$$= \frac{\left(\lim_{\theta\to 0}\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\times\frac{p\theta}{2}\right)^2}{\left(\lim_{\theta\to 0}\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\times\frac{q\theta}{2}\right)^2} = \frac{\left(1\times\frac{p\theta}{2}\right)^2}{\left(1\times\frac{q\theta}{2}\right)^2}$$

$$= \frac{\frac{p^2\theta^2}{4}}{\frac{q^2\theta^2}{4}} = p^2/q^2$$

xii)

$$\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$$

Solution:

$$\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta} \qquad \left(\frac{0}{0}\right) form$$

$$= \lim_{\theta\to0} \frac{1}{sin^3\theta}\left(\frac{sin\theta}{cos\theta} - sin\theta\right)$$

$$= \lim_{\theta\to0} \frac{1}{\sin^3\theta}(sin\theta - sin\theta cos\theta)$$

$$= \lim_{\theta\to0} \frac{sin\theta}{sin^3\theta}(1 - cos\theta) = \lim_{\theta\to0} 1/\sin^2\theta\ (1 - cos\theta)$$

$$\lim_{\theta\to0} \frac{1 - cos\theta}{1 - \cos^2\theta} \qquad \lim_{\theta\to0} \frac{1 - cos\theta}{(1 - cos\theta)(1 + cos\theta)}$$

$$= \frac{1}{1 + cos\theta} = \frac{1}{1+1} = \frac{1}{2}$$

Q4.express each limit in terms of e

i) $\lim_{n\to\infty} (1 + \ )^{2n}$

Solution:

$$\lim_{n\to\infty} (1 + \ )^{2n}$$

$$= [\lim_{n\to\infty} (1 + \ )^{n}]^{2} = e^2$$

ii) $\lim_{n\to\infty} (1 + {}^{1})^{\frac{n}{2}}$

Solution:

$$= [\lim_{n\to\infty}\ 1 + \quad ^{n}]^{1} = e^{\frac{1}{2}}$$

iii) $\lim_{n\to\infty} (1 + \ )^{n}$

Solution:

$$\lim_{n\to\infty} (1 + {}^{1})^{\frac{3n}{3}} = \lim_{n\to\infty} (1 + {}^{1})^{\frac{1}{3}}$$

$$= e^{\frac{1}{3}}$$

iv)

$$\lim_{n\to\infty} (1 - \ )^{n}$$

Solution:

$$\lim_{n\to\infty}\ 1 + (-\ )^{n} = [\lim_{n\to\infty}\ 1 + (-\ )^{-n}]^{-1}$$

$$= e^{-1}$$

v)

$$\lim_{n\to\infty} (1 + {}^{4})^{n}$$

Solution:

$$\lim_{n\to\infty} (1 + {}^{4})^{n}$$

$$= \lim_{n\to\infty} (1 + {}^{4})^{\frac{4n}{4}} = \lim_{n\to\infty} (1 + {}^{4})^{\frac{n}{4}\,4} = e^4$$

vi)

$\lim_{x\to0}(1 + 3x)^{\frac{2}{x}}$

Solution:

$\lim_{x\to0}(1 + 3x)^{\frac{2}{x}}$

$$= \lim_{x\to0}(1 + 3x)^{\frac{2}{x}\times\frac{3}{3}} = \lim_{x\to0}(1 + 3x)^{\frac{6}{3x}}$$

$$= [\lim_{x\to0}(1 + 3x)^{\frac{1}{3x}}]^{6} = e^6$$

vii)

$$\lim_{x\to0}(1 + 2x^2)^{\frac{1}{x^2}}$$

Solution:

$$\lim_{x\to0}(1 + 2x^2)^{\frac{1}{x^2}}$$

$$= \lim_{x\to0}(1 + 2x^2)^{\frac{2}{2x^2}} = \lim_{x\to0}(1 + 2x^2)^{\frac{1}{2x^2}\,2} = e^2$$

viii)

$$\lim_{h\to0}(1 - 2h)^{\frac{1}{h}}$$

Solution:

$$= \lim_{h\to0}(1 - 2h)^{\frac{1}{h}} = \lim_{h\to0}(1 + (-2h))^{\frac{1}{h}}$$

$$= \lim_{h\to0}\ 1 + (-2h)^{\frac{-2}{-2h}} = [\lim_{h\to0}\ 1 + (-2h)^{\frac{1}{-2h}}]^{-2}$$

$$= e^{-2}$$

ix)

$$\lim_{x\to0}(\text{———})^{x}$$

Solution:

$$\lim_{x\to0}(\text{———})^{x}$$

$$= \lim_{x\to0}(\text{———})^{-x} = \lim_{x\to0}(\text{—} + 1)^{x(-1)}$$

$$= [\lim_{x\to0}(\text{—} + 1)^{x}]^{-1} = e^{-1}$$

(*x*)

$$\lim_{x\to0}\left(\frac{e^{\frac{1}{x}} - 1}{e^{x} + 1}\right), x < 0$$

Solution:

$$\lim_{x\to0}\left(\frac{e^{\frac{1}{x}} - 1}{e^{x} + 1}\right)$$

Since $x < 0, so\ let\ x = -t\ where\ t > o$

$as\ x \to 0\ , t \to 0$

$$so\ \lim_{x\to0}\left(\frac{e^{\frac{1}{x}} - 1}{e^{x} + 1}\right) = \lim_{t\to0}\left(\frac{e^{\frac{1}{t}} - 1}{e^{t} + 1}\right)$$

$$= \lim \frac{e^{\frac{1}{1}} - 1}{(e^{\frac{1}{t}} + 1)} = \frac{e^{\frac{1}{1}} - 1}{(e^{\frac{1}{0}} + 1)} = \frac{e^{1} - 1}{e^{\infty} + 1} = \frac{\infty}{\infty + 1}$$

$$= \frac{0 - 1}{0 + 1} = -\frac{1}{1} = -1$$

xi)

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x>0$$

Solution:

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$=\lim_{x\to 0}\frac{e^{\frac{1}{x}}\left(1-\frac{1}{e^{\frac{1}{x}}}\right)}{e^{\frac{1}{x}}\left(1+\frac{1}{e^{\frac{1}{x}}}\right)}$$

$$=\frac{\left(1-\frac{1}{e^{0}}\right)}{\left(1+\frac{1}{e^{0}}\right)}=\frac{e^{\infty}}{1+e^{\infty}}=\frac{\infty}{1+\infty}=\frac{1-0}{}=1$$

The left hand limit:

if $\lim_{x\to a} f(x) = L$ *it means* $f(x)$ *takes value* L *as* x *approaches to a from the left side of "a" (i.e from* $-\infty$ *to a) then* $\lim_{x\to a} f(x) = L$ *is called left hand limit.*

The Right hand limit:

if $\lim_{x\to a} f(x) = L$ *it means* $f(x)$ *takes value* L *as* x *approaches to a from the right side of* a(*i.e from a to* ∞) *then* $\lim_{x\to a} f(x) = L$ *is called right hand limit.*

Existence of Limit of function(criteria)

$\lim_{x\to a} f(x) = L$ *if and only if*

$$\lim_{x\to a} f(x) = \lim_{x\to a} f(x) = L$$

i.e $L.H.S = R.H.S$

Continuous Function:

A function f is said to be continuous at a number $x = o$ *if*

i) $f(a)$ *is defined* ii) $\lim_{x\to a} f(x)$ *exist.* iii) $\lim_{x\to a} f(x) = f(a)$

Discontinuous function:

A function $f(x)$ is said to be discontinuous at $x = a$ if $\lim_{x\to a} f(x) \neq f(a)$

- *if* $f(x)$ *is not defined at* $x = a$ *then* $f(x)$ *is called discontinuous*
- Any function which does not satisfied at least one of three conditions of continuous is called discontinuous.

Exercise 1.4

Q1. Determine the left hand limit and the right hand limit and then find the limit of the following functions when $x \to c$

i) $\boldsymbol{f(x) = 2x^2 + x - 5,\ c = 1}$

Solution:

L.H.S

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x^2 + x - 5) = 2(1)^2 + 1 + 5$$
$$= -2$$

R.H.S

$$\lim_{x\to +} f(x) = \lim_{x\to 1^+}(2x^2 + x - 5) = 2(1)^2 + 1 + 5$$
$$= -2$$

As $L.H.S = R.H.S$

So,

$\lim_{x\to 1} f(x) = -2$

ii)

$f(x) = \frac{}{x-3}, c = -3$

Solution:

L.H.S

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} x^2 - 9$$

$$\lim_{x\to -3} (x-3)(x+3) = \lim_{x\to -3} x + 3 = -3 + 3 = 0$$

R.H.S

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} x^2 - 9$$

$$\lim_{x\to -3} (x-3)(x+3) = \lim_{x\to -3} x + 3 = -3 + 3 = 0$$

As $L.H.S = R.H.S$

So,

$$\lim_{x\to -3} f(x) = 0$$

iii)

$f(x) = |x-5|, c = 5$

Solution:

L.H.S

$$\lim_{x\to 5} f(x) = \lim_{x\to 5}|x-5| = 5 - 5 = 0$$

R.H.S

$$\lim_{x\to 5} f(x) = \lim_{x\to 5}|x-5| = 5 - 5 = 0$$

As

$L.H.S = R.H.S$

So

$$\lim_{x\to 5} f(x) = 0$$

Q2. ***Discuss the continuous of $f(x)$ at $x = c$***

i)

$$\boldsymbol{2x+5 \quad if\ x \le 1}$$
$$\boldsymbol{4x+1 \quad if\ x > 2}$$

Solution:

L.H.S

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(2x+5) = 2(2) + 5 = 9$$

R.H.S

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(4x+1) = 4(2) + 1 = 9$$

At $x = 2$

$f(x) = 2x + 5$

⇨ $f(2) = 2(2) + 5 = 9$

As $L.H.S = R.H.S$ *so*

$$\lim_{x\to 2} f(x) = 9$$

⇨ $\lim_{x\to 2} f(x) = f(x)$ *is continous at* $x = 2$

ii)

$$f(x)=\begin{cases}3x-1 & if\ x<1\\ 4 & if\ x=1,\ \ c=1\\ 2x & if\ x>1\end{cases}$$

Solution:

L.H.S

$$\lim_{x\to1} f(x)=\lim_{x\to1^-}(3x-1)=3(1)-1=2$$

R.H.S

$$\lim_{x\to1} f(x)=\lim_{x\to1}(2x)=2(1)=2$$

$$at\ x=1, f(x)=4\Rightarrow f(1)=4$$

$$as\ L.H.S=R.H.S\ so\ \lim_{x\to1} f(x)\ exist.$$

But $\lim_{x\to1} f(x)\neq f(1)$hence f(x) is discontinuous.

Q3.if $f(x)=\begin{cases}3x & if\ x\le-2\\ x^2-1 & if\ -2<x<2\\ 3 & if\ x\ge2\end{cases}$

Discuss continuity at $x=2$ ***and*** $x=-2$

Solution:

i)

$x=2$

$$L.H.S;.\ \lim_{x\to2} f(x)=\lim_{x\to2}(x^2-1)$$

$$=(2)^2-1=3$$

$$R.H.S\ \lim_{x\to2} f(x)=\lim_{x\to2} 3=3$$

$$at\ x=2, f(x)=3\Rightarrow f(2)=3$$

$$\because L.H.S=R.H.S\ so\ \lim_{x\to2} f(x)\ exist.$$

$$so\ \lim_{x\to2} f(x)=f(2)$$

$$hence\ f\ is\ continous\ at\ x=2$$

ii) $x=-2$

$$R.H.S;.\ \lim_{x\to-2} f(x)=\lim_{x\to-2}(x^2-1)$$

$$=(-2)^2-1=3$$

$$L.H.S\ \lim_{x\to-2} f(x)=\lim_{x\to-2} 3x=3(-2)=-6$$

$$at\ x=-2, f(x)=3x\Rightarrow f(-2)=3(-2)=-6$$

$$\because L.H.S\neq R.H.S\ so\ .$$

$$hence\ f(x) is\ discontinous\ at\ x=-2$$

Q4. If $f(x)=\{^{x+2,x\le-1}$ ***if and "c"***

so that $\lim_{x\to-1} f(x)$ ***exist.***

solution:

L.H.S

$$\lim_{x\to-1} f(x)=\lim_{x\to-1}(x+2)=-1+2=1$$

$$R.H.S=\lim_{x\to-1} f(x)=\lim_{x\to-1}(c+2)=c+2$$

Given that $\lim_{x\to-1} f(x)\ exist\ .so$

$$L.H.S=R.H.S$$

- ⇨ $1+c+2$
- ⇨ $1-2=c$
- ⇨ $c=-1$

Q5. Find the value of m and n, so that given function f **is continuous** $at\ x=3$

$$\begin{cases}mx\ if\ x<3\\ n\ if\ x=3\\ -2x+9\ if\ x>3\end{cases}$$

Solution:

$$L.H.S=\lim_{x\to3^-} f(x)=\lim_{x\to3^-}(mx)=3m$$

$$R.H.S=\lim_{x\to3^+} f(x)=\lim_{x\to3^+}(-2x+9)$$

$$=-2(3)+9=3$$

$$at\ x=3\ f(x)=n\ \Rightarrow f(3)=n$$

Given that $f(x) is\ continuous\ so\ L.H.S=R.H.S$

- ⇨ $3m=3$
- ⇨ $m=1$

We know that for a continuous function

$$\lim_{x\to3} f(x)=\lim_{x\to3} f(x)=f(3)$$

$$3m=3=n$$

- ⇨ $n=3, m=1$

i)

$$\begin{cases}mx\ if\ x<3\\ x^2\ if\ x\ge3\end{cases}$$

Solution:

$$=\lim_{x\to3} f(x)=\lim_{x\to3}(mx)=3m$$

$$R.H.S=\lim_{x\to3} f(x)=\lim_{x\to3}(x^2)$$

$$=(3)^2=9$$

$$at\ x=3\ f(x)=x^2\Rightarrow f(3)=(3)^2=9$$

Given that $f(x) is\ continuous\ so\ L.H.S=R.H.S$

- ⇨ $3m=9$
- ⇨ $m=3$

Q6. If $f(x)=\begin{cases}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\\ k\ \ ,x=2\end{cases}$

Find value of k ***so that*** f ***is continuos.***

Solution:

$$atx=2\ f(x)=k\ \ \Rightarrow f(2)=k$$

Now $\lim_{x\to2} f(x)=\lim_{x\to2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\ \ (\frac{0}{0})\ form$

$$\lim_{x\to2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\cdot\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$\lim_{x\to2}\frac{2x+5-x-7}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$=\lim\frac{}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$=\lim\frac{}{(\sqrt{2x+5}+\sqrt{x+7})}$$

$$\frac{1}{(\sqrt{2(2)+5}+\sqrt{2+7})}\quad\frac{1}{6}$$

$\because$ *given function is continuous at* $x=2$so

$$\lim_{x\to2} f(x)=f(2)$$

- ⇨ $\frac{1}{}=k\Rightarrow k=1/6$