MATHEMATICS 11 INTERMEDIATE PART 1

Bilal Article

Chapter 8.

Mathematical Induction and Binomial Theorem

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Chapter#8 Class 1st **Mathematical Induction and Binomial Theorem**

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Mathematical Induction

A method of testing formulas, theorem, statements and properties is called Mathematical induction method.

- This method is particularly used in series sum.

Principle of mathematical induction.

If a proposition $S(n)$ satisfies the following two conditions.

$C-1 \quad S(n) is\ true\ for\ n=1$

$C-2 \quad S(n)\ is\ true\ for\ n=k$

$\Rightarrow S(n)\ is\ true\ for\ n=k+1$

Then $S(n)$ is true for all positive integral valves of n.

Principle of Extended Mathematical induction:

Sometimes we want to prove formulas or results which are true for all integer n greater than or equal to some integer $i\ i.e; n \geq i \quad where\ i \neq 1$

In such cases we check formulas extended mathematical induction.

Exercise 8.1

Use mathematical induction to prove the following formulae for every positive integer n

Question # 1.

$$1+5+9+\cdots+(4n-3)=n(2n-1)$$

Solution. Suppose $S(n): 1+5+9+\cdots+(4n-3)=n(2n-1)$

Put $n=1$

$$S(1): 1=1(2(1)-1) \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that the condition $S(n)$ is true for $n=k$

$$S(k): 1+5+9+\cdots+(4k-3)=k(2k-1) \rightarrow (i)$$

The Statement for $n=k+1$

$$S(K+1): 1+5+9+\cdots+(4(k+1)-3)=(k+1)(2(k+1)-1)$$

$$\Rightarrow \quad 1+5+9+\cdots+(4k+1)=(k+1)(2k+2-1)$$

$$\Rightarrow \quad 1+5+9+\cdots+(4k+1)=(k+1)(2k+1) \rightarrow (ii)$$

Adding $4k+1$ on both sides of (i),we have

$$1+5+9+\cdots+(4k-3)+4k+1=k(2k-1)+4k+1$$

$$1+5+9+\cdots+(4k+1)=2k^2-k+4k+1$$

$$1+5+9+\cdots+(4k+1)=2k^2+3k+1$$

$$=2k^2+2k+k+1$$

$$=2k(k+1)+1(k+1)$$

$$=(k+1)(2k+1)$$

Thus condition $S(K+1)$ is true if $S(K)$ is true, So condition II is Satisfied and Hence $S(n)$ is true for all positive integer n.

Question # 2.

$$1+3+5+\cdots+(2n-1)=n^2$$

Solution. Suppose $S(n): 1+3+5+\cdots+(2n-1)=n^2$

Put $n=1$

$$S(1): 1=1^2 \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that the condition $S(n)$ is true for $n=k$

$$S(k): 1+3+5+\cdots+(2k-1)=k^2 \rightarrow (i)$$

The Statement for $n=k+1$

$$S(K+1): 1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$$

$$\Rightarrow \quad 1+3+5+\cdots+(2k+1)=(k+1)^2$$

Adding $2k+1$ on both sides of (i),we have

$$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)$$
$$1+3+5+\cdots+(2k+1)=k^2+2k+1^2$$
$$1+3+5+\cdots+(2k+1)=(k+1)^2$$

Thus condition $S(K+1)$ is true if $S(K)$ is true, So condition II is Satisfied and Hence $S(n)$ is true for all positive integer n.

Question # 3.

$$1+3+5+\cdots+(2n-1)=n^2$$

Solution. Suppose $S(n): 1+3+5+\cdots+(2n-1)=n^2$

Put $n=1$

$$S(1): 1=1^2 \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that the condition $S(n)$ is true for $n=k$

$$S(k): 1+3+5+\cdots+(2k-1)=k^2 \rightarrow (i)$$

The Statement for $n=k+1$

$$S(K+1): 1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$$
$$\Rightarrow \quad 1+3+5+\cdots+(2k+1)=(k+1)^2$$

Adding $2k+1$ on both sides of (i),we have

$$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)$$
$$1+3+5+\cdots+(2k+1)=k^2+2k+1^2$$
$$1+3+5+\cdots+(2k+1)=(k+1)$$

Thus condition $S(K+1)$ is true if $S(K)$ is true, So condition II is Satisfied and Hence $S(n)$ is true for all positive integer n.

Question #4.

Prove that $1+2+4+\cdots+2^{n-1}=2^n-1$

Solution. Suppose that

$$S(n): 1+2+4+\cdots+2^{n-1}=2^n-1$$

Put $n=1$

$$S(1): 1=2^1-1 \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1+2+4+\cdots+2^{k-1}=2^k-1 \rightarrow (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): 1+2+4+\cdots+2^{k+1-1}=2^{k+1}-1$$
$$\Rightarrow \quad 1+2+4+\cdots+2^k=2^{k+1}-1$$

Adding 2^K on both sides of equation (i), we have

$$1+2+4+\cdots+2^{k-1}+2^k=2^k-1+2^k$$
$$\Rightarrow \quad 1+2+4+\cdots+2^k=2^k(2)-1$$
$$\Rightarrow \quad 1+2+4+\cdots+2^k=2^{k+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question #5.

Prove that $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$

Solution. Suppose that

$$S(n): 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$$

Put $n=1$

$$S(1): 1=2\left[1-\frac{1}{2}\right] \Rightarrow 1=2\left[\frac{1}{2}\right] \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} = 2\,[1 - \frac{1}{2^k}] \rightarrow (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k+1-1}} = 2\,[1 - \frac{1}{2^{k+1}}]$$

$$\Rightarrow\ 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 2 - \frac{2}{2^{k+1}}$$

$$\Rightarrow\ 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 2 - \frac{1}{2^k}$$

Adding ${}_{2^k}$ on both sides of equation (i), we have

$1 + {}^1 + {}^1 + \cdots + {}_2{}^1{}_1 + {}^1 = 2\,[1 - {}^1] + {}^1$

$$\Rightarrow\ 1 + {}_2 + {}_4 + \cdots + {}_{2^k} = 2 - {}_{2^k} + {}_{2^k}$$

$$\Rightarrow\ 1 + {}_2 + {}_4 + \cdots + {}_{2^k} = 2 - {}_{2^k}(2 - 1)$$

$$\Rightarrow\ 1 + {}_2 + {}_4 + \cdots + {}_{2^k} = 2 - {}_{2^k}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 6

$$\mathbf{2 + 4 + 6 + \cdots + 2n = n(n+1) \rightarrow (i)}$$

Solution:

For $n = 1$

$$2(1) = 1(1+1) \Rightarrow 2 = 2$$
$$\Rightarrow (i)\ is\ true\ for\ n = 1$$
$$C - 1\ is\ satisfied.$$
$$Suppose\ (i) is\ true\ for\ n = k\ \ i.e;$$
$$2 + 4 + 6 + \cdots + 2k = k(k+1) \rightarrow (ii)$$
$$we\ shall\ prove\ taht\ (i) is\ true\ for\ n = k+1\ \ i.e$$
$$2 + 4 + 6 + \cdots + 2k + 2(k+1) = (k+1)(k+1+1)$$
$$2 + 4 + 6 + \cdots + 2k + 2(k+1) = (k+1)(k+2)$$
$$L.H.S = 2 + 4 + 6 + \cdots + 2k + 2(k+1)$$
$$= k(k+!) + 2(k+1)\ \ by\ (ii)$$
$$= (k+1)(k+2) = R.H.S$$
$$\Longrightarrow (i) is\ true\ for\ n = k+1,\quad C - 2$$
$$is\ satisfied. hence\ (i)\ is\ true\ for\ all\ integers\ n.$$

Question #7.

Prove that $\mathbf{2 + 6 + 18 + \cdots + 2 \times 3^{n-1} = 3^n - 1}$

Solution. Suppose that

$$S(n): 2 + 6 + 18 + \cdots + 2 \times 3^{n-1} = 3^n - 1$$

Put $n = 1$

$$S(1): 2 = 3^1 - 1 \Rightarrow 2 = 2$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): 2 + 6 + 18 + \cdots + 2 \times 3^{k-1} = 3^k - 1 \rightarrow (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): 2 + 6 + 18 + \cdots + 2 \times 3^{k+1-1} = 3^{k+1} - 1$$
$$\Rightarrow\ 2 + 6 + 18 + \cdots + 2 \times 3^k = 3^{k+1} - 1$$

Adding 2×3^k on both sides of equation (i), we have

$$2 + 6 + 18 + \cdots + 2 \times 3^{k-1} + 2 \times 3^k = 3^k - 1 + 2 \times 3^k$$
$$\Rightarrow\ 2 + 6 + 18 + \cdots + 2 \times 3^k = 3^k(3) - 1$$

$$\Rightarrow\ 2+6+18+\cdots+2\times 3^{k}=3^{k+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question #8.

Prove that $\mathbf{1\times 3+2\times 5+3\times 7+\cdots+n\times(2n+1)=\frac{n(n+1)(4n+5)}{6}}$

Solution. Suppose that

$$S(n): 1\times 3+2\times 5+3\times 7+\cdots+n\times(2n+1)=n(n+1)(4n+5)$$

Put $n=1$

$$S(1): 1\times 3 = 1(2)(9) \Rightarrow 3=3$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1)=k(k+1)(4k+5)\ \rightarrow (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2(k+1)+1)=(k+1)(k+1+1)(4(k+1)+5)$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=(k+1)(k+2)(4k+9)$$

Adding $(k+1)(2k+3)$ on both sides of equation (i), we have

$$1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1)+(k+1)(2k+3)=k(k+1)(4k+5)+(k+1)(2k+3)$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[k(4k+5)+6(2k+3)]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[4k^{2}+5k+12k+18]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[4k^{2}+17k+18]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[4k^{2}+8k+9k+18]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[4k(k+2)+9(k+2)]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=k+1\,[(k+2)(4k+9)]$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+\cdots+(k+1)(2k+3)=(k+1)(k+2)(4k+9)$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question No.9

$$\mathbf{\times\ \ \times 3\ \ 3\ \ 4\ \ +\ \ =}$$
$$\mathbf{1\ \ 2+2\ \ +\ \ \times\ \ +\cdots\ \ n\times(n+1)\ \ n(n+1)(n+2)}$$

Solution:

$$\mathbf{1\times 2+2\times 3+3\times 4+\cdots+n\times(n+1)=n(n+1)(n+2)}\ \rightarrow (\boldsymbol{i})$$

For $n=1$

$$1\times(1+1) = 1(1+1)(1+2)$$

$$\Rightarrow 2=2$$

$$\Rightarrow (i)\ is\ true\ for\ n=1\ \ ,\ C-1\ is\ true\ satisfied. Suppose\ (i)\ \ is\ true\ for\ n=k\ \ i.e,$$

$$1\times 2+2\times 3+\cdots+k\times(k+1)=\frac{k(k+1)(k+2)}{3}\rightarrow (ii)$$

We shall prove that (i) is true for $n=k+1\ \ i.e;$

$$1\times 2+2\times 3+\cdots+k\times(k+1)+(k+1)\times(k+1+1)=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

$$1\times2+2\times3+\cdots+k\times(k+1)+(k+1)\times(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$$

$$L.H.S=1\times2+2\times3+\cdots+k\times(k+1)+(k+1)\times(k+2)$$

$$=\frac{k(k+1)(k+2)}{3}+(k+1)\times(k+2)\ by\ (ii)$$

$$=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$$

$$=\frac{(k+1)(k+2)\{k+3\}}{}$$

$$=\frac{(k+1)(k+2)(k+3)}{}=R.H.S$$

$$\Rightarrow (i)\ is\ true\ for\ n=k+1$$

$$C-2\ is\ satisfied\ .Hence\ (i) is\ true\ \ for\ all\ integers\ n.$$

Question No.10

$$\mathbf{1\times2+3\times4+5\times6+\cdots+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{}}$$

Solution:

$$1\times2+3\times4+5\times6+\cdots+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{}\rightarrow(i)$$

For $n=1$

$$C-1\ is\ satisfied. Suppose\ (i)\ is\ true\ for\ n=k\ i.e$$

$$1\times2+3\times4+\cdots+\ (2k-1)\times2k=\frac{k(k+1)(4k-1)}{}\rightarrow(ii)$$

We shall prove that (i) *is true for* $n=k+1$ *i.e*

$$1\times2+3\times4+\cdots+(2k-1)\times2k+(2(k+1)-1)\times2(k+1)=\frac{(k+1)(k+1+1)(4(k+1)-1)}{}$$

$$1\times2+3\times4+\cdots+(2k-1)\times2k+(2k+1)\times2(k+1)=\frac{(k+1)(k+2)(4k+3)}{}$$

$$L.H.S=1\times2+3\times4+\cdots+(2k-1)\times2k+(2k+1)\times2(k+1)$$

$$=\frac{k(k+1)(4k-1)}{}+(2k+1)\times2(k+1)\ by\ (ii)$$

$$\frac{k(k+1)(4k-1)+6(2k+1)(k+1)}{3}$$

$$=\frac{(k+1)\{4k^2-k+12k+6\}}{}$$

$$\frac{(k+1)\{4k^2+8k+3k+6\}}{3}$$

$$=\frac{(k+1)\{4k^2+8k+3k+6\}}{}$$

$$=\frac{(k+1)\{4k(k+2)+3(k+2)\}}{}$$

$$=\frac{(k+1)(k+2)(4k+3)}{}=R.H.S$$

$$\Rightarrow (i) is\ true\ for\ n=k+1, C-2\ is\ satisfied. Hence\ (i) is\ true\ for\ all\ integral\ n.$$

Question #11.

Prove that $\frac{}{1\times2}+\frac{}{2\times3}+\frac{}{3\times4}+\cdots+\frac{}{n(n+1)}=1-\frac{}{n+1}$

Solution. Suppose that

$$S(n):\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Put $n=1$

$$S(1):\frac{1}{1(2)}=1-\frac{1}{2}\Rightarrow\frac{1}{2}=\frac{1}{2}$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k):\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}\to(i)$$

The Statement for $n=k+1$ becomes

$$S(k+1):\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{(k+1)(k+1+1)}=1-\frac{1}{k+1+1}$$

$$\Rightarrow\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$$

Adding $\frac{1}{(k+1)(k+2)}$ on both sides of equation (i), we have

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}\left[1-\frac{1}{(k+2)}\right]$$

$$\Rightarrow\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}\left[\frac{k+2-1}{(k+2)}\right]$$

$$\Rightarrow\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 12.

$$\mathbf{\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}}$$

Solution. Suppose that

$$S(n):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Put $n=1$

$$S(1):\frac{1}{1\times3}=\frac{1}{2+1}\Rightarrow\frac{1}{3}=\frac{1}{3}$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}----(i)$$

The Statement for $n=k+1$ becomes

$$S(k+1):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{k+1}{2(k+1)+1}$$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Adding $\frac{1}{(2k+1)(2k+3)}$ on both sides of equation (i), we have

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}\left[k+\frac{1}{(2k+3)}\right]$$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}\frac{2k^2+3k+1}{(2k+3)}$$

$$\Rightarrow\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}\left[\frac{2k^2+2k+k+1}{(2k+3)}\right]$$

$$\Rightarrow \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}[\frac{2k(k+1)+1(k+1)}{(2k+3)}]$$

$$\Rightarrow \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}\left[\frac{(2k+1)(k+1)}{(2k+3)}\right]$$

$$\Rightarrow \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{(k+1)}{(2k+3)}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 13.

$$\frac{\mathbf{1}}{\mathbf{2\times 5}}+\frac{\mathbf{1}}{\mathbf{5\times 8}}+\frac{\mathbf{1}}{\mathbf{8\times 11}}+\cdots+\frac{\mathbf{1}}{\mathbf{(3n-1)(3n+2)}}=\frac{\boldsymbol{n}}{\mathbf{2(3n+2)}}$$

Solution. Suppose that

$$S(n): 2\times 5 + 5\times 8 + 8\times 11 + \cdots + (3n-1)(3n+2) = 2(3n+2)$$

Put $n=1$

$$S(1): 2\times 5 = 2(3+2) \Rightarrow 10 = 10$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 2\times 5 + 5\times 8 + 8\times 11 + \cdots + \frac{1}{(3k-1\ \ 3k+2)} = 2(3k+2)$$

For $n=k+1$ *then staement is*

$$S(k+1): 2\times 5 + 5\times 8 + 8\times 11 + \cdots + (3(k+1)-1)(3(k+1)+2) = 2(3(k+1)+2)$$

$$\frac{1}{2\times 5}\quad \frac{1}{5\times 8}\quad \frac{1}{8\times 11}\quad \frac{1}{(3k+3-1)(3k+3+2)}\quad \frac{k+1}{2(3k+3+2)}$$

$$\frac{1}{2\times 5}\quad \frac{1}{5\times 8}\quad \frac{1}{8\times 11}\quad \frac{1}{(3k+2)(3k+5)}\quad \frac{k+1}{2(3k+5)}$$

$$\textit{adding both sides}\ (3k+2)(3k+5)\ \textit{in}\ (i)$$

$$2\times 5 + 5\times 8 + 8\times 11 + \cdots + (3k-1)(3k+2) + (3k+2)(3k+5)$$

$$\frac{k}{2(3k-2)}\quad \frac{1}{(3k+2)(3k+5)}\quad \frac{1}{(3k+2)}\ \frac{k}{2}\ \frac{1}{3k+5}$$

$$\frac{k}{3k+2}\quad \frac{3k^2+5k+2}{2(3k+5)}\quad \frac{1}{(3k+2)}\quad \frac{3k^2+3k+2k+2}{2(3k+5)}$$

$$\frac{1}{(3k+2)}\quad \frac{3k(k+1)+2(k+1)}{2(3k+5)}\quad \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)}\quad \frac{k+1}{2(3k+5)}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 14.

$$\boldsymbol{r}+\boldsymbol{r}^{\mathbf{2}}+\boldsymbol{r}^{\mathbf{3}}+\cdots+\boldsymbol{r}^{\boldsymbol{n}}=\boldsymbol{r}(\mathbf{1}-\boldsymbol{r}^{\boldsymbol{n}})$$

Solution. Suppose that

$$S(n): r+r^2+r^3+\cdots+r^n = r(1-r^n)$$

Put $n=1$

$$S(1): r=\frac{r(1-r)}{1-r}\Rightarrow r=r$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): r+r^2+r^3+\cdots+r^k=\frac{r(1-r^k)}{1-r}\rightarrow (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r(1-r^{k+1})}{1-r}$$

Adding r^{k+1} on both sides of equation (i), we have

$$r + r^2 + r^3 + \cdots + r^k + r^{k+1} = \frac{r(1-r^k)}{1-r} + r^{k+1}$$

$$r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r - r^{k+1} + r^{k+1} - r^{k+2}}{1-r}$$

$$\Rightarrow \; r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r - r^{k+2}}{r}$$

$$\Rightarrow \; r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r(1-r^{\;+1})}{r}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 15.

$$\boldsymbol{a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = {}_{2}[2a+(n-1)d]}$$

Solution. Suppose that

$$S(n): a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = {}_{2}[2a+(n-1)d]$$

Put $n = 1$

$$S(1): a = {}_{2}[2a + (\;1 - 1)\;d] \Rightarrow a = a$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): a + (a+d) + (a+2d) + \cdots + [a+(k-1)d] = {}^{k}[2a+(k-1)d] \rightarrow (i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): a + (a+d) + (a+2d) + \cdots + [a+(k+1-1)d] = {}^{k+1}[2a+(k+1-1)d]$$

$$a + (a+d) + (a+2d) + \cdots + [a+kd] = {}^{k+1}[2a+kd]$$

Adding $[a+kd]$ on both sides of equation (i), we have

$$a + (a+d) + (a+2d) + \cdots + [a+(k-1)d] + [a+kd] = {}^{k}[2a+(k-1)d] + [a+kd]$$

$$a + (a+d) + (a+2d) + \cdots + [a+kd] = {}^{1}[2ka + k(k-1)d + a + kd]$$

$$\Rightarrow \; a + (a+d) + (a+2d) + \cdots + [a+kd] = {}_{2}[2ka + k^2d - kd + 2a + 2kd]$$

$$\Rightarrow \; a + (a+d) + (a+2d) + \cdots + [a+kd] = {}_{2}[2ka + k^2d + 2a + kd]$$

$$\Rightarrow \; a + (a+d) + (a+2d) + \cdots + [a+kd] = {}^{1}[k(2a+kd) + 1(2a+kd)]$$

$$\Rightarrow \; a + (a+d) + (a+2d) + \cdots + [a+kd] = {}^{1}[(2a+kd)(k+1)]$$

$$\Rightarrow \; a + (a+d) + (a+2d) + \cdots + [a+kd] = {}^{(k+1)}[(2a+kd)]$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 16.

$$\boldsymbol{1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1}$$

Solution. Suppose that

$$S(n): 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

Put $n = 1$

$$S(1): 1 \cdot 1! = (1+1)! - 1 \Rightarrow 1 = 2! - 1 \Rightarrow 1 = 2 - 1 \Rightarrow 1 = 1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1 ----(i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (k+1)\cdot(k+1)! = (k+1+1)! - 1$$

$$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (k+1)\cdot(k+1)! = (k+2)! - 1$$

Adding $(k+1)\cdot(k+1)!$ on both sides of equation (i), we have

$$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + k\cdot k! + (k+1)\cdot(k+1)! = (k+1)! - 1 + (k+1)\cdot(k+1)!$$

$$\Rightarrow 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (k+1)\cdot(k+1)! = (k+1)!\,(1+k+1) - 1$$

$$\Rightarrow 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (k+1)\cdot(k+1)! = (k+1)!\,(k+2) - 1$$

$$\Rightarrow 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (k+1)\cdot(k+1)! = (k+2)! - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 17.

$$\boldsymbol{a_n = a_1 + (n-1)d} \qquad \boldsymbol{When\ a_1, a_1+d, a_1+2d, \dots form\ an\ A.P.}$$

Solution. Suppose that

$$S(n): a_n = a_1 + (n-1)d$$

Put $n = 1$

$$S(1): a_1 = a_1 + (1-1)d \Rightarrow a_1 = a_1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): a_k = a_1 + (k-1)d ----(i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): a_{k+1} = a_1 + (k+1-1)d$$

$$a_{k+1} = a_1 + kd$$

Adding d on both sides of equation (i), we have

$$a_k + d = a_1 + (k-1)d + d$$

$$\Rightarrow a_{k+1} = a_1 + (k-1+1)d$$

$$\Rightarrow a_{k+1} = a_1 + (k)d$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 18.

$$\boldsymbol{a_n = a_1 r^{n-1}} \qquad \boldsymbol{When\ a_1, a_1 r, a_1 r^2, \dots form\ an\ G.P.}$$

Solution. Suppose that

$$S(n): a_n = a_1 r^{n-1}$$

Put $n = 1$

$$S(1): a_1 = a_1 r^{1-1} \Rightarrow a_1 = a_1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): a_k = a_1 r^{k-1} \to (i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): a_{k+1} = a_1 r^{k+1-1}$$

$$a_{k+1} = a_1 r^k$$

Multiplying r on both sides of equation (i), we have

$$a_k r = a_1 r^{k-1} r$$

$$\Rightarrow a_{k+1} = a_1 r^{k-1+1}$$

$$\Rightarrow a_{k+1} = a_1 r^k$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 19.

$$\boldsymbol{1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}}$$

Solution. Suppose that

$$S(n): 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Put $n = 1$

$$S(1): 1^1 = \frac{1(4\cdot 1^2 - 1)}{3} \Rightarrow 1^1 = \frac{3}{3} \Rightarrow 1 = 1$$

Thus condition I is satisfied.
Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1^2+3^2+5^2+\cdots+(2k-1)^2=\frac{k(4k^2-1)}{3} \qquad \rightarrow (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): 1^2+3^2+5^2+\cdots+(2(k+1)-1)^2=\frac{(k+1)(4(k+1)^2-1)}{3}$$

$$1^2+3^2+5^2+\cdots+(2k+2-1)^2=(k+1)(4(k^2+2k+1)-1)$$

$$1^2+3^2+5^2+\cdots+(2k+1)^2=(k+1)(4k^2+8k+4-1)$$

$$1^2+3^2+5^2+\cdots+(2k+1)^2=(k+1)(4k^2+8k+3)$$

$$1^2+3^2+5^2+\cdots+(2k+1)^2=4k^3+8k^2+3k+4k^2+8k+3$$

$$1^2+3^2+5^2+\cdots+(2k+1)^2=4k^3+12k^2+11k+3$$

Multiplying $(2k+1)^2$ on both sides of equation (i), we have

$$1^2+3^2+5^2+\cdots+(2k-1)^2+(2k+1)^2=k(4k^2-1)+(2k+1)^2$$

$$\Rightarrow 1^2+3^2+5^2+\cdots+(2k+1)^2=k(4k^2-1)+3(2k+1)^2$$

$$\Rightarrow 1^2+3^2+5^2+\cdots+(2k+1)^2=4k^3-k+3(4k^2+4k+1)$$

$$\Rightarrow 1^2+3^2+5^2+\cdots+(2k+1)^2=4k^3-k+12k^2+12k+3$$

$$\Rightarrow 1^2+3^2+5^2+\cdots+(2k+1)^2=4k^3+12k^2+11k+3$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 20.

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$$

Solution. Suppose that

$$S(n): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$$

Put $n=1$

$$S(1): \binom{3}{3}=\binom{1+3}{4} \Rightarrow \binom{3}{3}=\binom{4}{4} \Rightarrow 1=1$$

Thus condition I is satisfied.
Now Suppose that $S(n)$ is true for $n=k$

$$S(k): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}=\binom{k+3}{4} \qquad \rightarrow (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+1+2}{3}=\binom{k+1+3}{4}$$

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+3}{3}=\binom{k+4}{4}$$

Multiplying $\binom{k+3}{3}$ on both sides of equation (i), we have

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+3}{4}+\binom{k+3}{3}$$

Since $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$

$$\Rightarrow \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+3}{3}=\binom{k+3+1}{4}$$

$$\Rightarrow \binom{}{3} + \binom{}{3} + \binom{5}{3} + \cdots + \binom{k+3}{3} = \binom{k+4}{4}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer

Question # 21.

Prove by Mathematical INDUCTION that for all positive integral values of n.

(i). n^2+n is divisible by 2

Solution. Suppose that

$$S(n): n^2+n$$

Put $n=1$

$$S(1): 1^2+1=2$$

Clearly $S(1)$ Is divisible by 2 .Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): k^2+k$$

Then there exit a quotient Q Such that

$$k^2+k=2Q$$

The Statement for $n=k+1$ becomes

$$S(k+1): (k+1)^2+(k+1)$$
$$=k^2+2k+1+k+1$$
$$=k^2+k+2+2k$$
$$=2Q+2(1+k)$$
$$=2[Q+(1+k)]$$

Clearly $S(k+1)$ is divisible by 2

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integers.

ii) 5^n-2^n $\textit{is divisible by}$ 3

solution:

$$\mathbf{5^n-2n\ \textit{is divisible by}\ 3 \rightarrow (i)}$$
$$5^n-2^n=5^1-2^1=5-2=3$$

Which is divisible by 3 $C-1$ $\textit{is satisfied. suppose}$ (i) $\textit{is true fon}$ $n=k$ $\textit{i.e};$

$$5^k-2^k \textit{ is divisible by } 3$$
$$\Rightarrow\ 5^k-2^k=3Q\rightarrow(ii)$$

We shall prove that (i) $\textit{is true for}$ $n=k+1$ $\textit{i.e}$

$$5^{k+1}-2^{k+1} \textit{ is divisible by } 3$$

Now

$$5^{k+1}-2^{k+1} \textit{ is divisible by } 3$$

Now

$$5^{k+1}-2^{k+1}=5^k.5^1-2^k.2^1$$
$$5^k(3+2)-2^k.2$$
$$=3.5^k+2.5^k-2.2^k$$
$$=3.5^k+2.5^k-2.2^k$$
$$=3.5^k+2(5^k-2^k)$$
$$3.5^k+2(3Q)\ \textit{by}\ (ii)$$
$$=3(5^k+2Q)$$

Which is clearly divisible by 3 $C-2$ $\textit{is satisfied. Hence}$ (i) is satisfied for all integers n.

(iii) 5^n-1 $\textit{is divisible by}$ 4

Solution:

$$5^n-1\ \textit{ is divisible by } 4\rightarrow(i)$$
$$\textit{for } n=1$$
$$5^n-1=5^1-1=5-1=4 \textit{ which is divisible by } 4.$$
$$C-1 \textit{ is satisfied.}$$
$$\textit{Suppose } (i) \textit{is true for } n=k\ \textit{i.e}$$
$$5^k-1=4Q\rightarrow(ii)$$
$$\textit{we shall prove that } (i) \textit{is true for } n=k+1\ \textit{i.e};$$
$$5^{k+1}-1=5^k.5^1-1$$
$$=5^k(4+1)-1$$

$$4.5^k + 1.5^k - 1$$
$$4.5^k + (5^k - 1)$$
$$4.5^k + 4Q \text{ by (ii)}$$
$$4(56k + Q)$$
Which is clearly divisible by 4.

$C - 2$ *is satisfied, Hence* (i) *is true for all* $+ve$ *integers* n.

(iv) $\mathbf{8 \times 10^n - 2}$ ***is divisible by 6***

Solution:

$$8 \times 10^n - 2 \text{ is divisible by } 6 \to (i)$$
For n=1
$$8 \times 10^n - 2 = 8 \times 10^1 - 2 = 80 - 2 = 78 = 6 \times 13$$
Which is divisible by 6 $C - 1$ *is satisfed.*

Suppose (i) is true for $n = k + 1$ i.e;
$$8 \times 10^{k+1} - 2 \text{ is divisible by } 6$$
Now
$$8 \times 10^{k+1} - 2 = 8 \times 10^k - 2$$
$$= 80 \times 10^k - 2$$
$$= (72 + 8) \times 10^k - 2$$
$$= 6 \times 12 \times 10^k + 6Q \quad by\ (ii)$$
$$= 6\{12 \times 10^k + Q\}$$
Which is clearly divisible by 6. $C - 2$ is satisfied . Hence (i) is true for all +ve integers n.

(v) $\boldsymbol{n^3 - n}$ ***is divisible by 6***

Solution:

$$n^3 - n \text{ is divisible by } 6 \to (i)$$
For n=1
$$n^3 - n = (1)^3 - 1 = 1 - 1 = 0$$
Which is divisible by 6, $C - 1$ *is satisfied. Suppose* (i) *is true for* $n = k$ *i.e*
$$k^3 - k \text{ is divisible by } 6$$
$$\Rightarrow k^3 - k = 6Q \to (ii)$$
We shall prove that (i) is true for n=k+1
$$(k + 1)^3 - (k + 1) \text{ is divisible by } 6$$
Now
$$(k + 1)^3 - (k + 1) = k^3 + 1 + 3k^2 + 3k - k - 1$$
$$= (k^3 - k) + 3k(k + 1)$$
$$= 6Q + 3k(k + 1)\ by\ (ii)$$
$$= 6Q + 3(2P) \because \quad k(k + 1) \text{ is an even}$$
$$= 6Q + 6P$$
$$= 6(Q + P)$$
Which is clearly divisible by 6.

$C - 2$ *is satisfied. Hence* (i) *is true for all integers* n.

Question # 22.

$$\mathbf{3 + 3^2 + \cdots + 3^n = 2\,(1 - 3^n)}$$

Solution. Suppose that

$$S(n): 3 + 3^2 + \cdots + 3^n = 2\,(1 - 3^n)$$
Put $n = 1$
$$S(1): \frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3}\right) \Rightarrow \frac{1}{3} = \frac{1}{2}\left(\frac{2}{3}\right) \Rightarrow \frac{1}{3} = \frac{1}{3}$$
Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$
$$S(k): \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right) \to (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{k+1}} = \frac{1}{2}(1 - \frac{1}{3^{k+1}})$$

Multiplying $\frac{1}{3^{k+1}}$ on both sides of equation (i), we have

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}(1 - \frac{1}{3^k}) + \frac{1}{3^{k+1}}$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3\quad 1} = \frac{1}{2}(1 - \frac{1}{3^k}) + \frac{1}{3 \cdot 3^k}$$

$$\Rightarrow 3 + 3^2 + \cdots + 3^{k+1} = 2\,(1 - 3^k + 3 \cdot 3^k)$$

$$\Rightarrow 3 + 3^2 + \cdots + \frac{1}{3\quad 1} = 2\,(1 - 3 \cdot 3^k)$$

$$\Rightarrow 1 + 3^2 + \cdots + 3^{k+1} = 1\,(1 - 3^{k+1})$$

$$\Rightarrow 1 + 3^2 + \cdots + 3^{k+1} = 3 \quad \frac{1}{2.3^{k+1}}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 23.

$$\mathbf{1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} \cdot n^2 = (-1)^{n-1} \cdot n(n+1)}$$

Solution. Suppose that

$$S(n): 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} \cdot n^2 = (-1)^{n-1} \cdot n(n+1)$$

Put $n = 1$

$$S(1): 1 = (-1)^0 \cdot 1(1+1) \Rightarrow 1 = 2 \Rightarrow 1 = 1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} \cdot k^2 = (-1)^{k-1} \cdot k(k+1) \quad ----(i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k+1-1} \cdot (k+1)^2 = (-1)^{k+1-1} \cdot (k+1)(k+1+1)$$

$$1^2 - 2_2 + 3^2 - 4^2 + \ldots + (-1)^k \cdot (k+1)^2 = (-1)^k \cdot (k+1)(k+2)$$

Multiplying $(-1)^k \cdot (k+1)^2$ on both sides of equation (i), we have

$$1^2 - 2_2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} \cdot k^2 + (-1)^k \cdot (k+1)^2 = (-1)^{k-1} \cdot k(k+1) + (-1)^k \cdot (k+1)^2$$

$$\Rightarrow 1_2 - 2^2 + 3_2 - 4^2 + \ldots + (-1)^k \cdot (k+1)^2 = (-1)^{k-1} \cdot (k+1)\,(k - 2(k+1))$$

$$\Rightarrow 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^k \cdot (k+1)^2 = (-1)^{k-1} \cdot (k+1)\,(k - 2k - 2)$$

$$\Rightarrow 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^k \cdot (k+1)^2 = (-1)^{k-1} \cdot (k+1)\,(-k-2)$$

$$\Rightarrow 1_2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^k \cdot (k+1)^2 = (-1)^k \cdot (k+1)(k+2)$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 24.

$$\mathbf{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)}$$

Solution. Suppose that

$$S(n): 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)$$

E4r4r55gcf4rd66f65rrdr6tg5rtf8j

$$S(1): 1^3 = 1^2 \cdot (2 \cdot 1^2 - 1) \Rightarrow 1 = (2 - 1) \Rightarrow 1 = 1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n =$ k

$$S(k): 1^3+3^3+5^3+\ldots+(2k-1)^3 = k^2(2k^2-1) ----(i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): 1^3+3^3+5^3+\ldots+(2(k+1)-1)^3 = (k+1)^2(2(k+1)^2-1)$$

$$1^3+3_3+5^3+\ldots+(2k+2-1)^3 = (k^2+2k+1)(2(k^2+2k+1)-1)$$

$$1^3+3_3+5^3+\ldots+(2k+1)^3 = (k^2+2k+1)(2k^2+4k+2-1)$$

$$1^3+3_3+5^3+\ldots+(2k+1)^3 = (k^2+2k+1)(2k^2+4k+1)$$

$$1^3+3^3+5^3+\ldots+(2k+1)^3 = 2k^4+4k^3+k^2+4k^3+8k^2+2k+2k^2+4k+1$$

$$1^3+3^3+5^3+\ldots+(2k+1)^3 = 2k^4+8k^3+11k^2+6k+1$$

Multiplying$(2k+1)^3$ on both sides of equation (i), we have

$$1^3+3^3+5^3+\ldots+(2k-1)^3+(2k+1)^3 = k^2(2k^2-1)+(2k+1)^3$$

$$\Rightarrow \quad 1_3+3^3+5^3+\cdots+(2k+1)^3 = 2k^4-k^2+8k^3+1+12k^2+6k$$

$$\Rightarrow \quad 1^3+3^3+5^3+\cdots+(2k+1)^3 = 2k^4+8k^3+11k^2+6k+1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 25.

$$\boldsymbol{x+1 \; is \; the \; factor \; of \; x^{2n}-1; x \neq -1}$$

Solution. Suppose that

$$S(n): x^{2n}-1$$

Put $n = 1$

$$S(1): x^2-1 = (x+1)(x-1)$$

Clearly $(x+1)$ is the factor of $S(1)$.Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): x^{2k}-1$$

Then there exist Quotient Q such that

$$x^{2k}-1 = (x+1)Q \rightarrow (i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): x^{2(k+1)}-1$$
$$= x^{2(k+1)}-1$$

Adding and Subtracting x^{2k}

$$= x^{2k+2}-x^{2k}+x^{2k}-1$$
$$= x^{2k}(x^2-1)+(x+1)Q \quad using\ (i)$$
$$= x^{2k}(x+1)(x-1)+(x+1)Q$$
$$= (x+1)(x^{2k}(x-1)+Q)$$

Clearly $(x+1)$ is the factor of $S(k+1)$.

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 26.

$$\boldsymbol{x-y \; is \; the \; factor \; of \; x^n-y_n; x \neq y}$$

Solution. Suppose that

$$S(n): x^n-y^n$$

Put $n = 1$

$$S(1): x^1-y^1 = x-y$$

Clearly $(x-y)$ is the factor of $S(1)$.Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): x^k-y^k$$

Then there exist Quotient Q such that

$$x^k-y^k = (x-y)Q --\rightarrow (i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): x^{k+1}-y^{k+1}$$
$$= x^{k+1}-y^{k+1}$$

Adding and Subtracting xy^k

$$= x^{k+1}-xy^k+xy^k-y^{k+1}$$
$$= x(x^k-y^k)+y^k\,(x-y)$$
$$= x(x-y)Q+y^k(x-y) \qquad using\ (i)$$

$$= (x-y)(xQ+y^k)$$

Clearly $(x-y)$ is the factor of $S(k+1)$.

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 27.

$$\boldsymbol{x+y \text{ is the factor of } x^{2n-1}+y^{2n-1}; x \neq y}$$

Solution.

$$x+y \text{ is a factor of } x^{2n-1}+y^{2n-1} \rightarrow (i) \qquad (x \neq -y)$$

For $n=1$

$$x^{2n-1}+y^{2n-1} = x^{2(1)-1}+y^{2(1)-1} = x+y$$

Clearly $x+y$ is a factor of $x+y$

$$C-1 \text{ is satisfied. Suppose } (i) \text{ is true for } n=k \text{ i.e}$$
$$x+y \text{ is a factor of } x^{2n-1}+y^{2n-1}$$
$$\Rightarrow x^{2n-1}+y^{2n-1} = (x+y)Q \rightarrow (ii)$$

We shall prove that (i) is true

$$for\ n=k+1$$
$$x+y \text{ is a factor of } x^{2n-1}+y^{2n-1}$$
$$x^{2n-1}+y^{2n-1} = x^{2(k+1)-1}+y^{2(k+1)-1}$$
$$= x^{2k+2-1}+y^{2k-1+2}$$
$$= x^{2k-1}.x^2+y^{2k-1}.y^2$$
$$= x^{2k-1}.x^2+x^2y^{2k-1}-x^2y^{2k-1}+y^{2k-1}.y^2$$
$$= x^2(x^{2k-1}+y^{2k-1})-y^{2k-1}(x^2-y^2)$$
$$= x^2Q(x+y)-y^{2k-1}(x^2-y^2) \text{ by } (ii)$$
$$= (x+y)\{x^2Q-y^{2k-1}(x-y)\}$$
$$(x+y) \text{ is a factor of } x^{2(k+1)-1}$$
$$C-2 \text{ is satisfied of } x^{2(k+1)-1}+y^{2(k+1)-1}$$
$$C-2 \text{ is satisfied. Hence } (i) \text{ is true for all } +ve \text{ integers } n.$$

Question # 28. Use mathematical induction to show that

$$\boldsymbol{1+2+2^2+\cdots+2^n = 2^{n+1}-1}$$

for all non-negative integers n.

Solution.

Suppose that

$$S(n): 1+2+2^2+\cdots+2^n = 2^{n+1}-1$$

Put $n=1$

$$S(1): 1 = 2^{1+1}-1 \Rightarrow 1 = 2-1 \Rightarrow 1=1$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1+2+2^2+\cdots+2^k = 2^{k+1}-1 --- (i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): 1+2+2_2+\cdots+2^{k+1} = 2^{k+1+1}-1$$
$$S(k+1): 1+2+2^2+\cdots+2^{k+1} = 2^{k+2}-1$$

Adding 2^{k+1} in both sides of (i)

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$$
$$1+2+2^2+\cdots+2^{k+1} = 2^{k+1}(1+1)-1$$
$$1+2+2^2+\cdots+2^{k+1} = 2^{k+1}(2)-1$$
$$1+2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-1$$
$$1+2+2^2+\cdots+2^{k+1} = 2^{k+2}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 29. If A and B are square matrices and, AB = BA then show by mathematical induction that $AB^n = B^nA$ for any positive integer.

Solution. Suppose that

$$S(n): AB^n = B^nA$$

Put $n=1$

$$S(1): AB^1 = B^1A \Rightarrow AB = BA$$

$S(1) is\ true\ as\ we\ have\ given\ AB\ = BA$.Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): AB^k = B^kA \dashrightarrow (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): AB^{k+1} = B^{k+1}A$$

Post-multiplying (i) by B

$$(AB^k)B = (B^kA)B$$
$$A(B^kB) = B^k(AB) \quad By\ Associative\ Law$$
$$A(B^{k+1}) = B^k(BA) \quad Given\ AB = BA$$
$$A(B^{k+1}) = (B^kB)A \quad By\ Associative\ Law$$
$$AB^{k+1} = B^{k+1}A$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 30. Prove by Principle of mathematical induction that $n^2 - 1$ is divisible by 8 when n is and odd positive integer.

Solution. Suppose that

$$S(n): n^2 - 1$$

Put $n = 1$

$$S(1): 1^2 - 1 = 0$$

Clearly it is divisible by 8.Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): k^2 - 1$$

Then there exist Quotient Q such that

$$k^2 - 1 = 8Q \dashrightarrow (i)$$

The Statement for $n = k + 2$ becomes

$$S(k+2): (k+2)^2 - 1$$
$$= k^2 + 4k + 4 - 1$$
$$= k^2 - 1 + 4k + 4$$
$$= 8Q + 4(k+1) \quad using\ (i)$$

Since k is an odd number then $k + 1$ is an even number then there exist an integer P such that

$$K + 1 = 2P$$

Then

$$= 8Q + 4(2P)$$
$$= 8Q + 8P$$
$$= 8(Q + P)$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 31 Use the principle of mathematical induction to prove that $\ln x^n = n\ lnx$ for any integral $n \geq 0$ if x is a positive number.

Solution. Suppose that

$$S(n): \ln x^n = n\ lnx$$

Put $n = 1$

$$S(1): lnx^1 = 1lnx \Rightarrow lnx = lnx$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): \ln x^k = k\ lnx \dashrightarrow (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): \ln x^{k+1} = (k+1)\ lnx$$

Now adding lnx on both sides of (i)

$$\ln x^k + lnx = k\ lnx + lnx$$
$$lnx^{k+1} = (k+1)lnx$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 32.$n! > 2^n - 1$ $for\ integral\ values\ of\ n \geq 4$.

Solution. Suppose that

$$S(n): n! > 2^n - 1$$

Put $n = 1$

$$S(1): 4! > 2^4 - 1 \Rightarrow 24 > 16 - 1 \Rightarrow 24 > 15$$

Thus condition I is satisfied.
Now Suppose that $S(n)$ is true for $n = k$

$$S(k): k! > 2^k - 1 \rightarrow (i)$$

The Statement for $n = k + 1$ becomes

$$S(k+1): (k+1)! > 2^{k+1} - 1$$

Multiplying both sides of (i) by $k + 1$

$$k!\,(k+1) > (2^k - 1)(k+1)$$
$$k!\,(k+1) > (2^k - 1)(k - 1 + 2)$$

$$(k+1)! > 2^k k - 2^k - k + 2^{k+1} - 1$$
$$(k+1)! > 2^{k+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 33 $n^2 > n + 3 \ for\ integral\ valves\ of\ n \geq 3$

Solution:

$$n^2 > n + 3 \quad \forall\ n \geq 3 \rightarrow (i)$$
$$for\ n = 3$$
$$(3)^2 > 3 + 3 \Rightarrow 9 > 6$$
$$\Rightarrow (i)\ is\ true\ for\ n = 3,\ \ C - 1$$

Is satisfied.
Suppose (i) is true for $n = k$ $i.e$

$$k^2 > k + 3 \ \ \forall\ k \geq 3 \rightarrow (ii)$$

We shall prove that (i) $is\ true$
For $n = k + 1$ $i.e$

$$(k+1)^2 > (k+1) + 3$$
$$\Rightarrow (k+1)^2 > k + 4$$

Now adding $2k + 1\ both\ sides\ of\ (ii)$

$$2k + 1 + k^2 > 2k + 1 + k + 3$$
$$\Rightarrow (k+1)^2 > k + 4 + 2k$$
$$\Rightarrow (k+1)^2 > k + 4 + 2k$$
$$\Rightarrow (k+1)^2 > k + 4 \quad as\ \ 2k > 0$$
$$\Rightarrow (i)\ is\ true\ for\ n = k + 1\ \ C - 2\ \ is\ satisfied. Hence\ (i) is\ true\ for\ all\ n \geq 3$$

Question # 34 $\boldsymbol{4^n > 3^n + 2^{n-1}}$ $\boldsymbol{for\ integarl\ \ valves\ of\ n \geq 2}$

Solution:

$$4^n > 3^n + 2^{n-1} \qquad \forall\ \ n \geq 2 \rightarrow (i)$$

For $n = 2$

$$4^2 > 3^2 + 2^{2-1}$$
$$\Rightarrow 16 > 9 + 2$$
$$\Rightarrow 16 > 11$$
$$\Rightarrow (i)\ is\ true\ for\ n = 2\ \ , C - 1\ \ is\ satisfied\ . Suppose\ (i) is\ true\ for\ n = k\ i.e$$
$$k^n > 3^n + 2^{k-1} \qquad \forall k \geq 2 \rightarrow (ii)$$

We shall prove that (i) is true for $n = k + 1$ $i.e$

$$4^{k+1} > 3^{k+1-1}$$
$$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k$$

Now x $(ii)\ by\ 4$

$$4.4^k > 4(3^k + 2^{k-1})$$
$$4^{k+1} > 4.3^k + 4.2^{k-1}$$
$$\Rightarrow 4^{k+1} > 4.3^k + 4.2^{k-1}$$
$$\Rightarrow 4^{k+1} > (3+1).3^k + (2+2).2^{k-1}$$

$$\Rightarrow 4^{k+1} > 3.3^k + 3^k + 2.2^{k-1} + 2.2^{k-1}$$
$$\Rightarrow 4^{k+1} > 3^{k+1} + 3^k + 2^{k-1}{}_{+}{}^{1} + 2^{k-1+1}$$
$$\Rightarrow 4^{k+1} > 3^{k+1} + 3^k + 2^k + 2^k$$
$$\Rightarrow 4^{k+1} > 3^{k+1} + (3^k + 2^k) + 2^k$$
$$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k \qquad As \;\; 3^k + 2^k > 0 \;\; \forall k \geq 2$$
$$\Rightarrow (i)\ is\ true\ for\ n = k+1\ , C-2\ is\ satisfied. Hence\ (i) is\ true\ for\ all\ n \geq 2$$

Question # 35 $\boldsymbol{3^n < n! \quad for\ intehral\ valves\ of\ n > 6}$

Solution:

$$\boldsymbol{3^n < n!\ \ \forall n > 6 \rightarrow (i)}$$

For $n = 7$

$$3^7 > 7!\ \ \Rightarrow 2187 < 5040$$
$$\Rightarrow (i)\ is\ true\ for\ n = 7\ \ , C-1\ is\ satisfied\ Suppose\ (i) is\ true\ for\ n = k\ i.e;$$
$$3^k < k!\ \ \forall k > 6 \rightarrow (ii)$$

We shall prove that (i) $is\ true\ for\ n = k+1\ \ i.e$

$$3^{k+1} < (k+1)!$$

Now × (i) $by\ 3\ we\ get$

$$3.3^k < 3k! \qquad As\ 3 < k+1\ \ \forall\ k > 6$$
$$\Rightarrow 3^{k+!} < (k+1)k!$$
$$\Rightarrow 3^{k+1} < (k+1)!$$

⇒ (i) is true for $n = k+1$ $, C-2$ is satisfied. Hence (i) is true for all $n > 6$

Question # 36 $\boldsymbol{n! > n^2 \quad for\ integarl\ valves\ of\ n \geq 4}$

Solution:

$$n! > n^2 \qquad \forall n \geq 4 \rightarrow (i)$$

For $n = 4$

$$4! > (4)^2$$
$$\Rightarrow 24 > 16$$
$$\Rightarrow (i) is\ true\ for\ n = 4\ \ , C-1\ is\ satisfied. Suppose\ (i) is\ true\ for\ n = k\ \ i.e$$

For $n = k\ \ i.e$

$$k! > k^2 \quad \forall\ k \geq 4\ \rightarrow (ii)$$

We shall prove that (i) $is\ true$ for $n = k+1\ i.e;$

$$(k+1)! > (k+1)^2$$
$$Now\ \ \times (ii)\ by\ \ (k+1)\ we\ get$$
$$(k+1)! > k!\,(k+1)k^2$$
$$\Rightarrow \quad (k+1)! > (k+1)(k+1) \quad As\ k^2 > k+1\ \ \backslash for\ k \geq 4$$
$$\Rightarrow (k+1)!\ > (k+1)^2$$

⇒ $(i) is\ \ true\ for\ n = k+1$ $, C-2$ is satisfied. Hence (i) is true for all integral of $n > 1-1$

Question # 37. $\boldsymbol{3+5+7+\cdots+(2n+5) = (n+2)(n+4)}$ **for integral values of** $\boldsymbol{n \geq -1}$.

Solution. Suppose that

$$S(n): 3+5+7+\cdots+(2n+5) = (n+2)(n+4)$$

Put $n = -1$

$$S(1): 3 = (-1+2)(-1+4) \Rightarrow 3 = (1)(3) \Rightarrow 3 = 3.$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n = k$

$$S(k): 3+5+7+\cdots+(2k+5) = (k+2)(k+4) --- (i)$$

The Statement for $n = k+1$ becomes

$$S(k+1): 3+5+7+\cdots+(2(k+1)+5) = (k+1+2)(k+1+4)$$
$$3+5+7+\cdots+(2k+2+5) = (k+3)(k+5)$$
$$3+5+7+\cdots+(2k+7) = (k+3)(k+5)$$

Multiplying both sides of (i) by $(2k+7)$

$$3+5+7+\cdots+(2k+5)+(2k+7) = (k+2)(k+4)+(2k+7)$$
$$3+5+7+\cdots+(2k+7) = k^2+2k+4k+8+2k+7$$

$$3+5+7+\cdots+(2k+7)=k^2+8k+15$$
$$3+5+7+\cdots+(2k+7)=k^2+5k+3k+15$$
$$3+5+7+\cdots+(2k+7)=k(k+5)+3(k+5)$$
$$3+5+7+\cdots+(2k+7)=(k+5)(k+3)$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

Question # 38. $\mathbf{1+nx\le(1+x)^n}$ **for integral values of** $\mathbf{n\ge -1}$.

Solution. Suppose that

$$S(n): 1+nx\le(1+x)^n$$

Put $n=2$

$$S(2): 1+2x\le(1+x)^2 \Rightarrow 1+2x\le 1+2x+x^2.$$

Thus condition I is satisfied.

Now Suppose that $S(n)$ is true for $n=k$

$$S(k): 1+kx\le(1+x)^k ---(i)$$

The Statement for $n=k+1$ becomes

$$S(k+1): 1+(k+1)x\le(1+x)^{k+1}$$

Multiplying both sides of (i) by $(1+x)$

$$(1+kx)(1+x)\le(1+x)^k(1+x)$$
$$1+x+kx+kx^2\le(1+x)^{k+1}$$
$$1+(k+1)x\le(1+x)^{k+1}$$

Thus $S(k+1)$ is true if $S(k)$ is true, So condition II is satisfied and $S(n)$ is true for all positive integer values of n.

“Binomial theorem “

Statement:

$let\ '\boldsymbol{a}'\ and$ "x" be two real numbers and "n" $be\ a\ natural\ numbers\ then$

$$(a+x)^n=\binom{}{0}a^n+\binom{}{1}a^{n-1}x^{n-1}+\binom{}{2}a^{n-2}x^2+\cdots+\binom{}{r-1}a^{n-(r-1)}.x^{r-1}+\binom{}{r}a^{n-r}x^r+\cdots$$
$$+\binom{}{n-1}ax^{n-1}+\binom{}{n}x^n$$

Proof:

We prove that it by mathematical induction method consider

$$(a+x)^n=\binom{}{0}a^n+\binom{}{1}a^{n-1}x^{n-1}+\binom{}{2}a^{n-2}x^2+\cdots+\binom{}{r-1}a^{n-(r-1)}.x^{r-1}+\binom{}{r}a^{n-r}x^r+\cdots$$
$$+\binom{}{n-1}ax^{n-1}+\binom{}{n}x^n\rightarrow(i)$$

For $n=1$

$$(a+x)^1=\binom{1}{}a^1+\binom{}{1}a^{1-1}x^1$$
$$\Rightarrow a+x=1.a+1.a^ox \qquad \because \binom{1}{}=\binom{}{1}=1 \quad a^0=1$$
$$\Rightarrow a+x=a+x$$
$$\Rightarrow (i)\quad is\ true\ for\ n=1\quad C-1\ \ is\ satisfied.$$

Suppose (i) $is\ true\ for\ n=k\ i.e$

$$(a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^{k-1}+\binom{k}{2}a^{k-2}x^2+\cdots+\binom{k}{r-1}a^{k-(r-1)}.x^{r-1}+\binom{n}{r}a^{k-r}x^r+\cdots$$
$$+\binom{k}{k-1}ax^{k-1}+\binom{k}{k}x^k\rightarrow(ii)$$

We shall prove that (i) is true for $n=k+1$

For this multiplying eq.(2) by $(a+x)$ $we\ get$

$(a+x)(a+x)^k$

$$= (a+x)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{k-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)}.x^{r-1} + \binom{n}{r}a^{k-r}x^r + \cdots + \binom{k}{k-1}ax^{k-1} + \binom{k}{k}x^k\right]$$

$$\Rightarrow (a+x)^{k+1} = a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{k-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)}.x^{r-1} + \binom{n}{r}a^{k-r}x^r + \cdots + \binom{}{k-1}ax^{k-1} + \binom{}{k}x^k\right]$$

$$+x\left[\binom{}{0}a^k + \binom{}{1}a^{k-1}x^{k-1} + \binom{}{2}a^{k-2}x^2 + \cdots + \binom{}{r-1}a^{k-(r-1)}.x^{r-1} + \binom{}{r}a^{k-r}x^r + \cdots + \binom{}{k-1}ax^{k-1} + \binom{}{k}x^k\right]$$

$$(a+x)^{k+1} = \binom{}{o}a^{k+1} + \binom{}{1}a^kx^1 + \binom{}{2}a^{k-1}x^2 + \cdots + \binom{}{r-1}a^{k-r+2}x^{r-1} + \binom{}{r}a^{k-r+1}x^r + .. +$$

$$\binom{}{k-1}a^2x^{k-1} + \binom{}{k}ax^k$$

$$+\binom{}{0}a^kx^1 + \binom{}{1}a^{k-1}x^2 + \binom{}{2}a^{k-2}x^3 + \cdots + \binom{}{r-1}a^{k-r+1}x^r + \binom{}{r}a^{k-r}x^{r+1} + \cdots + \binom{}{k-1}ax^k$$

$$\binom{k}{}x^{k+1}$$

As we know that

$$\binom{n}{} = 1\ \binom{k}{} = 1, \binom{k+1}{} = 1 \Rightarrow \binom{k}{} = \binom{k+1}{}$$

$$\binom{}{r} + \binom{}{r-1} = \binom{n+1}{} \Rightarrow \binom{k}{} + \binom{}{o} = \binom{k+1}{}, \binom{k}{} + \binom{k}{} = \binom{k+1}{}$$

$$\binom{}{r} + \binom{}{r-1} = \binom{k\quad 1}{r}\ \ and\ \ \binom{}{k} + \binom{}{k-1} = \binom{k\quad 1}{k}$$

$$\binom{}{n} = 1 \Rightarrow \binom{k}{} = 1\ \ also\ \ \binom{k+1}{} = 1\quad \binom{k}{} = \binom{k+1}{}$$

$$Putting\ valves\ we\ have$$

$$(a+x)^{k+1} = \binom{k\quad 1}{0}a^{k+1} + \binom{k+1}{}a^kx^1 + \binom{k+1}{}a^{k-1}x^2 + \cdots + \binom{k+1}{}a^{k-r+1}x^r + \cdots + \binom{}{k+1}x^{k+1}$$

$$(a+x)^{k+1} = \binom{k\quad 1}{0}a^{k+1} + \binom{k+1}{}a^{k+1-1}x^1 + \binom{k+1}{}a^{k+1-2}x^2 + \cdots + \binom{k+1}{}a^{k+1-r}x^r + \cdots + \binom{}{k+1}x^{k+1}$$

$$\Rightarrow (1) is\ true\ for\ n = k+1, C-2\ is\ satisfied$$

$Hence\ (1) is\ true\ for\ all\ natural\ numbers\ n$

Binomial Expression:

A polynomial consisting of two terms is called Binomial or Binomial expression

$$e.g\ ; x-2y, a+b, 3x+5y\quad e.t.c$$

- The expression of $(a+b)^n$ *for small valves of n can be* obtained by direct calculation such as

$$(a+b)' = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 - a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

here

(i) Right sides of these equations are called Binomial expansion.

(ii) *The exponents* 1,2,3,4 *are called indices (plural of index) to expand any binomial* Expansion for higher valves of n. we uses of expansion named as Binomial theorem.

Remember:

i In the expansion of $(a+x)^n$ *there are* $n+1$ *terms.*

$i.e$ *one term more than the exponent*)

ii The exponent of "a" decrease from n to zero. While exponent of 'x' *increase fromzero to n*

iii In the expansion of $(a+x)^n$ *the sum of exponents of* "a"and "x" is equal to n.

iv In the expansion of $(a+x)^n$ the term $\binom{ }{r} a^{n-r}x^r$ is called $(r+1)th$ term i.e $T_{+1} = \binom{ }{r} a^{n-r}.x^r$

v It is also called general term. The successive terms can be obtained by putting $r = 0,1,2,3,\dots,n$

vi In binomial expansion the coefficients from the beginning and end are same.

$$i.e \quad \binom{ }{o} = \binom{ }{n}$$

vii $\binom{ }{o},\binom{ }{1},\binom{ }{2},\dots,\binom{ }{r},\dots.\binom{ }{n}$ are called binomial coefficients.

The Middle term in the expansion of $(a+x)^n$

For $(a+x)^n$

Total number of terms is $n+1$

Case I: (n is even):

If n is even, then total number of terms $= n+1(odd)$

Now, middle term $= {}^{n} + 1 = {}^{n+2}$

$$e.g \text{ in the expansion of } (a+x)^6 \text{ here } n = 6 \ (\because n \text{ is even})$$

Total terms $= 6+1 = 7$

Middle term$= {}_{2} + 1 = 3+1 = 4$

So 4th term will be its middle terms.

Case II (n is odd):

If n is odd, then total number of terms $= n+1(even)$

Now, middle term $= \left({}^{n+1}\right)$ *and* $\left({}^{n+3}\right)$

$$e.g \text{ in the expansion of } (a+x)^5 \text{ here } n = 5 \ (\because n \text{ is odd})$$

Total terms $= 5+1 = 6$

Middle term$= {}^{5+1} = {}^{6} = 3$

Middle term$= {}^{5+3} = {}^{8} = 4$

So 3rd and 4th term will be its middle terms.

Some Deductions from the binomial expansion of $(a+x)^n$

i We know that

$$(a+x)^n = \binom{ }{o} a^n + \binom{ }{1} a^{n-1}x^1 + \binom{ }{2} a^{n-2}.x^2 + \binom{ }{3} a^{n-3}x^3 + \cdots + \binom{ }{n} x^n \to (i)$$

By putting $a = 1$ in (i)

$$(1+x)^n = \binom{n}{ } + \binom{n}{ } x^1 + \binom{n}{ } x^2 + \binom{n}{ } x^3 + \cdots + \binom{n}{ } x^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

ii By butting $a = 1$ *and replace* x *by* $(-x)$

$$(1-x)^n = 1 - nx + \frac{n(\ \ -1)}{n2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + (-1)^n x^n$$

iii To find Sum of Binomial coefficients: by putting $a = 1, x = 1$ *in* (i)

$$(1+1)^n = \binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$2^n = sum$ of Binomial coefficients

iv Sum of even coefficients equals to sum of odd coefficients

$$(a+x)^n = \binom{n}{o} a^n + \binom{n}{1} a^{n-1}x^1 + \binom{n}{2} a^{n-2}.x^2 + \binom{n}{3} a^{n-3}x^3 + \cdots + \binom{n}{n} x^n$$

By putting $a = 1, x = -1$

$$(1-1)^n = \binom{n}{} + \binom{}{1}(-1)^1 + \binom{}{2}(-1)^2 + \binom{n}{}(-1)^3 + \cdots + \binom{n}{}(-1)^n$$

$$\Rightarrow \binom{}{0} - \binom{}{1} + \binom{}{2} - \binom{}{3} + \binom{}{4} \ldots + \binom{}{n-1}(-1)^{n-1} + \binom{}{n}(-1)^n = 0$$

If n is odd positive integer

$$\binom{}{\mathbf{0}} + \binom{}{\mathbf{2}} + \binom{}{\mathbf{4}} + \cdots + \binom{}{\boldsymbol{n}} = \binom{}{\mathbf{1}} + \binom{}{\mathbf{3}} + \binom{}{\mathbf{5}} + \cdots + \binom{}{\boldsymbol{n}-\mathbf{1}}$$

We conclude that

Sum of even coefficient= Sum of odd coefficients

Exercise 8.2

Question # 1. Using binomial theorem, expand the following:

(i). $(a+2b)^5$

Solution.

$$(a+2b)^5=\binom{5}{0}a^5(2b)^0+\binom{5}{1}a^{5-1}(2b)^1+\binom{5}{2}a^{5-2}(2b)^2+\binom{5}{3}a^{5-3}(2b)^3+\binom{5}{4}a^{5-4}(2b)^4+\binom{5}{5}a^{5-5}(2b)^5$$

$$(a+2b)^5=(1)a^5+(5)a^4(2b)+(10)a^3(4b^2)+(10)a^2(8b^3)+(5)a^1(16b^4)+(1)(32b^5)$$

$$(a+2b)^5=a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+b^5$$

(ii). $\left(_2-_{x^2}\right)^6$

Solution.

$$\left(\text{—}-_{x^2}\right)^6=\binom{}{0}\left(\text{—}\right)^6\left(-_{x^2}\right)^0+\binom{}{1}\left(\text{—}\right)^{6-1}\left(-_{x^2}\right)^1+\binom{}{2}\left(\text{—}\right)^{6-2}\left(-_{x^2}\right)^2+\binom{}{3}\left(\text{—}\right)^{6-3}\left(-_{x^2}\right)^3$$
$$+\binom{}{4}\left(\text{—}\right)^{6-4}\left(-_{x^2}\right)^4+\binom{}{5}\left(\text{—}\right)^{6-5}\left(-_{x^2}\right)^5+\binom{}{6}\left(\text{—}\right)^{6-6}\left(-_{x^2}\right)^6$$

$$\left(\text{—}-_{x^2}\right)^6=(1)\left(\text{—}\right)^6+(6)\left(\text{—}\right)^5\left(-_{x^2}\right)^1+(15)\left(\text{—}\right)^4\left(-_{x^2}\right)^2+(20)\left(\text{—}\right)^3\left(-_{x^2}\right)^3+(15)\left(\text{—}\right)^2\left(-_{x^2}\right)^4$$
$$+(6)\left(\text{—}\right)^1\left(-_{x^2}\right)^5+(1)\left(-_{x^2}\right)^6$$

$$\left(_2-_{x^2}\right)^6=(1)_{64}+(6)_{32}\left(-_{x^2}\right)+(15)_{16}\left(\text{—}\right)+(20)\left(_8\right)\left(-_{x^6}\right)+(15)\left(_4\right)\left(\text{—}\right)+(6)\left(\text{—}\right)\left(-_x{}^2\right)$$
$$+(1)\left(\text{—}\right)$$

x	2 ^{6}	x^6	$3x^3$	15	20	60	96	64
2	x^2	64	8	4	x^3	x^6	x^9	x^{12}

(iii). $\left(3a-_{3a}\right)^4$

Solution.

$$\left(3a-\text{—}\right)^4=\binom{}{0}(3a)^4+\binom{}{1}(3a)^3\left(-_{3a}\right)^1+\binom{}{2}(3a)^2\left(-_{3a}\right)^2+\binom{}{3}(3a)\left(-_{3a}\right)^3+\binom{}{4}\left(-_{3a}\right)^4$$

$$=1(81a^4)-4(27a^3)\left(\text{—}\right)+6(9a^2)\left(_{9a^2}\right)^2-4(3a)\left(_{27a^3}\right)^3+{}_{81a^4}^{\;4}$$

$$=81a^4-36a^2x+6x^2-{}_{9a^2}^{\;3}+{}_{81a^4}^{\;4}$$

(iv). $\left(2a-_a\right)^7$

Solution.

$$\left(2a-_a\right)^7=\binom{}{0}(2a)^7+\binom{}{1}(2a)^6\left(-_a\right)+\binom{}{2}(2a)^5\left(-_a\right)^2+\binom{}{3}(2a)^4\left(-_a\right)^3+\binom{}{4}(2a)^3\left(-_a\right)^4+\binom{}{5}(2a)^2\left(-_5\right)^5$$
$$+\binom{7}{6}(2a)\left(-\frac{x}{a}\right)^6+\binom{7}{7}\left(-\frac{x}{a}\right)^7$$

$$=1(128a^5)-7(64a^6)\left(\frac{x}{a}\right)+21(32a^5)\left(\frac{x^2}{a^2}\right)-35(16a^4)\left(\frac{x^3}{a^3}\right)+35(8a^3)\left(\frac{x^4}{a^4}\right)-21(4a^2)\left(\frac{x^5}{a^5}\right)+7(2a)\left(\frac{x^6}{x^6}\right)-\frac{x^7}{a^7}$$

$$=128a^7-448a^5x+672a^3x^2-560ax^2+280\frac{x^4}{a}-84\frac{x^5}{a^3}+14\frac{x^6}{a^5}-\frac{x^7}{a^7}$$

(v). $(\frac{x}{2y}-\frac{2y}{x})^8$

Solution.

$$(\frac{}{2y}-\frac{}{x})^8 = (_0)(\frac{}{2y})^3 + (_1)(\frac{}{})^7(-\frac{}{x}) + (_2)(\frac{}{})^6(-\frac{}{})^2 + (_3)(\frac{}{2y})^5(-\frac{}{x})^3 + (_4)(\frac{}{})^4(-\frac{}{x})^4$$

$$+(_5)(\frac{}{2y})^3(-\frac{}{x})^5 + (_6)(\frac{}{})^2(-\frac{}{x})^6 + (_7)(\frac{}{})^1(-\frac{}{x})^7 + (_8)(\frac{}{2y})^7(-\frac{}{x})^8$$

$$= 1(\frac{}{256y^8})^8 - 8(\frac{}{128y^7})^7(\frac{}{x}) + 28(\frac{}{64y^8})^6(\frac{}{x^2})^2 - 56(\frac{}{32y^5})^5(\frac{}{x^5})^3 + 70(\frac{}{16y^2})^4(\frac{}{x^2})^2$$

$$-56(\frac{x}{3})(32y^5) + 28(\frac{x}{2})(64y) - 8(\frac{x}{})(128y^7) + 256y^8$$

$$\frac{x^8}{256y^8} \quad \frac{x^6}{8y^6} \quad \frac{7x^4}{4y^4} \quad \frac{14x^2}{y^2} \quad \frac{22y^2}{x^2} \quad \frac{448y^4}{x^4} \quad \frac{512y^6}{x^6} \quad \frac{256y^8}{x^8}$$

(vi). $(\sqrt{x}-\sqrt{a})^6$

Solution.

$$(\sqrt{x}-\sqrt{a})^6 = (_0)(\sqrt{x})^6(-\sqrt{a})^0 + (_1)(\sqrt{x})^{6-1}(-\sqrt{a})^1 + (_2)(\sqrt{x})^{6-2}(-\sqrt{a})^2 + (_3)(\sqrt{x})^{6-3}(-\sqrt{a})^3$$

$$+(_4)(\sqrt{x})^{6-4}(-\sqrt{a})^4 + (_5)(\sqrt{x})^{6-5}(-\sqrt{a})^5 + (_6)(\sqrt{x})^{6-6}(-\sqrt{a})^6$$

$$(\sqrt{x}-\sqrt{a})^6 = (1)(\sqrt{x})^6 + (6)(\sqrt{x})^5(-\sqrt{a})^1 + (15)(\sqrt{x})^4(-\sqrt{a})^2 + (20)(\sqrt{x})^3(-\sqrt{a})^3\ 2$$

$$+(15)(\sqrt{x})(-\sqrt{a})^4 + (6)(\sqrt{x})^1(-\sqrt{a})^5 + (1)(-\sqrt{a})^6$$

$$(\sqrt{x}-\sqrt{a})^6 = (1)(\;)^3 - (6)(\sqrt{x})^4 + (15)(\sqrt{x})^2 - (20)(\sqrt{x})^0 + (15)(\sqrt{a})^2 - (6)(\sqrt{a})^4 + (1)(\;)^3$$

$$(\sqrt{x}-\sqrt{a})^6 = x^3 - 6x^2 + 15x - 20 + 15a - 6a^2 + a^3$$

Question # 2
Calculate the following by means of binomial theorem:

(i).$(0.97)^3$

Solution.

$$(0.97)^3 = (1-0.03)^3$$

$$(0.97)^3 = (_0)(1)^3 + (_1)(1)^2(-0.03) + (_2)(1)(-0.03)^2 + (_3)(-0.03)^3$$

$$(0.97)^3 = (1)(1) + (3)(1)(-0.03) + (3)(1)(0.0009) + (1)(-0.000024)$$

$$(0.97)^3 = 1 - 0.09 - 0.0027 - 0.000027$$

$$(0.97)^3 = 0.912673.$$

(ii). $(2.02)^4$

Solution.

$$(2.02)^4 = (2+0.02)^4$$

$$= \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(0.02)^1 + \binom{4}{2}(2)^2(0.02)^2 + \binom{4}{3}(2)^1(0.02)^3 + \binom{4}{4}(0.02)^4$$

$$= 1(16) + 4(8)(0.02) + 6(4)(0.0004) + 4(2)(0.000008) + 1(0.00000016)$$

$$16 + 0.64 + 0.0096 + 0.000064 + 0.00000016$$

$$16.64966416$$

(iii).$(9.98)^4$

Solution.

$$(10-0.02)^4 = \binom{4}{0}(10)^4 + \binom{4}{1}(10)^3(-0.02) + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3 + \binom{4}{4}(-0.02)^4$$

$$(9.98)^4 = (1)(10000) + (4)(1000)(-0.02) + (6)(100)(0.0004) + (4)(1000)(-0.000008) + (1)(0.00000016)$$

$$(9.98)^4 = 10000 - 80 + 0.24 - 0.00032 + 0.00000016$$

$$(9.98)^4 = 9920.23968$$

(iv). $(21)^5$

Solution.

$$(21)^5 = (20+1)^5$$

$$= \binom{5}{0}(20)^5 + \binom{5}{1}(20)^4 + \binom{5}{2}(20)^3(1)^2 + \binom{5}{3}(20)^2(1)^3 + \binom{5}{4}(20)(1)^4 + \binom{5}{5}(1)^5$$

$$= 1(320000) + 5(160000) + 10(8000) + 10(400) + 5(20) + 1$$

$$= 3200000 + 800000 + 80000 + 4000 + 100 + 1$$

$$= 4084101$$

Question # 3
Expand and simplify the following:

(i). $(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4$

Solution.

$$(a+\sqrt{2}x)^4 = \binom{4}{0}(a)^4(\sqrt{2}x)^0 + \binom{4}{1}(a)^{4-1}(\sqrt{2}x)^1 + \binom{4}{2}(a)^{4-2}(\sqrt{2}x)^2 + \binom{4}{3}(a)^{4-3}(\sqrt{2}x)^3 + \binom{4}{4}(a)^{4-4}(\sqrt{2}x)^4$$

$$(a+\sqrt{2}x)^4 = a^4 + (4)(a)^3(\sqrt{2}x)^1 + (6)(a)^2(\sqrt{2}x)^2 + (4)(a)^1(\sqrt{2}x)^3 + (1)(\sqrt{2}x)^4$$

$$(a+\sqrt{2}x)^4 = a^4 + 4\sqrt{2}a^3x + 12a^2x^2 + 8\sqrt{2}ax^3 + 4x^4 \rightarrow (i)$$

Replacing $\sqrt{2}$ by $-\sqrt{2}$ in equation (i).

$$(a-\sqrt{2}x)^4 = a^4 - 4\sqrt{2}a^3x + 12a^2x^2 - 8\sqrt{2}ax^3 + 4x^4 \rightarrow (ii)$$

Adding (i) and (ii), we have

$(a+\sqrt{2}x)^4+(a-\sqrt{2}x)^4$

$$= a^4+4\sqrt{2}a^3x+12a^2x^2+8\sqrt{2}ax^3+4x^4+a^4-4\sqrt{2}a^3x+12a^2x^2-8\sqrt{2}ax^3+4x^4$$

$$(a+\sqrt{2}x)^4+(a-\sqrt{2}x)^4 = a^4+12a^2x^2+4x^4+a^4+12a^2x^2+4x^4$$

$$= 2a^4+8x^4+24a^2x^2$$

$$= 2\{a^4+12a^2x^2+4x^4\}$$

(ii). $\mathbf{(2+\sqrt{3})^5+(2-\sqrt{3})^5}$

Solution.

$$(2+\sqrt{3})^5 = \binom{5}{0}(2)^5+\binom{5}{1}(2)^4(\sqrt{3})^1+\binom{5}{2}(2)^3(\sqrt{3})^2+\binom{5}{3}(2)^2(\sqrt{3})^3+\binom{5}{4}(2)^1(\sqrt{3})^4+\binom{5}{5}(2)^0(\sqrt{3})^5$$

$$= 1(32)+5(16)(\sqrt{3})+10(8)(3)+10(4)(3\sqrt{3})+5(2)(9)+1(9\sqrt{3})$$

$$\Rightarrow \mathbf{(2+\sqrt{3})^5 = 32+80\sqrt{3}+240+120\sqrt{3}+90+9\sqrt{3}} \rightarrow (\boldsymbol{i})$$

$$replace\ \sqrt{\mathbf{3}}\ by -\sqrt{\mathbf{3}}\ \ we\ get$$

$$(2-\sqrt{3})^5 = 32-80\sqrt{3}+240-120\sqrt{3}+90-9\sqrt{3} \rightarrow (ii)$$

Adding (i) and (ii)

$$(2+\sqrt{3})^5+(2-\sqrt{3})^5 = 2\{32+240+90\}$$

$$= 2(362) = 724$$

(iii). $(2+i)^5-(2-i)^5$

Solution.

$$(2+i)^5 = \binom{5}{0}(2)^5(i)^0+\binom{5}{1}(2)^{5-1}(i)^1+\binom{5}{2}(2)^{5-2}(i)^2+\binom{5}{3}(2)^{5\_3}(i)^3+\binom{5}{4}(2)^{5-4}(i)^4+\binom{5}{5}(2)^{5-5}(i)^5$$

$$(2+i)^5 = (1)(32)+(5)(2)^5(i)^1+(10)(2)^3(i)^2+(10)(2)^2(i)^3+(5)(2)^1(i)^4+(i)^5$$

$$(2+i)^5 = 32+80i+80i^2+40i^3+10i^4+i^5 --- (i)$$

Replacing i *by* $-i$ *in equation (i).*

$$(2-i)^5 = 32-80i+80i^2-40i^3+10i^4-i^5 --- (ii)$$

Subtracting (i) and (ii)

$$(2+i)^5-(2-i)^5 = 32+80i+80i^2+40i^3+10i^4+i^5-32+80i-80i^2+40i^3-10i^4+i^5$$

$$(2+i)^5-(2-i)^5 = 80i+40i^3+i^5+80i+40i^3+i^5$$

$$(2+i)^5-(2-i)^5 = 160i+80i^3+2i^5$$

$$(2+i)^5-(2-i)^5 = 160i-80i+2i$$

$$(2+i)^5-(2-i)^5 = 82i$$

(iv). $(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3$

Solution. $(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3$

Suppose that $\sqrt{x^2-1} = t$

$$(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3 = (x+t)^3+(x-t)^3$$

$$(x+\sqrt{x^2-1})^3+(x+\sqrt{x^2-1})^3 = (x^3+3x^2t+3t^2x+t^3)+(x^3-3at+3t^2x-t^3)$$

$$(x+\sqrt{x^2-1})^3+(x+\sqrt{x^2-1})^3=x^3+3tx^2+3t^2x+t^3+x^3-3at+3t^2x-t^3$$

$$(x+\sqrt{x^2-1})^3+(x+\sqrt{x^2-1})^3=2x^3+6t^2x$$

Replace $t=\sqrt{x^2-1}$

$$(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3=2x^3+6\left(\sqrt{x^2-1}\right)^2x$$

$$(x+\sqrt{x^2}-1)^3+(x-\sqrt{x^2}-1)^3=2x^3+6(x^2-1)x$$

$$(x+\sqrt{x^2}-1)^3+(x-\sqrt{x^2}-1)^3=2x^3+6x^3-6x$$

$$(x+\sqrt{x^2}-1)^3+(x-\sqrt{x^2}-1)^3=8x^3-6x$$

$$=2x(4x^2-3)$$

Question # 4
Expand the following in ascending powers of x :

(i). $(2+x-x^2)^4$

Solution. $(2+x-x^2)^4$

Put $t=2+x$

$$(2+x-x^2)^4=(t-x^2)^4$$

$$(2+x-x^2)^4=\binom{}{0}(t)^4(-x^2)^0+\binom{}{1}(t)^{4-1}(-x^2)^1+\binom{}{2}(t)^{4-2}(-x^2)^2+\binom{}{3}(t)^{4-3}(-x^2)^3$$
$$+\binom{}{4}(t)^{4-4}(-x^2)^4$$

$$(2+x-x^2)^4=(1)(t)^4+(4)(t)^3(-x^2)^1+(6)(t)^2(x^4)+(4)(t)^1(-x^6)+(1)x^8$$

$$(2+x-x^2)^4=t^4-4t^3x^2+6t^2x^4-4tx^6+x^8\rightarrow(i)$$

Now

$$t^4=(2+x)^4$$

$$t^4=\binom{}{0}(2)^4(x)^0+\binom{}{1}(2)^{4-1}(x)^1+\binom{}{2}(2)^{4-2}(x)^2+\binom{}{3}(2)^{4-3}(x)^3+\binom{}{4}(2)^{4-4}(x)^4$$

$$t^4=(1)(16)+(4)(2)^3(x)^1+(6)(2)^2(x)^2+(4)(2)^1(x)^3+(1)(x)^4$$

$$t^4=16+32x+24x^2+8x^3+x^4$$

$$t^3=(2+x)^3=(2)^3+3(2)^2(x)+3(2)(x)^2+x^3$$

$$t^3=8+12x+6x^2+x^3$$

$$t^2=(2+x)^2=4+4x+x^2$$

Putting all values in (i),we have

$$(2+x-x^2)^4=16+32x+24x^2+8x^3+x^4-4(8+12x+6x^2+x^3)x^2+6(4+4x+x^2)^2x^4-4(2+x)x^6$$
$$+x^8$$

$$(2+x-x^2)^4=16+32x+24x^2+8x^3+x^4-32x^2-48x^3-24x^4+x^5+24+24x+6x^2-8x^6-x^7+x^8$$

$$(2+x-x^2)^4=16+32x-8x^2-40x^3+x^4+20x^5-2x^6-4x^7+x^8$$

(ii). $\boldsymbol{(1-x+x^2)^4}$

Solution.

$$(1-x+x^2)^4$$

$$=\binom{4}{0}(1-x)^4+\binom{4}{1}(1-x)^3(x^2)^1+\binom{4}{2}(1-x)^2(x^2)^2+\binom{4}{3}(1-x)^1(x^2)^3+\binom{4}{4}(1-x)^0(x^2)^4$$

$$=(1-x)^4+4x^2(1-x)^3+6x^4(1-x)^2+4x^6(1-x)+x^8$$

$$1-4x+6x^2-4x^3+x^4+4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)+4x^6-(4x^7)+x^8$$

$$1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$$

$$1-4x+(6+4)x^2+(-4-12)x^3+(1+12+6)x^4+(-4-12)x^5+(6+4)x^6-4x^7+x^8$$

$$1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

(iii). $(1-x-x^2)^4$

Solution..

$$(1-x-x^2)^4$$

$$=(_{0})(1-x)^4+(_{1})(1-x)^3(-x^2)^1+(_{2})(1-x)^2(-x^2)^2+(_{3})(1-x)^1(-x^2)^3+(_{4})(1-x)^0(-x^2)^4$$

$$=(1-x)^4-4x^2(1-x)^3+6x^4(1-x)^2-4x^6(1-x)+x^8$$

$$1-4x+6x^2-4x^3+x^4-4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)-4x^6+(4x^7)+x^8$$

$$1-4x+6x^2-4x^3+x^4-4x^2+12x^3-12x^4+4x^5+6x^4-12x^5+6x^6-4x^6+4x^7+x^8$$

$$1-4x+(6-4)x^2+(-4+12)x^3+(1-12+6)x^4+(4-12)x^5+(6-4)x^6+4x^7+x^8$$

$$1-4x+2x^2+8x^3+5x^4-8x^5+2x^6+4x^7+x^8$$

Question # 5

Expand the following in descending powers of x :

(i). $(x^2+x-1)^3$

$$=(_{0})(x^2)^3+(_{1})(x^2)^2(x-1)^1+(_{2})(x^2)^1(x-1)^2+(_{3})(x-1)^3$$

$$=x^6+3x^4(x-1)+3x^2(x^2+1-2x)+(x^3-1-3x^2+3x)$$

$$=x^6+3x^5-3x^4+3x^4+3x^2-6x^3+x^3-1-3x^2+3x$$

$$=x^6+3x^5+(-3+3)x^4+(-6+1)x^3+(3-3)x^2+3x-1$$

$$=x^6+3x^5-5x^3+3x-1$$

(ii). $\left(x-1-\frac{}{x}\right)^3$

Solution. $\left(x-1-\frac{}{x}\right)^3$

Suppose that $t=x-1$ *then*

$$\left(t-\frac{1}{}\right)^3=t^3-\frac{3t^2}{}+\frac{3t}{}-\frac{1}{}$$

$$\left(t-\frac{}{x}\right)^3=t^3-\frac{}{x}\cdot t^2+\frac{}{x^2}\cdot t-\frac{}{x^3}\rightarrow(i)$$

Now

$$t^3=(x-1)^3=x^3-3x^2+3x-1$$

$$t^2=(x-1)^2=x^2-2x-1$$

Putting all values in (i), we have

$$\left(x-1-\frac{1}{x}\right)^3 = x^3-3x^2+3x-1-\frac{3}{x}\cdot(x^2-2x-1)+\frac{3}{x^2}\cdot(x-1)-\frac{1}{x^3}$$

$$\left(x-1-\frac{1}{x}\right)^3 = x^3-3x^2+3x-1-3x+6-\frac{3}{x}+\frac{3}{x}-\frac{3}{x^2}-\frac{1}{x^3}$$

$$\left(x-1-\frac{1}{x}\right)^3 = x^3-3x^2+5-\frac{3}{x^2}-\frac{1}{x^3}$$

Question # 6

Find the term involving:

(i).x^4 *in the expansion of* $(3-2x)^7$

Solution. Since

$$T_{r+1} = \binom{n}{r} a^{n-r}x^r$$

Here $a = 3, x = -2x, n = 7$ *so, we have*

$$T_{r+1} = \binom{7}{r}(3)^{7-r}(-2x)^r$$

$$T_{r+1} = \binom{7}{r}(3)^{7-r}(-2)^r(x)^r$$

For term involving x^4 we must have $x^r = x^4 \Rightarrow r = 4$

$$T_5 = \binom{7}{4}(3)^{7-4}(-2)^4(x)^4$$

$$T_5 = (35)(3)^3(2)^4(x)^4$$

$$T_5 = 15120x^4$$

(ii) x^{-2} *in the expansion of* $\left(x-\frac{2}{x^2}\right)^{13}$

Solution. Since

$$T_{r+1} = \binom{n}{r} a^{n-r}x^r$$

Here $a = x, x = -\frac{2}{x^2}, n = 13$ *so, we have*

$$T_{r+1} = \binom{13}{r}(x)^{13-r}\left(-\frac{2}{x^2}\right)^r$$

$$T_{r+1} = \binom{13}{r}(-2)^r(x)^{13-r-2r}$$

$$T_{r+1} = \binom{13}{r}(-2)^r(x)^{13-3r}$$

For term involving x^{-2} we must have $x^{13-3r} = x^{-2} \Rightarrow 13-3r = -2$

$$-3r = -2-13$$

$$-3r = -15$$

$$r = 5$$

$$T_6 = \binom{13}{6}(-2)^5(x)^{-2}$$

$$T_6 = (1287)(-32)(x)^{-2}$$

$$T_6 = -41184x^{-2}$$

(iii). a^4 *in the expansion of* $\left(\frac{2}{x}-a\right)^9$

Solution.

$$let\ T_{r+1} = \binom{n}{r} a^{n-r}.b^r$$

$$\because T_{r+1} = \binom{n}{r} a^{n-r}$$

$$\Rightarrow T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r$$

$$T_{\ +1} = \binom{}{r} 2^{9-r}.\left(-\right)^{9-r}(-a)^r$$

$$for\ required\ result\ \ r = 4$$

$$\Rightarrow T_{\ +1} = \binom{}{4} 2^{9-4}\left(-\right)^{9-4}(-a)^4$$

$= 126(2)^5.\left(\frac{}{x}\right)^5 a^4$,

$$\Rightarrow T\ \ = 126(32).\frac{}{x^5} a^4$$

$$T\ = 4032a^4$$

(iv). y^3 *in the expansion of* $(x - \sqrt{y})^{11}$

Solution.

.let $T_{\ +1}$ *be the required term*

$$\because T_{\ +1} = \binom{}{r} a^{n-r} b^r$$

$$\Rightarrow T_{\ +1} = \binom{11}{} a^{11-r}.(-\sqrt{y})^r$$

$$T_{\ +1} = \binom{11}{} x^{11-r}(-1)^r\ y^{\frac{}{2}}$$

For put $\frac{}{2} = 3$

$$\Rightarrow r = 6$$

$$T_{\ +1} = \binom{11}{} x^{11\_6}(-1)^6 y^{\frac{}{2}}$$

$$\Rightarrow T\ \ = 462x^5(1)y^3 = 44462x^5y^3$$

Question # 7
Find the coefficient of;

(i). x^5 *in the expansion of* $\left(x^2 - \frac{}{2x}\right)^{10}$

Solution. Since

$$T_{\ +1} = \binom{}{r} a^{n-r}x^r$$

Here $a = x^2, x = -\frac{}{2x}, n = 10\ so, we\ have$

$$T_{r+1} = \binom{10}{r}(x^2)^{1}{}_{0}{}^{-r}\left(-\frac{3}{2x}\right)^r$$

$$T_{r+1} = \binom{10}{r}(x)^{20\_2r}\left(-\frac{3}{2}\right)^r x^{-r}$$

$$T_{r+1} = \binom{10}{r} (x)^{20-2r-r}(-\frac{3}{2})^r$$

$$T_{r+1} = \binom{10}{r} (x)^{20-3r}(-\frac{3}{2})^r$$

For term involving x^5 we must have $x^{20-3r} = x^5 \Rightarrow 20 - 3r = 5$

$$-3r = -20 + 5$$

$$-3r = -15$$

$$r = 5$$

$$T = \binom{10}{} (x)^5(-\,_2)^5$$

$$T = (252)(x)^5(-\,^{243})$$

$$T = -\,^{15309}\,x^5$$

Hence coefficient of $x^5 = -\,^{15309}$.

(ii).x^n *in the expansion of* $(x^2 - \,_x)^{2n}$

Solution. Since

$$T_{+1} = (_r)\,a^{n-r}x^r$$

Here $a = x^2, x = -\,_x, n = 2n$ *so, we have*

$$T_{+1} = \binom{2n}{} (x^2)^{2n-r}(-\,_x)^r$$

$$T_{+1} = \binom{2n}{} (x)^{4n-2r}(-1)^r x^{-r}$$

$$T_{+1} = \binom{2n}{} (x)^{4n-3r}(-1)^r$$

For term involving x^{4n-3r} we must have $x^{4n-3r} = x^n \Rightarrow 4n - 3r = n$

$$3n = 3r$$

$$n = r$$

So

$$T_{+1} = \binom{2n}{} (x)^n(-1)^n$$

$$T_{+1} = \,_{n!\,n!}\,(x)^n(-1)^n$$

$$T_{+1} = (-1)^n\,_{n!\,n!}\,(x)^n$$

Hence coefficient of $x^5 = (-1)^n \frac{!}{n!n!}$.

Question # 8

Find 6*th* term in the expansion of $(x^2 - \frac{3}{2x})^{10}$

Solution. Since

$$T_{r+1} = \binom{n}{r} a^{n-r}x^r$$

Here $a = x^2 , x = -\frac{3}{2x} , n = 10 , r = 5 so , we\ have$

$$T_6 = \binom{10}{5} (x^2)^{10-5} (-\frac{3}{2x})^5$$

$$T_6 = \binom{10}{5} (x^2)^5 (-\frac{3}{2x})^5$$

$$T = (252)x_1^{\ 0} (- \ _{32x^5})$$

$$T \ = \ - \ 15309 \ x^5$$

Question # 9
Find the term independent of *x* in the following expansions..

(i). $(x - \ _{x})^{10}$

Solution.

$$Let\ T_{\ +1}\ be\ the\ required\ term$$

$$\because T_{\ +1} = (_{r})\ a^{n-r}.b^r$$

$$\Rightarrow T_{\ +1} = (^{10})\ a^{10-r}\ (- \ _{x})^r$$

$$T_{\ +1} = (^{10})\ (-2)^r . x^{10-r}. x^{-r}$$

$$\Rightarrow T_{\ +1} = (^{10})\ (-2)^r . x^{10-\ _{2}r}$$

For required result put

$$10 - 2r = 0 \Rightarrow 2r = 10 = r = 5$$

$$\Rightarrow T_{\ +1} = (^{10})\ (-2)^5 . x^{10-\ _{2}(_{5})}$$

$$= 252(-32)x^0$$

$$T \ = -8064(1) = -8064$$

(ii). $(\sqrt{x} - \ _{2x^2})^{10}$

Solution. Since

$$T_{\ +1} = (_{r})\ a^{n-r} x^r$$

Here $a = \sqrt{x} , x = \ _{2x^2} , n = 10\ so , we\ have$

$$T_{\ +1} = \binom{10}{r} (\sqrt{x})^{10-r} (\frac{}{2x^2})^r$$

$$T_{r+1} = \binom{10}{r} (x)^{\frac{10-r}{2}} (\frac{1}{2})^r x^{-2r}$$

$$T_{r+1} = \binom{10}{r} (x)^{\frac{10-r}{2}-2r} (\frac{1}{2})^r$$

$$T_{r+1} = \binom{10}{r} (x)^{\frac{10-5r}{2}} \left(\frac{1}{2}\right)^r$$

For term independent of x we must have $x^{\frac{10-5r}{2}} = x^0 \Rightarrow \frac{10-5r}{2} = 0$

$$5r = 10$$

$$r = 2$$

So

$$T\ = \binom{10}{} (x)^0 \left(\ \right)^2$$

$$T\ = (45)\left(\ \right)$$

$$T\ = \ 45$$

(iii). $(1 + x^2)^3 \left(1 + \frac{}{x^2}\right)^4$

Solution. $(1 + x^2)^3 \left(1 + \frac{1}{}\right)^4 = (1 + x^2)^3 \left(\frac{1+x^2}{}\right)^4$

$$(1 + x^2)^3 \left(1 + \frac{}{x^2}\right)^4 = x^{-8}(1 + x^2)^3(1 + x^2)^4$$

$$(1 + x^2)^3 \left(1 + \frac{}{x^2}\right)^4 = x^{-8}(1 + x^2)^7$$

Since

$$T_{\ +1} = x^{-8} \binom{}{r} a^{n-r} x^r$$

Here $a = 1, x = x^2, n = 7\ so, we\ have$

$$T_{\ +1} = x^{-8} \binom{}{r} (1)^{7-r} (x^2)^r$$

$$T_{\ +1} = \binom{}{r} x^{2r-8}$$

For term independent of x we must have $x^{2r-8} = x^0 \Rightarrow 2r - 8 = 0$

$$2r = 8$$

$$r = 4$$

So

$$T\ = \binom{}{4} x^0$$

$$T\ = 35$$

Question # 10

Determine the middle term in the following expensions:

(i). $\left(\frac{2}{x} - \frac{}{2}\right)^{12}$

Solution

Since $n = 12\ ia\ an\ even\ so\ middle\ term\ is\ \frac{n+2}{2} = \frac{12+2}{2} = 7$

Therefor $r + 1 = 7 \Rightarrow r = 6$

Here $a = \frac{1}{x}, x = \frac{-x^2}{2}, n = 12\ so, we\ have$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{6+1} = \binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$T_{6+1} = (924)\left(\frac{1}{x}\right)^6\left(\frac{x^{12}}{64}\right)$$

$$T_{6+1} = {}^{231}\ x^6$$

Thus the middle term of the given expansion is ${}^{231}x^6$.

(ii). $({}_2\, x - {}_{3x})^{11}$

Solution

Since $n = 11$ *ia an odd so middle term are* ${}^{n+1} = {}^{11+1} = 6$ *and* ${}^{n+3} = {}^{11+3} = 7$

So for the First middle term

Here $a = {}_2\,x, x = -{}_{3x}\ , n = 11$ *so , we have*

$$T_{\ +1} = ({}_r)\, a^{n-r}x^r$$

$$T_{\ +1} = (^{11})\,(\text{—}x)^{11-5}(-{}_{3x})^5$$

$$T\ \ = (^{11})\,(\text{—}x)^6(-{}_{3x})^5$$

$$T\ \ = (^{11})\,({}_2\,x)^6\ (-{}_{3x})^5$$

$$T\ \ = 462\,(\text{——})^6\ (-{}^1)^5 .(\text{—})^5$$

$$T\ \ = 462.(\text{—})^6 .x^6(-1)^5\,({}_3)^5\ x^{-5}$$

$$T\ \ = -{}^{462}\ .3^1x$$

$$\Rightarrow T\ \ = {}^{-462(3)}\ {}_x = {}^{-231(3)}\ {}_x$$

$$\Rightarrow T\ \ = -{}^{693}\ x$$

For 7th term put $r = 6$

$$T_{\ +1} = (\ {}_6\ )\,({}_2\,x)^{11-6} .(-{}_{3x})^6$$

$$= 462.{}^5_{2^5}\,x^5(-1)^6.{}_{3^6x^6}$$

$$= \frac{462}{32(3x)} = \frac{231}{48x} = \frac{77}{16x}$$

(iii). $(2x - \frac{1}{2x})^{2m+1}$

Solution

Since $n = 2m + 1$ *ia an odd so middle term are* $\frac{n+1}{2} = \frac{2m+1+1}{2} = m + 1$ *and* $\frac{n+3}{2} = \frac{2m+1+3}{2} = m + 2$

So for the First middle term

Here $a = 2x, x = -\frac{1}{2x}$, $n = 2m + 1$, $r = m$ so , we have

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{m+1} = \binom{2m+1}{} (2x)^{2m+1} \_^{m} (-\frac{1}{2x})^m$$

$$T_{m+1} = \binom{2m+1}{} (2x)^{m+1} (- \ _{2x})^{m}$$

$$T_{m+1} = \binom{2m+1}{} (x)^{m+1} \_^{m} (-1)^m$$

$$T_{m+1} = \binom{2m+1}{} (2x)\ (-1)^m$$

$$= (\ \_\ 1)^m \ \ m!\,(m+1\ !^{!}\ (2x)^1$$

$$T_{m+1} = 2(-1)^m \ \ m!\,(m+1)!\ x$$

Now for the 2nd middle term

Here $a = 2x, x = -\ _{2x}$, $n = 2m + 1$, $r = m + 1$ so , we have

$$T\ _{+1} = (_r)\ a^{n-r} x^r$$

$$T_{m+1+1} = \binom{2m+1}{} (2x)^{2m+1-m-1} (-\ _{2x})^{m+1}$$

$$T_{m+2} = \binom{2m+1}{} (2x)^{2m-m} (-\ _{2x})^{m+1}$$

$$\frac{(2m+1)!}{(m+1)(2m+1-m-1)!} \quad ^{2m+1-m-1} \quad ^{m}{}_{+1}$$

$$= (-1)^{m+1} \ (m+1)!\,m!\ (2x)^m.(2x)^{-m-1}$$

$$= (\ \_\ 1)^{m+1}\ (m+1)m!\ (\ x)^{-1}\ _2$$

$$^{m+2} \quad \frac{(2m+1)!}{(m+1)!\,m!} \ \frac{1}{2x}$$

$$^{m+2} \quad \frac{1}{2} \quad \frac{(2m+!)!}{m!\,(m+1)!} \ \frac{1}{x}$$

Important Note:

$$\because (\boldsymbol{a} + \boldsymbol{b})^3 = \boldsymbol{a}^3 + 3\boldsymbol{a}^2\boldsymbol{b} + 3\boldsymbol{a}\boldsymbol{b}^2 + \boldsymbol{b}^3$$

And $(\boldsymbol{b} + \boldsymbol{a})^3 = \boldsymbol{b}^3 + 3\boldsymbol{a}\boldsymbol{b}^2 + 3\boldsymbol{a}^2\boldsymbol{b} + \boldsymbol{a}^3$

Now $3\boldsymbol{a}^2\boldsymbol{b} = \boldsymbol{3rd\ term\ from\ the\ end\ in\ the\ expension\ of}\ (\boldsymbol{b} + \boldsymbol{a})^3$

So we conclude that

Required term from the end in the expansion of $(\boldsymbol{a} + \boldsymbol{b})^{\boldsymbol{n}}$ $\boldsymbol{is\ equal\ to\ required\ term\ from\ beging}$ **in the expansion of** $(\boldsymbol{b} + \boldsymbol{a})^{\boldsymbol{n}}$

Question # 11

Find $(2n+1)th$ term of the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$ Solution.

Here $a = x, x = -\frac{1}{2x}$ $, n = 3n\,, r = 2n$

Number of term from the end= $2n+1$

To make it from beginning we take $a = \frac{}{2x}$ $, x = x \; and \; r+1 = 2n+1 \Rightarrow r = 2n$

As

$$T_{\;+1} = \binom{}{r} a^{n-r} x^{r}$$

$$T_{\;n+1} = \binom{}{2n} \left(-\frac{}{2x}\right)^{3n-2n} (x)^{2n}$$

$$T_{\;n+1} = \binom{}{2n} \left(-\frac{}{2x}\right)^{n} (x)^{2n}$$

$$T_{\;n+1} = \binom{}{2n} \left(-\frac{}{2}\right)^{n} x^{-n} (x)^{2n}$$

$$T_{\;n+1} = \binom{}{2n} \left(-\frac{}{2}\right)^{n} (x)^{2n-n}$$

$$T_{\;n+1} = \binom{}{2n} \left(-\frac{}{2}\right)^{n} (x)^{n}$$

$$T_{\;n+1} = \frac{}{(2n)!\,(3n-2n)!} \left(-\frac{}{2}\right)^{n} (x)^{n}$$

$$_{2n+1} \quad \frac{(-1)^n}{2^n} \quad \frac{(3n)!}{(2n)!\,(n)!}$$

Question # 12

Show that the middle term of $(1+x)^{2n}$ is $\frac{1\cdot3\cdot5\cdot\ldots\;2n-1)}{!}$ $2^n x^n$.

Solution. Since n is even so the middle term is $\frac{2n+2}{} = n+1$

Here $a = 1, x = x\,, n = 2n\,, r+1 = n+1 \Rightarrow r = n$

As

$$T_{\;+1} = \binom{2n}{} (1)^{2n-n} x^n$$

$$T_{\;+1} = \binom{2n}{} x^n$$

$$_{r+1} = \frac{}{n!\,(2n-n)!} x^n$$

$$_{r+1} = \frac{!}{n!\,(n)!} x^n$$

$$T_{r+1} = \frac{(2n)(2n-1)(2n-2)\ldots 5\cdot4\cdot3\cdot2\cdot1}{n!\,(n)!} x^n$$

$$T_{r+1} = \frac{[(2n)(2n-2)(2n-4)\ldots\cdot4\cdot2][(2n-1)(2n-3)(2n-5)\cdot\ldots\cdot5\cdot3\cdot1]}{n!\,(n)!} x^n$$

$$T_{r+1} = \frac{2^n[(n)(n-1)(n-2)\dots\cdot 3\cdot 2\cdot 1](2n-1)(2n-3)(2n-5)\cdot\dots\cdot 5\cdot 3\cdot 1]}{n!\,(n)!}x^n$$

$$T_{r+1} = \frac{2^n n!\,[(2n-1)(2n-3)(2n-5)\cdot\dots\cdot 5\cdot 3\cdot 1]}{n!\,(n)!}x^n$$

$$T_{r+1} = \frac{5\cdot 3\cdot 1\dots(2n-1)}{n!}2^n x^n$$

Question # 13
Show that:

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = 2^{n-1}$$

Solution.

Consider

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \;---\;(i)$$

Put $x = 1\ in\ (i)$

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n-1} + \binom{n}{n} \;---\;(ii)$$

Put $x = -1\ in\ (i)$

$$(1-1)^n = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \binom{n}{4}(-1)^4 + \dots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \binom{n}{4}(-1)^4 + \dots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

If we consider n is even then

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \dots - \binom{n}{n-1} + \binom{n}{n}$$

$$\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}\right] = \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right]$$

Using it in equation (ii), we have

$$2^n = \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right] + \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right]$$

$$2^n = 2\left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right]$$

$$\frac{2^n}{2} = \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right]$$

$$2^{n-1} = \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right]$$

$$\left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}\right] = 2^{n-1}$$

Hence Proved.

Question # 14
Show that:

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \frac{1}{5}\binom{n}{4} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

Solution.

$$L.H.S = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \frac{1}{5}\binom{n}{4} + \dots + \frac{1}{n+1}\binom{n}{n}$$

$$L.H.S = 1 + \frac{1}{2}\frac{n!}{1!\,(n-1)!} + \frac{1}{3}\frac{n!}{2!\,(n-2)!} + \frac{1}{4}\frac{n!}{3!\,(n-3)!} + \cdots + \frac{1}{n+1} \cdot 1$$

$$L.H.S = \frac{n+1}{n+1}[1 + \frac{1}{2}\frac{n!}{1!\,(n-1)!} + \frac{1}{3}\frac{n!}{2!\,(n-2)!} + \frac{1}{4}\frac{n!}{3!\,(n-3)!} + \cdots + \frac{1}{n+1} \cdot 1]$$

$$L.H.S = \frac{1}{n+1}[n+1 + \frac{1}{2}\frac{(n+1)n!}{1!\,(n-1)!} + \frac{1}{3}\frac{(n+1)n!}{2!\,(n-2)!} + \frac{1}{4}\frac{(n+1)n!}{3!\,(n-3)!} + \cdots + \frac{1}{n+1} \cdot (n+1)]$$

$$L.H.S = \frac{1}{n+1}[n+1 + \frac{1}{2}\frac{(n+1)n!}{1!\,(n-1)!} + \frac{1}{3}\frac{(n+1)n!}{2!\,(n-2)!} + \frac{1}{4}\frac{(n+1)n!}{3!\,(n-3)!} + \cdots + 1]$$

$$L.H.S = \frac{1}{n+1}[n+1 + \frac{(n+1)!}{2!\,(n-1)!} + \frac{(n+1)!}{3!\,(n-2)!} + \frac{(n+1)!}{4!\,(n-3)!} + \cdots + 1]$$

$$L.H.S = \frac{1}{n+1}[n+1 + \frac{(n+1)!}{2!\,(n+1-2)!} + \frac{(n+1)!}{3!\,(n+1-3)!} + \frac{(n+1)!}{4!\,(n+1-3)!} + \cdots + 1]$$

$$L.H.S = \frac{1}{n+1}[\binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{4} + \binom{n+1}{5} + \cdots + \binom{n+1}{n+1}]$$

$$L.H.S = \frac{1}{n+1}[-1 + 1 + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{4} + \binom{n+1}{5} + \cdots + \binom{n+1}{n+1}]$$

$$L.H.S = \frac{1}{n+1}[-1 + \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{4} + \binom{n+1}{5} + \cdots + \binom{n+1}{n+1}]$$

$$L.H.S = \frac{1}{n+1}[-1 + 2^{n+1}]$$

$$L.H.S = \frac{2^{n+1} - 1}{n+1} R.H.S$$

Hence Proved.

The Binomial theorem when the index "n" is a negative integer or a fraction:

when n is negative integer or a fraction, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \infty \; provide \; |x| + \cdots$$

This is called Binomial Series.

Note:

1. In Binomial Series

First term $T = 1$

Second term$T = nxx$

$Third\ Term = T = n(n-1)\, x^2$

Fourth term $T = n(n-1)(n-2)\, x^3$

Similarly

General term $= T_{+1} = n(n-1)(n-2)\ldots(n-r+1)\, x^r$

2. the symbols $(_o)\ (_1)\ (_2)\ etc$ are meaningless when n is negative of a fraction.

Exercise 8.3

Question No.1 Expand the following $upto$ 4 terms, taking the valves of x such that the expansion in each case is valid.

(i) $(1-x)^{1/2}$

Solution:

$$(1-x)^{1/2} = 1 + \left(\frac{1}{2}\right)(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2 - \frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots$$

Note:

The expansion of $(1-x)^{\frac{1}{2}}$ is valid if $|x| < 1$

ii. $(1+2x)^{-1}$

Solution:

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + \cdots$$

$$= 1 - 2x + 4x^2 - 8x^3 + \cdots$$

Note:

The expansion of $(1+2x)^{-1}$ $is\ valid\ if$ $|2x| < 1 \Rightarrow 2|x| < 1 \Rightarrow |x| < \frac{1}{2}$

iii. $(1+x)^{-\frac{1}{3}}$

Solution:

$$(1+x)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2 + \frac{1}{6}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$$

Note:

The expansion $(1+x)^{-\frac{1}{3}}$ is valid if $|x|<1$

iv. $(4-3x)^{\frac{1}{2}}$

Solution:

$$(4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{1/2} = 2\left(1-\frac{3}{4}\right)^{1/2}$$

$$= 2\left\{1+\frac{}{2}\left(-\frac{}{4}x\right)+\frac{1\,(\;-1)}{2!}\left(-\frac{}{4}x\right)^{2}+\frac{1\,(\;-1)(\;-2)}{3!}\left(-\frac{}{4}x\right)^{3}+\cdots\right\}$$

$$= 2\left\{1-\frac{}{8}x+\frac{}{2}\left(\frac{}{2}\right)\left(-\frac{}{2}\right)\left(\;x^{2}\right)+\frac{}{6}\left(\;\right)\left(-\frac{}{2}\right)\left(-\frac{}{2}\right)\left(-\frac{}{64}\right)+\cdots\right\}$$

$$= 2\left\{1-\frac{}{8}x-\frac{}{128}x^{2}-\frac{}{1024}x^{3}+\cdots\right\}$$

$$= 2-\frac{}{4}x-\frac{}{64}x^{2}-\frac{}{512}x^{3}-\cdots$$

Note:

The expansion of $(4-3x)^{1/2}$ *is* valid if $|\frac{}{4}x|<1$

$$\Rightarrow \frac{3}{}|x|<1 \Rightarrow |x|<\frac{4}{}$$

(v) $(8-2x)^{-1}$

Solution:

$$(8-2x)^{-1} = \left[8\left(1-\frac{}{8}x\right)\right]^{-1} = 8^{-1}\left(1-\frac{}{4}x\right)^{-1} = \frac{}{8}\left(1-\frac{}{4}x\right)^{-1}$$

$$= \frac{1}{}\left\{1+(-1)\left(-\frac{1}{}x\right)+\frac{(-1)(-2)}{}\left(-\frac{1}{}x\right)^{2}+\frac{(-1)(-2)(\;\;3)}{}\left(-\frac{1}{}x\right)^{3}+\cdots\right\}$$

$$= \frac{}{8}\left\{1+\frac{}{4}x+\frac{}{16}x^{2}+\frac{}{64}x^{3}+\cdots\right\}$$

$$= \frac{}{8}+\frac{}{32}x+\frac{}{128}x^{2}+\frac{}{512}x^{3}+\cdots$$

Note:

The expansion of $(8-2x)^{-1}$ *is* valid if f$|\frac{}{4}x|<1 \Rightarrow \frac{}{4}|x|<1 \Rightarrow |x|<4$

(vi) $(2-3x)^{-2}$

Solution:

$$(2-3x)^{-2} = \left[2\left(1-\frac{}{2}x\right)\right]^{-2} = 2^{-2}\left(1-\frac{}{2}x\right)^{-2} = \frac{}{4}\left(1-\frac{}{2}x\right)^{-2}$$

$$= \frac{1}{}\left\{1+(-2)\left(-\frac{3}{}x\right)+\frac{(-2)(-3)}{}\left(-\frac{3}{}x\right)^{2}+\frac{(-2)(-3)(\;\;4)}{}\left(-\frac{3}{}x\right)^{3}+\cdots\right\}$$

$$= \frac{1}{}\left\{1+3x+\frac{27}{}x^{2}+\frac{27}{}x^{3}+\cdots\right\}$$

$$= \frac{1}{}+\frac{3}{}x+\frac{27}{}x^{2}+\frac{27}{}x^{3}+\cdots$$

Note:

The expansion of $(2-3x)^{-2}$ *is valid if* $|\frac{}{2}x|<1 \Rightarrow \frac{}{2}|x|<1 \Rightarrow |x|<\frac{}{3}$

(vii) $\frac{(1-x)^{-1}}{}$

Solution:

$$\frac{(1-x)^{-1}}{(1+x)^{2}} = (1-x)^{-1}(1+x)^{-2}$$

$$= \left\{1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^{2}+\frac{(-1)(-2)(-3)}{3!}(-x)^{3}+\cdots\right\}\left\{1+(-2)x+\frac{(-2)(-3)}{2!}x^{2}+\frac{(-2)(-3)(-4)}{3!}x^{3}+\cdots\right\}$$

$$= \{1+x+x^{2}+x^{3}+\cdots\}\{1-2x+3x^{2}-4x^{3}+\cdots\}$$

$$= 1(1-2x+3x^2-4x^3+\cdots)+x(1-2x+3x^2+\cdots)+x^2(1-2x+\cdots)+x^3(1-\cdots)+\cdots$$
$$= 1-2x+3x^2-4x^3+x-2x^2+3x^3+x^2-2x^3+x^3+\cdots$$
$$= 1-2x+x+3x^2-2x^2+x^2-4x^3+3x^3-2x^3+x^3+\cdots$$
$$= 1-x+2x^2-2x^3+\cdots$$

Note:

The expansion of $(1-x)^{-1}$ *and* $(1+x)^{-2}$ *are valid if* $|x|<1$ *thus expansion of* $(1-x)^{-1}(1+x)^{-2}$ is valid if $|x|<1$

(viii) $\frac{\sqrt{1+2x}}{1-x}$

Solution:

$$\frac{\sqrt{1+2x}}{} = (1+2x)^{\frac{1}{2}}(1-x)^{-1}$$

$$= \{1+\frac{1}{2}(2x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(2x)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2x)^3+\cdots\}\times\{1+(-1)(-x)+\frac{}{2!}(-x)^2+\frac{}{3!}(-x)^3+\cdots\}$$

$$= \{1+x+\frac{}{2}(\;)(-\frac{}{2})(4x^2)+\frac{}{6}(\frac{}{2})(-\frac{}{2})(-\frac{}{2})+\cdots\}\{1+x+x^2+x^3+\cdots\}$$
$$= \{1+x+x^2+x^3+\cdots.\}\times\{1+x+x^2+x^3+\cdots\}$$

$$= 1(1+x+x^2+x^3+\cdots)+x(1+x+x^2+\cdots)-\frac{}{2}x^2(1+x+\cdots)+\frac{}{2}(1+\cdots)+\cdots$$
$$= 1+x+x+x^2+x^2-\frac{1}{}x^2+x^3+x^3+\cdots$$
$$= 1+2x+(2-\frac{1}{})x^2+2x^3+\cdots$$
$$= 1+2x+\frac{3}{}x^2+2x^3+\cdots$$

Note:

The expansion of $(1+2x)^{\frac{1}{}}$ *is valid if* $|2x|<1 \Rightarrow 2|x|<1 \Rightarrow |x|<\frac{}{2}$ and the expansion of $(1-x)^{-1}$ *is* valid if $|x|<\frac{}{2}$

Thus expansion of $\frac{\sqrt{1+2x}}{}$ *is valid if* $|x|<\frac{1}{}$

(ix) $\frac{(4+2x)^{\frac{1}{2}}}{}$

Solution:

$$\frac{(4+2x)^{\frac{}{2}}}{} = [4(1+\frac{2}{}x)]^{\frac{}{2}}(2-x)^{-1} = 4^{\frac{1}{}}(1+\frac{1}{}x)^{\frac{}{2}}[2(1-\frac{1}{}x)]^{-1}$$

$$= 2(1+\frac{}{2}x)^{\frac{1}{}}.2^{-1}(1-\frac{}{2}x)^{-1} = (1+\frac{}{2}x)^{\frac{1}{}}(1-\frac{}{2}x)^{-1}$$

$$\{1+\frac{1}{}(\;x)+\frac{2(\quad 1)}{2!}(\;x)^2+\frac{2(2-\quad(2-2)}{3!}(\;x)^3\}\{1+(-1)(-\frac{1}{}x)+\frac{(-1\quad -2)}{2!}(-\frac{1}{}x)^2+\frac{(-1)(-2)(-3)}{}(-\frac{1}{}x)^3+\cdots\}$$

$$= \{1+\frac{1}{}x+\frac{1}{}(\;)(-\frac{1}{})(\;x^2)+\frac{1}{}(\;)(-\frac{1}{})(-\;)+\cdots\}\{1+\frac{1}{}x+\frac{1}{}x^2+\frac{1}{}x^3+\cdots\}$$

$$= \{1+\frac{}{4}x-\frac{}{32}x^2+\frac{}{128}x^3+\cdots\}\{1+\frac{}{2}x+\frac{}{4}x^2+\frac{}{8}x^3+\cdots\}$$

$$= 1(1+\frac{}{2}x+\frac{}{4}x^2+\frac{}{8}x^3+\cdots)+\frac{}{4}x(1+\frac{}{2}x+\frac{}{4}x^2+\cdots)-\frac{}{32}x^2(1+\frac{}{2}x)+\frac{}{128}x^3(1+\cdots)+\cdots$$

$$= 1+\frac{}{2}x+\frac{}{4}x^2+\frac{}{8}x^3+\frac{}{4}x+\frac{}{8}x^2+\frac{}{16}x^3-\frac{}{32}x^2-\frac{}{64}x^3+\frac{}{128}x^3+\cdots$$

$$= 1+(\frac{1}{2}+\frac{1}{4})x+(\frac{1}{4}+\frac{1}{8}-\frac{1}{32})x^2+(\frac{1}{8}+\frac{1}{16}-\frac{1}{74}+\frac{1}{128})x^3+\cdots$$

$$1+\frac{2+1}{4}x+\frac{8+4-1}{32}x^2+\frac{16+8-1+!}{128}x^3+\cdots$$

$$1+\frac{3}{4}x+\frac{11}{32}x^2+\frac{23}{128}x^3+\cdots$$

Note:

The expansion $(4+2x)^{\frac{1}{2}}$ *is validd if* $|\frac{1}{2}x| < 1 \Rightarrow \frac{1}{2}|x| < 1 \Rightarrow |x| < 2$ *and the expansion of* $(2-x)^{-1}$ is valid if $|\frac{1}{2}x| < 1 \Rightarrow \frac{1}{2}|x| < 1 \Rightarrow |x| < 2$

Thus expansion of $\frac{(4+2x)^{1/2}}{2-x^x}$ is valid if $|x| < 2$

(x)

$(1+x-2x^2)^{\frac{1}{2}}$

solution:

$$[1+(x-2x^2)]^{\frac{1}{2}} = 1 + \frac{1}{2}(x-2x^2) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(x-2x^2)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(x-2x^2)^3 + \cdots$$

$$= 1 + \frac{1}{2}x - x^2 + \frac{1}{2}(\frac{1}{2})(-\frac{1}{2})[x^2+4x^4-4x^3] + \frac{1}{6}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})[x^3 - 3x^2(2x^2) + 3x(2x^2)^2 - (2x^2)^3 + \cdots]$$

$$= 1 + \frac{}{2}x - x^2 + \frac{}{8}(x^2+4x^4-4x^3) + \frac{}{16}(x^3-6x^4+12x^5-8x^6) + \cdots$$

$$= 1 + \frac{}{2}x - x^2 - \frac{}{8}x^2 - \frac{}{2}x^4 + \frac{}{2}x^3 + \frac{}{16}x^3 - \frac{}{8}x^4 + \frac{}{4}x^5 - \frac{}{2}x^6 + \cdots$$

$$= 1 + \frac{}{2}x + (-1-\frac{}{8})x^2 + (\frac{}{2}+\frac{}{16})x^3 + (-\frac{}{2}-\frac{}{8})x^4 + \frac{}{4}x^5 - \frac{}{2}x^6 + \cdots$$

$$= 1 + \frac{1}{}x - \frac{9}{}x^2 + (\quad)x^3 + (\quad)x^4 + \frac{3}{}x^5 - \frac{1}{}x^6 + \cdots$$

$$= 1 + \frac{}{2}x - \frac{}{8}x^2 + \frac{}{16}x^3 - \frac{}{8}x^4 + \frac{}{4}x^5 - \frac{}{2}x^6 + \cdots$$

$$= 1 + \frac{}{2}x - \frac{}{8}x^2 + \frac{}{16}x^3 - \cdots$$

Note :

The expansion of $[1+(x-2x^2)]^{\frac{1}{}}$ is valid if $|x-2x^2| < 1$

$$\Rightarrow x - 2x^2 < 1 \quad or \quad -(x-2x^2) < 1$$

$$\Rightarrow x - 2x^2 - 1 < 0 \quad or - x + 2x^2 - 1 < 0$$

$$-2x^2 + x - 1 < 0 \quad or \quad 2x^2 - x - 1 < 0$$

$$solving - 2x^2 + x - 1 = 0 \quad solving \quad 2x^2 - x - 1 = 0$$

$$\frac{-1 \pm \sqrt{(1)^2 - 4(-2)(-1)}}{2(-2)} \quad or \quad 2x^2 - 2x + x - 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-8}}{} \quad or \quad 2x(x-1) + 1(x-1) = 0$$

$$\Rightarrow x = \frac{-1 \pm \quad -7}{4} \quad (rejected\ being\ complex) \quad \Rightarrow (x-1)(2x+1) = 0$$

$$\Rightarrow x - 1 = 0, 2x + 1 = 0 \Rightarrow x = 1 \quad , x = -\frac{1}{}$$

The expansion of $[1+(x-2x\quad)]^{\frac{1}{2}}$ is valid if $-\frac{}{2} < |x| < 1$

(xi) $(1-2x+3x^2)^{\frac{}{2}}$

Solution:

$$[1-2x+3x^2]^{\frac{}{2}} = 1 + \frac{-}{2}(3x^2-2x) + \frac{1(\quad-1)}{2!}(3x^2-2x^2) + \frac{1(\quad-1)(\quad-2)}{3!}(3x^2-2x)^3 + \cdots$$

$$= 1 + \frac{3}{2}x^2 - x + \frac{1}{2}(\frac{1}{2})(-\frac{1}{2})(9x^4-12x^3+4x^2) + \frac{1}{6}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})[(3x^2)^3 - 3(3x^2)(2x) + 3(3x^2)(2x)^2 - (2x)^3 + \cdots]$$

$$= 1 + \frac{3}{2}x^2 - x - \frac{1}{8}(9x^4-12x^3+4x^2) + \frac{1}{16}(27x^6-54x^5+36x^4-8x^3) + \cdots$$

$$= 1 + \frac{3}{2}x^2 - x - \frac{9}{8}x^4 + \frac{3}{2}x^3 - \frac{1}{2}x^2 + \frac{27}{16}x^6 - \frac{27}{8}x^5 + \frac{9}{4}x^4 - \frac{1}{2}x^3 + \cdots$$

$$= 1 - x + (\frac{3}{2} - \frac{1}{2})x^2 + (\frac{3}{2} - \frac{1}{2})x^3 + (\frac{9}{4} - \frac{9}{8})x^4 - \frac{27}{8}x^5 + \frac{27}{16}x^6 + \cdots$$

$$1 - x + x^2 + x^3 + \frac{9}{8}x^4 - \frac{27}{8}x^4 - \frac{27}{8}x^5 + \frac{27}{16}x^6 + \cdots$$

Note :

The expansion $(1 - 2x + 3x^2)^{\frac{1}{2}}$ is valid if $|3x^2 - 2x| < 1$

$3x^2 - 2x < 1$ or $-3(3x^2 - 2x) < 1$

$3x^2 - 2x - 1 < 0$ or $-3x^2 + 2x - 1 < 0$

$solving\ 3x^2 - 2x - 1 = 0$ or $3x^2 - 2x + 1 < 0$

$3x^2 - 3x + x - 1 = 0$ $\quad Solving\ 3x^2 - 2x + 1 = 0$

$3x(x - 1) + 1(x - 1) = 0$ $\quad \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{(2)(3)}$

$(x - 1)(3x + 1) = 0$ $\quad x = \frac{2 \pm \sqrt{4 - 12}}{}$

$x - 1 = 0 \quad , \quad 3x + 1 = 0$ $\quad x = \frac{2 \pm \quad -8}{6}$ $(rejected\ being\ complex)$

$x = 1 \quad x = - \frac{}{3}$

The expansion of $(1 - 2x + 3x^2)^{1}$ $is\ valid\ if\ -^{1} < x < 1$

Question No.22 using Binomial theorem find the value of the following to three places of decimal.

(i) $\sqrt{99}$

Solution:

$$\sqrt{99} = (99)^{\frac{1}{2}} = (100 - 1)^{\frac{1}{2}}$$

$$[100(1 - \frac{1}{100}] = (100)^{\frac{1}{2}}(1 - 0.01)^{\frac{1}{2}}$$

$$= 10\{1 + \frac{}{2}(-0.01) + \frac{(\ - 1)}{2!}(-0.01)^2 + \cdots\}$$

$$= 10\{1 - 0.005 + {}^{1}(\ )(-{}^{1})(0.0001) + \cdots\}$$

$$= 10\{1 - 0.005 - 0.0000125 - \cdots\}$$

$$= 10(1 - 0.0050125)$$

$$= 10(0.9949) = 9.95$$

(ii) $(.98)^{\frac{1}{2}}$

Solution:

$$(\mathbf{0.98})^{\frac{1}{2}} = (\mathbf{1 - 0.02})^{\frac{1}{2}}$$

$$= \mathbf{1} + \mathbf{1}(-\mathbf{0.02}) + \frac{(\ - \mathbf{1})}{}(-\mathbf{0.02})^2 + \cdots$$

$$= \mathbf{1 - 0.01} + {}^{\mathbf{1}}({}^{\mathbf{1}})(-{}^{\mathbf{1}})(\mathbf{0.00040}) + \cdots$$

$$= \mathbf{1 - 0.01} - {}^{\mathbf{1}}(\mathbf{0.00040}) + \cdots$$

$$= \mathbf{1 - 0.01 - 0.000050} + \cdots$$

$$= \mathbf{1 - 0.010050}$$

$$= \mathbf{0.989}$$

$$= \mathbf{0.99}$$

(iii)

$$(\mathbf{1.03})^{\frac{1}{3}}$$

Solution:

$$(\mathbf{1.03})^{\frac{1}{3}} = (\mathbf{1 + 0.03})^{\frac{1}{3}}$$

$$= \mathbf{1} + \mathbf{1}(\mathbf{0.03}) + \frac{(\ - \mathbf{1})}{}(\mathbf{0.03})^2 + \cdots$$

$$= \mathbf{1 + 0.010} + {}^{\mathbf{1}}(\ )(-{}^{\mathbf{2}})(\mathbf{0.0009}) + \cdots$$

$$= \mathbf{1 + 0.010} - \frac{}{\mathbf{9}}(\mathbf{0.0009}) + \cdots$$

$$= \mathbf{1 + 0.010 - 0.0001} + \cdots$$

$$= \mathbf{1.0099}$$

(iv)

$$\mathbf{3\sqrt{65}}$$

Solution:

$$\mathbf{3\sqrt{65}} = (\mathbf{65})^{\frac{1}{3}} = (\mathbf{64 + 1})^{\frac{1}{3}}$$

$$[\mathbf{64}(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{64}})]^{\frac{1}{3}} = (\mathbf{64})^{\frac{1}{3}}(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{64}})^{\frac{1}{3}}$$

$$= (\mathbf{4^3})^{\frac{1}{3}}(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{64}})$$

$$= \mathbf{4}\{\mathbf{1} + \frac{}{\mathbf{3}}(\ ) + \frac{(\ - \mathbf{1})}{\mathbf{2!}}(\ )^{\mathbf{2}} + \cdots\}$$

$$= \mathbf{4}\{\mathbf{1} + \frac{\mathbf{1}}{\mathbf{192}} + \frac{\mathbf{1}}{\mathbf{2}}(\frac{\mathbf{1}}{\mathbf{3}})(-\frac{\mathbf{1}}{\mathbf{3}})(\frac{\mathbf{1}}{\mathbf{4096}}) + \cdots\}$$

$$= \mathbf{4}\{\mathbf{1} + \mathbf{0.0052} - \frac{\mathbf{1}}{\mathbf{36864}} + \cdots\}$$

$$= \mathbf{4(1.0051) = 4.02}$$

(v)

$$\sqrt[4]{17}$$

Solution:

$$\sqrt[4]{17} = (17)^{\frac{1}{4}} = (16+1)^{\frac{1}{4}}$$
$$= \left[16\left(1+\frac{1}{16}\right)\right]^{\frac{1}{4}} = (16)^{\frac{1}{4}}\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$$
$$= (2^4)^{\frac{1}{4}}\left(1+\frac{1}{16}\right)^{\frac{1}{4}} = 2\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$$
$$= 2\left\{1+\frac{1}{4}\left(\frac{1}{16}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{1}{16}\right)^2+\cdots\right\}$$
$$= 2\left\{1+\frac{1}{64}+\frac{1}{2}\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(\frac{1}{256}\right)+\cdots\right\}$$
$$= 2\left\{1+0.015-\frac{3}{8192}+\cdots\right\}$$
$$= 2\{1+0.015-0.00036+\cdots\}$$
$$= 2(1.014) = 2.029 = 2.03$$

(vi)

$$\sqrt[5]{31}$$

Solution:

$$\sqrt[5]{31} = (31)^{\frac{1}{5}} = (32-1)^{\frac{1}{5}} = \left[32\left(1-\frac{1}{32}\right)\right]^{\frac{1}{5}}$$
$$= (32)^{\frac{1}{5}}\left[1-\frac{1}{32}\right]^{\frac{1}{5}} = (2^5)^{\frac{1}{5}}\left(1-\frac{1}{32}\right)^{\frac{1}{5}}$$
$$= 2\left(-\frac{1}{32}\right)^{\frac{1}{5}}$$
$$= 2\left\{1+\left(\frac{1}{5}\right)\left(-\frac{1}{32}\right)+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}\left(-\frac{1}{32}\right)^2+\cdots\right\}$$
$$= 2\left\{1-\frac{1}{160}+\frac{1}{2}\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\left(\frac{1}{1024}\right)+\cdots\right\}$$
$$= 2\left\{1-0.006-\frac{1}{12800}+\cdots\right\}$$
$$= 2\{1-0.006-0.00078+\cdots\}$$
$$= 2(0.993) = 1.987$$

(vii) $\frac{1}{\sqrt[3]{998}}$

Solution:

$$\frac{1}{\sqrt[3]{998}} = \frac{1}{(998)^{\frac{1}{3}}} = (998)^{-\frac{1}{3}}$$
$$= (1000-2)^{-\frac{1}{3}} = \left[1000\left(1-\frac{2}{1000}\right)\right]^{-\frac{1}{3}}$$
$$= (10^3)^{-\frac{1}{3}}\left(1-\frac{1}{500}\right)^{-\frac{1}{3}} = 10^{-1}\left(1-\frac{1}{500}\right)^{-\frac{1}{3}}$$
$$= \frac{1}{10}\left\{1+\left(-\frac{1}{3}\right)\left(-\frac{1}{500}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{1}{500}\right)^2\right\}$$
$$= \frac{1}{10}\left\{1+\frac{1}{1500}+\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(\frac{1}{250000}\right)+\cdots\right\}$$
$$= \frac{1}{10}\{1+0.0006+\cdots\} = \frac{1}{10}(1.0006) = 0.100$$

(viii) $\frac{1}{\sqrt[5]{252}}$

Solution:

$$\frac{1}{\sqrt[5]{252}} = \frac{1}{(252)^{\frac{1}{5}}} = (252)^{-\frac{1}{5}}$$
$$= (243+9)^{-\frac{1}{5}} = \left[243\left(1+\frac{9}{243}\right)\right]^{-\frac{1}{5}}$$
$$= (3^5)^{-\frac{1}{5}}\left(1+\frac{9}{243}\right)^{-\frac{1}{5}} 3^{-1}\left(1+\frac{1}{27}\right)^{-\frac{1}{5}}$$
$$= \frac{1}{3}\left\{1+\left(-\frac{1}{5}\right)\left(\frac{1}{27}\right)+\frac{\left(-\frac{1}{5}\right)\left(-\frac{1}{5}-1\right)}{2!}\left(\frac{1}{27}\right)^2+\cdots\right\}$$
$$= \frac{1}{3}\left\{1-\frac{1}{135}+\frac{1}{2}\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(\frac{1}{729}\right)+\cdots\right\}$$
$$= \frac{1}{3}\{1-0.007+0.000016+\cdots\}$$
$$= \frac{1}{3}(0.993) = 0.331$$

(ix) $\frac{\sqrt{7}}{\sqrt{8}}$

Solution:

$$\frac{\sqrt{7}}{\sqrt{8}} = \sqrt{\frac{7}{8}} = \left(1-\frac{1}{8}\right)^{\frac{1}{2}}$$
$$= 1+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{8}\right)^2+\cdots$$
$$= 1-\frac{1}{16}+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{64}\right)+\cdots$$
$$= 1-0.0062-\frac{1}{512}+\cdots$$
$$= 1-0.062-0.0019+\cdots$$
$$= 0.936$$

(x) $(0.998)^{-\frac{1}{3}}$

Solution:

$$(0.998)^{-\frac{1}{3}} = (1-0.002)^{-\frac{1}{3}}$$
$$= 1+\left(-\frac{1}{3}\right)(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-0.002)^2+\cdots$$
$$= 1+0.00066+\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)(0.00004)+\cdots$$
$$1+0.00066+\cdots$$
$$= 1.00066 = 1.001$$

(xi)

$$\frac{1}{6\sqrt{486}}$$

Solution:

$$\frac{1}{6\sqrt{486}} = \frac{1}{(486)^{\frac{1}{6}}} = (486)^{-\frac{1}{6}}$$

$$= (729-243)^{-\frac{1}{6}} = \left[729\left(1-\frac{243}{729}\right)\right]^{-\frac{1}{6}}$$

$$= (3^6)^{-\frac{1}{6}}\left(1-\frac{1}{3}\right)^{-\frac{1}{6}}$$

$$= 3^{-1}\left(1-\frac{1}{3}\right)^{-\frac{1}{6}} = \frac{1}{3}\left(1-\frac{1}{3}\right)^{-\frac{1}{6}}$$

$$= \frac{1}{3}\left\{1+\left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)+\frac{-\frac{1}{6}\left(-\frac{1}{6}-1\right)}{2!}\left(-\frac{1}{3}\right)^2+\cdots\right\}$$

$$= \frac{1}{3}\left\{1+\frac{1}{18}+\frac{1}{2}\left(-\frac{1}{6}\right)\left(-\frac{7}{6}\right)\left(\frac{1}{9}\right)+\cdots\right\}$$

$$= \frac{1}{3}\{1+0.05+0.01+\cdots\}$$

$$= \frac{1}{3}\{1.06\} = 0.3536$$

(xii)

$$(1280)^{\frac{1}{4}}$$

Solution:

$$(1280)^{\frac{1}{4}} = (1296-16)^{\frac{1}{4}}$$

$$= \left[1296\left(1-\frac{16}{1296}\right)\right]^{\frac{1}{4}} = (6^4)^{\frac{1}{4}}\left(1-\frac{1}{81}\right)^{\frac{1}{4}}$$

$$= 6\left\{1+\frac{1}{4}\left(-\frac{1}{81}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{1}{81}\right)^2+\cdots\right\}$$

$$= 6\left\{1-\frac{1}{324}+\frac{1}{2}\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(\frac{1}{6561}\right)+\cdots\right\}$$

$$6\{1-0.003-\cdots\}$$

$$= 6(0.997) = 5.98$$

Question No.3

Find the coefficients of x^n in the expansion of

i $\frac{1+x^2}{(1+x)^2}$

Solution:

$$\frac{1+x^2}{(1+x)^2} = (1+x^2)(1+x^{-2})$$

$$= (1+x^2)\{1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\cdots\}$$

$$= (1+x^2)\{1+(-1)^1 2x + (-1)^2 3x^2 + (-1)^3 4x^3+\cdots\}$$

Following the above pattern , we have

$$= (1+x^2)\{1+(-1)^1 2x+(-1)^2 3x^2+(-1)^3 4x^3+\cdots + (-1)^{n-1}(n-1)x^{n-2}+(-1)^{n-1}nx^{n-1} + (-1)^n(n+1)x^n+\cdots\}$$

The terms involving x^n in the expression of $(1+x^2)(1+x)^{-1}$ are

$$1(-1)^n(n+1)x^n = (-1)^n(n+!)x^n \ and$$

$$x^2(-1)^{n-2}(n-1)^{n-2} = (-1)^{n-2}(n-1)x^n$$

Therefore coefficients of x^n

$$= (-1)^n(n+1)+(-1)^{n-2}(n-1)$$

$$= (-1)^n\{n+!+(-1)^{-2}(n-1)\}$$

$$= (-1)^n\{n+!+n-1\} = (-1)^n(2n)$$

(ii) $\frac{(1+x)^2}{(1-x)^2}$

Solution:

$$\left(\frac{1+x}{1-x}\right)^2 = (1+x)^2(1-x)^{-2}$$

$$= (1+x)^2\{1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \cdots\}$$

$$= (1+x)^2\{1+(-2x+3x^2+4x^3+\cdots\}$$

Following the above pattern we have

$$= (1+x^2+2x)\{1+2x+3x^2+4x^3+\cdots+(n-1)x^{n-2} + nx^{n-1}+(n+1)x^n+\cdots\}$$

The terms involving x^n in the expansion of $(1+x)^2(1-x)^{-2}$ are $1(n+1)x^n = (n+1)x^n$

$$(2xnx^{n-1} = 2nx^n \ and$$

$$x^2(n-1)x^{n-2} = (n-1)x^n$$

Therefore coefficients of x^n

$$= n+1+2n+n-1 = 4n$$

(iii) $\frac{(1+x)^3}{(1-x)^2}$

Solution:

$$\frac{(1+x)^3}{(1-x)^2} = (1+x)^3(1-x)^{-2}$$

$$= (1+3x+3x^2+x^3)$$

$$\{1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3\}$$

Following the above pattern we have

$$= (1+3x+3x^2+x^3)\{1+2x+3x^2+4x^3+\cdots +(n-1)x^{n-3}+(n-1)x^n+\cdots\}$$

The terms involving $x^n are$

$$1.(n+1)x^n = (n+1)x^n$$

$$3x.nx^{n-1} = 3nx^n$$

$$3x^2.(n-1)x^{n-2} = 3(n-1)x^n$$

And $x^3(n-2)x^{n-3} = (n-2)x^n$

Therefore coefficients of x^n

$$= n+1+3n+3(n-1)+(n-2)$$

$$= n+1+3n+3n-3+n-2$$

$$= 8n-4 = 4(2n-1)$$

(iv)

$$\frac{(1+x)^2}{(1-x)^3}$$

Solution:

$$\frac{(1+x)^2}{(1-x)^3} = (1+x)^2(1-x)^{-3}$$

$$= (1+x^2+2x)\{1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^6 + \frac{(-3)(-4)(-5)}{3!}(-x)^3+\cdots\}$$

$$= (1+x^2+2x)\{1+3x+\frac{3\times 4}{2}x^2+\frac{4\times 5}{2}x^3+\cdots\}$$

Following the above pattern we have

$$= (1+2x+x^2)\{1+\frac{2\times 3}{2}x+\frac{3\times 4}{2}x^2+\frac{4\times 5}{2}x^3 + \cdots + \frac{n-1}{2}x^{n-2}+\frac{n(n+1)}{2}x^{n-1}+\frac{(n+1)(n+2)}{2}x^n+\cdots\}$$

The terms involving x^n are

$$1.\frac{(n+1)(n+2)}{2}x^n = \frac{(n+1)(n+2)}{2}x^n$$

$$x.\frac{n(n-1)}{2}x^{n-1} = \frac{2n(n-1)}{2}x^n$$

And $x^2\frac{(n-1)}{2}x^{n-2} = \frac{(n-1)n}{2}x^n$

Therefore coefficients of x^n

$$= \frac{(n+1)(n+2)}{2}+\frac{2n(n+1}{2}+\frac{(n-1)n}{2}$$

$$= \frac{n^2+2n+n+2+2n^2+2n+n^2-n}{2}$$

$$= \frac{4n^2+4n+2}{2} = \frac{2(2n^2+2n+1)}{2}$$

$$= 2n^2+2n+1$$

(v)

$$(1-x+x^2-x^3+\cdots)^2$$

Solution:

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots$$

$$\Rightarrow (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

Squaring both sides

$$\Rightarrow [(1+x)^{-1}]^2 = (1 - x + x^2 - x^3 + \cdots)^2$$

$$\Rightarrow (1 - x + x^2 - x^3 + \cdots)^2 = (1+x)^{-2}$$

$$= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$

$$= 1 + (-2)x + \frac{(-2)(-3)x^2}{2!} + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$

$$= 1 + (-1)^1 2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \cdots$$

Following the above pattern we have

$$1 + (-1)^1 2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \cdots + (-1)^n (n+1)x^n + \cdots$$

The coefficients of $x^n = (-1)^n(n+1)$

Question No.4 if x is so small that its square and higher powers can be neglected, then show that

(i)

$$\frac{1-x}{\sqrt{1+x}} \approx 1 - \frac{3}{2}x$$

Solution :

$$L.H.S = \frac{1-x}{\sqrt{1+x}}$$

$$= (1-x)(1+x)^{-\frac{1}{2}}$$

$$= (1-x)\{1 + (-\tfrac{1}{2})x + \cdots\}$$

$$= (1-x)(1 - \tfrac{1}{2}x + \cdots)$$

$$= 1 - (\tfrac{1}{2} + 1)x + \cdots$$

$$= 1 - \frac{3}{2}x + \cdots$$

$$\approx 1 - \frac{3}{2}x \approx R.H.S$$

(ii)

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$$

Solution:

$$L.H.S = \frac{\sqrt{1+2x}}{\sqrt{1-x}}$$

$$= (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$= \{1 + \tfrac{1}{2}(2x) + \cdots\}\{1 + (-\tfrac{1}{2})(-x) + \cdots\}$$

$$= (1+x)(1 + \tfrac{1}{2}x) + \cdots$$

$$= 1 + \frac{1}{2}x + x + \cdots$$

$$= 1 + (\tfrac{1}{2} + 1)x + \cdots$$

$$1 + \frac{3}{2}x + \cdots$$

$$\approx 1 + \frac{3}{2}x \approx R.H.S$$

(iii)

$$\frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{4+5x} \approx \frac{1}{4} - \frac{17}{284}x$$

Solution:

$$L.H.S = \frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{4+5x}$$

$$= \frac{[9(1+\frac{7}{9}x)]^{\frac{1}{2}} - [16(1+\frac{3}{16}x)]^{\frac{1}{4}}}{4+5x}$$

$$= \frac{[3^2(1+\frac{7}{9}x)]^{\frac{1}{2}} - [2^4(1+\frac{3}{16}x)]^{\frac{1}{4}}}{4(1+\frac{5}{4}x)}$$

$$= \frac{1}{4}[3(1+\tfrac{7}{9}x)^{\frac{1}{2}} - 2(1+\tfrac{3}{16}x)^{\frac{1}{4}}](1+\tfrac{5}{4}x)^{-1}$$

$$= \frac{1}{4}[3\{1 + \tfrac{1}{2}(\tfrac{7}{9}x) + \cdots\} - 2\{1 + \tfrac{1}{4}(\tfrac{3}{16}x) + \cdots\}] \times (1 + (-1)(\tfrac{5}{4}x) + \cdots)$$

$$= \frac{1}{4}[3(1 + \tfrac{7}{18}x + \cdots) - 2(1 + \tfrac{3}{64}x + \cdots)](1 - \tfrac{5}{4}x + \cdots)$$

$$= \frac{1}{4}[3 + \tfrac{7}{6}x - 2 - \tfrac{3}{32}x + \cdots](1 - \tfrac{5}{4}x + \cdots)$$

$$= \frac{1}{4}(1 + \frac{112-9}{96}x + \cdots)(1 - \frac{5}{4}x + \cdots)$$

$$= \frac{1}{4}(1 + \frac{103}{96}x + \cdots)(1 - \frac{5}{4}x + \cdots)$$

$$= \frac{1}{4}(1 + (\frac{-120+103}{96})x + \cdots)$$

$$= \frac{1}{4}(1 - \frac{17}{96}x + \cdots.)$$

$$= \frac{1}{4} - \frac{1}{4}(\frac{17}{96}x) + \cdots$$

$$\approx \frac{1}{4} - \frac{17}{384}x \approx R.H.S$$

Hence proved.

(iv)

$$\frac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \frac{25}{4}x$$

Solution:

$$L.H.S = \frac{\sqrt{4+x}}{(1-x)^3} = (4+x)^{\frac{1}{2}}(1-x)^{-3}$$

$$= [4(1+\tfrac{1}{4}x)]^{\frac{1}{2}}(1-x)^{-3}$$

$$= 2(1+\tfrac{1}{4}x)^{\frac{1}{2}}(1-x)^{-3}$$

$$= 2[1 + \tfrac{1}{2}(\tfrac{1}{4}x) + \cdots][1 + (-3)(-x) + \cdots]$$

$$= 2\left(1+\frac{1}{8}x+\cdots\right)(1+3x+\cdots)$$
$$= 2\left(1+3x+\frac{1}{8}x+\cdots\right)$$
$$= 2\left(1+\left(3+\frac{1}{8}\right)x+\cdots\right)$$
$$= 2\left(1+\frac{25}{8}x+\cdots\right)$$
$$= 2+\frac{25}{4}x+\cdots$$
$$\approx 2+\frac{25}{4}x \approx R.H.S$$

hence proved

(v)

$$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \approx 4\left(1-\frac{5}{6}x\right)$$

Solution:

$$= \frac{(1+x)^{\frac{1}{2}}\left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{3}{2}}}{\left[8\left(1+\frac{5}{8}x\right)\right]^{\frac{1}{3}}}$$
$$= \frac{(1+x)^{\frac{1}{2}}\left[2^2\left(1-\frac{3}{4}x\right)\right]^{\frac{3}{2}}}{\left[2^3\left(1+\frac{5}{8}x\right)\right]^{\frac{1}{3}}}$$
$$= \frac{(1+4)^{\frac{1}{2}}\left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{3}{2}}}{2\left(1+\frac{5}{8}x\right)^{\frac{1}{3}}}$$
$$= \frac{8}{2}(1+x)^{\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{\frac{3}{2}}\left(1+\frac{5}{8}x\right)^{-\frac{1}{3}}$$
$$= 4\left\{1+\frac{1}{2}x+\cdots\right\}\left\{1+\frac{3}{2}\left(-\frac{3}{4}x\right)+\cdots\right\}$$
$$\left\{1+\left(-\frac{1}{3}\right)\left(\frac{5}{8}x\right)+\cdots\right\}$$
$$= 4\left(1+\frac{1}{2}x+\cdots\right)\left(1-\frac{9}{8}x+\cdots\right)\left(1-\frac{5}{24}+\cdots\right)$$
$$= 4\left(1-\frac{9}{8}x+\frac{1}{2}x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$
$$= 4\left(1-\frac{5}{8}x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$
$$= 4\left(1-\frac{5}{24}x-\frac{5}{8}x+\cdots\right)$$
$$= 4\left(1-\left(\frac{5}{24}+\frac{5}{8}\right)x+\cdots\right)$$
$$= 4\left(1-\frac{5+15}{24}x+\cdots\right)$$
$$= 4\left(1-\frac{20}{24}x+\cdots\right)$$
$$= 4\left(1-\frac{5}{6}x+\cdots\right)$$
$$\approx 4\left(1-\frac{5}{6}x\right) \approx R.H.S$$

Hence proved

(vi)

$$\frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{1/3}} \approx \frac{3}{2}-\frac{61}{48}x$$

Solution:

$$L.H.S = \frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}}$$
$$= \frac{(1-x)^{\frac{1}{2}}\left[9\left(1-\frac{4}{9}x\right)\right]^{\frac{1}{2}}}{\left[8\left(1+\frac{3}{8}x\right]\right)^{\frac{1}{3}}}$$
$$= \frac{(1-x)^{\frac{1}{2}}\left[3^2\left(1-\frac{4}{9}x\right)\right]^{\frac{1}{2}}}{\left[2^3\left(1+\frac{3}{8}\right)\right]^{\frac{1}{3}}}$$
$$= \frac{(1-x)^{\frac{1}{2}}.3\left(1-\frac{4}{9}x\right)^{1/2}}{2\left(1+\frac{3}{8}x\right)^{1/3}}$$
$$= \frac{3}{2}(1-x)^{\frac{1}{2}}\left(1-\frac{4}{9}x\right)^{\frac{1}{2}}\left(1+\frac{3}{8}x\right)^{-\frac{1}{3}}$$
$$= \frac{3}{2}\left\{1-\frac{1}{2}x+\cdots\right\}\left\{1+\frac{1}{2}\left(-\frac{4}{9}x\right)+\cdots\right\}$$
$$\left\{1+\left(-\frac{1}{3}\right)\left(\frac{3}{8}x\right)+\cdots\right\}$$
$$= \frac{3}{2}\left\{1-\frac{1}{2}x+\cdots\right\}\left\{1-\frac{2}{9}x+\cdots\right\}\left\{1-\frac{1}{8}x+\cdots\right\}$$
$$= \frac{3}{2}\left\{1-\frac{2}{9}x-\frac{1}{2}x+\cdots\right\}\left\{1-\frac{1}{8}x+\cdots\right\}$$
$$= \frac{3}{2}\left(1-\left(\frac{2}{9}+\frac{1}{2}\right)x+\cdots\right)\left(1-\frac{1}{8}x+\cdots\right)$$
$$= \frac{3}{2}\left(1-\frac{1}{8}x-\frac{13}{18}x+\cdots\right)$$
$$= \frac{3}{2}\left(1-\left(\frac{1}{8}+\frac{13}{18}\right)x+\cdots\right)$$
$$= \frac{3}{2}\left(1-\frac{9+52}{72}x+\cdots\right)$$
$$= \frac{3}{2}\left(1-\frac{61}{72}x+\cdots\right)$$
$$= \frac{3}{2}-\frac{3}{2}\left(\frac{61}{72}\right)x+\cdots$$
$$\approx \frac{3}{2}-\frac{61}{48}x \approx R.H.S$$

Hence proved.

(vii)

$$\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-x)^{1/3}} \approx 2-\frac{1}{12}x$$

Solution:

$$L.H.S = \frac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$= \frac{(4-x)^{\frac{1}{2}}}{(8-x)^{\frac{1}{3}}} + \frac{(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$= 1 + \frac{[4(1-\frac{1}{4}x)]^{\frac{1}{2}}}{[8(1-\frac{1}{8}x)]^{\frac{1}{3}}}$$

$$= 1 + \frac{[2^2(1-\frac{1}{4}x)]^{\frac{1}{2}}}{[2^3(1-\frac{1}{8}x)]^{\frac{1}{3}}}$$

$$= 1 + \frac{2+(1-\frac{1}{4}x)^{1/2}}{2+(1-\frac{1}{8}x)^{\frac{1}{3}}}$$

$$= 1 + (1-\frac{1}{4}x)^{\frac{1}{2}}(1-\frac{1}{8}x)^{-\frac{1}{3}}$$

$$= 1 + (1+\frac{1}{2}(-\frac{1}{4}x)+\cdots)(1+(-\frac{1}{3})(-\frac{1}{8}x)+\cdots)$$

$$= 1 + (1+\frac{1}{8}x+\cdots)(1+\frac{1}{24}x+\cdots)$$

$$= 1 + (1+\frac{1}{24}x-\frac{1}{8}x+\cdots)$$

$$= 1 + (1+(\frac{1}{24}-\frac{1}{8})x+\cdots)$$

$$= 1 + (1+\frac{(1-3)}{24}x+\cdots)$$

$$= 1 + 1 - \frac{2}{24}x+\cdots$$

$$\approx 2 - \frac{1}{12}x \approx R.H.S$$

Hence proved.

Question No.5 if x is so small that its cube and higher power can be neglected , then show that

$$(i)\ \sqrt{1-x-2x^2} \approx 1-\frac{1}{2}x-\frac{9}{8}x^2$$

Solution:

$$L.H.S = \sqrt{1-x-2x^2}$$

$$(1-x-2x^2)^{\frac{1}{2}}$$

$$= [1-(x+2x^2)]^{\frac{1}{2}}$$

$$= 1+\frac{1}{2}\{-(x+2x^2)\}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\{-(x+2x^2\}^2+\cdots$$

$$= 1-\frac{1}{2}x-x^2+\frac{1}{2}(\frac{1}{2})(-\frac{1}{2})((x+2x^2))^2+\cdots$$

$$= 1-\frac{1}{2}x-x^2-\frac{1}{8}(x^2+4x^3+4x^4)+\cdots$$

$$= 1-\frac{1}{2}x-x^2-\frac{1}{8}x^2-\cdots$$

$$= 1-\frac{1}{2}x-(1+\frac{1}{8})x^2+\cdots.$$

$$= 1-\frac{1}{2}x-\frac{9}{8}x^2 \approx R.H.S$$

Hence proved.

(ii)

$$\sqrt{\frac{1+x}{1-x}} \approx 1+x+\frac{1}{2}x^2$$

Solution:

$$L.H.S = \sqrt{\frac{1+x}{1-x}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$= \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$= \{1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\cdots\}\{1+(-\frac{1}{2}x+\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^2+\cdots)\}$$

$$= \{1+\frac{1}{2}x+\frac{1}{2}(\frac{1}{2})(-\frac{1}{2}x^2)+\cdots\}\{1+\frac{1}{2}x+\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^2+\cdots\}$$

$$= \{1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots\}\{1+\frac{1}{2}x+\frac{3}{8}x^2+\cdots\}$$

$$= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2+\cdots$$

$$= 1+(\frac{1}{2}+\frac{1}{2})x+(\frac{3}{8}+\frac{1}{4}-\frac{1}{8})x^2+\cdots$$

$$= 1+x+(\frac{3+2-1}{8})x^2+\cdots = 1+x+\frac{4}{8}x^2+\cdots$$

$$= 1+x+\frac{1}{2}x^2+\cdots \approx 1+x+\frac{1}{2}x^2+\cdots = R.H.S$$

Hence proved

Question No.6 if x is very nearly equal to 1, then prove that $Px^p - qx^q \approx (p-q)x^{p+q}$

Solution:

$$L.H.S = Px^p - qx^q$$

Let $x = 1+h$ where h is so small that its square and higher power can be neglected, so

$$L.H.S = p(1+h)^p - q(1+h)^q$$

$$= p\{1+ph+\cdots\} - q\{1+qh+\cdots\}$$

$$\{p+p^2h+\cdots\} - \{q+q^2h+\cdots\}$$

$$= (p-q)+(p^2h-q^2h)+\cdots$$

$$= (p-q)+(p-q)(p+q)h+\cdots$$

$$= (p-q)(p-q)(p+q)h+\cdots$$

$$= (p-q)\{1+(p+q)h+\cdots\}$$

$$\approx (p-q)\{1+(p+q)h\}$$

$$\cong (p-q)(1+h)^{p+q} \quad \because (1+x)^n = 1+nx$$

$$\approx (p-q)x^{p+q} \quad \because x = 1+h$$

$$\approx R.H.S$$

Hence proved.

Question No.7

If $p-q$ is small when compared with p and q , *show that*

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p(2n+1)q} \approx \frac{(p+q)^{\frac{1}{n}}}{2q}$$

Solution:

$$L.H.S = \frac{(2n+1)p+(2n-1)q}{(2n-1)p(2n+1)q}$$

Let $p-q=h \Rightarrow p=q+h$ where h is so small that its squares and higher powers can be neglected, so

$$L.H.S = \frac{(2n+1)p+(2n-1)q}{(2n-1)p(2n+1)q}$$

$$= \frac{(2n+1)q+(2n+1)h+(2n-1)q}{(2n-1)q+(2n-1)h+(2n+1)q}$$

$$= \frac{(2n+!+2n-1)q+(2n+!)h}{(2n-1+2n+1)q+(2n-1)h}$$

$$= \frac{4nq+(2n+1)h}{4nq+(2n-1)h}$$

$$= \frac{4nq\{1+\frac{(2n+1)}{4nq}h\}}{4nq\{1+\frac{(2n-1)}{4nq}h\}}$$

$$= \{1+(\frac{2n+1}{4nq})h\}\{1+(\frac{2n-1}{4nq})h\}^{-1}$$

$$= \{1+(\frac{(2n-1)}{2nq})h\}\{1-(\frac{2n-1}{4nq})h\}$$

$$= 1-(\frac{2n-1}{4nqq})h(\frac{2n+1}{4nq})h+\cdots$$

$$= 1+\frac{h}{4nq}\{-2n+1+2n+1\}$$

$$= 1+\frac{2h}{4nq}-\cdots = 1+\frac{h}{2nq}-\cdots$$

$$= 1+\frac{1}{h}(\frac{h}{2q})-\cdots \approx \{1+\frac{h}{2q}\}^{\frac{1}{n}}$$

$$\approx \{1+\frac{p-q}{2q}\}-\cdots\{1+\frac{h}{2q}\}-\cdots$$

$$\approx \{1+\frac{p-q}{2q}\}^{\frac{1}{n}} \because h=p-q$$

$$\approx \{\frac{2q++p-q}{2q}\} \approx (\frac{p+q}{2q})^{\frac{1}{h}}$$

$$\approx R.H.S$$

Hence proved

Question No.8

Show that

$$[\frac{n}{2(n+N)}]^{\frac{1}{2}} \approx \frac{8m}{9n-N} - \frac{n+N}{4n}$$

Where n and N are nearly equal.

Solution:

$$L.H.S = [\frac{n}{2(n+N)}]^{\frac{1}{2}}$$

$$= [\frac{n}{2(n+n+h}]^{\frac{1}{2}} = [\frac{n}{2(2n+h)}]^{\frac{1}{2}}$$

$$= [\frac{n}{4n+2h}]^{\frac{1}{2}} = [\frac{n}{4n(1+\frac{2h}{4n})}]^{\frac{1}{2}}$$

$$= [\frac{n}{2^2(1+\frac{2h}{4n})}]^{\frac{1}{2}}$$

$$= \frac{1}{2(1+\frac{h}{2n})^{\frac{1}{2}}} = \frac{1}{2}(1+\frac{hn}{2})^{-\frac{1}{2}}$$

$$= \frac{1}{2}\{1+(-\frac{1}{2})(\frac{h}{2n})+\cdots\}$$

$$= \frac{1}{2}\{1-\frac{h}{4n}+\cdots\}$$

$$= \frac{1}{2}-\frac{h}{8n}+\cdots$$

$$\approx \frac{1}{2}-\frac{h}{2} \rightarrow (i)$$

$$R.H.S = \frac{8m}{9n-N} - \frac{n+nN}{4}$$

$$\because N = n+h$$

$$= \frac{8n}{9n-(n+h)} - \frac{n+n+h}{4n}$$

$$= \frac{8n}{9n-n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n(1-\frac{h}{8n})} - \frac{2n(1+\frac{h}{2n})}{4n}$$

$$= \frac{1}{(1-\frac{h}{8n})} - \frac{1+\frac{h}{2n}}{2}$$

$$= (1-\frac{h}{8n})^{-1} - \frac{1}{2}(1+\frac{h}{2n})$$

$$= -\frac{1}{2}-\frac{h}{4n}+1+(-1)(\frac{h}{8n})+\cdots$$

$$= -\frac{1}{2}-\frac{h}{4n}+1+\frac{h}{8n}+\cdots$$

$$= (1-\frac{1}{2})+(\frac{1}{8}-\frac{1}{4})\frac{h}{n}+\cdots$$

$$= \frac{1}{2}+(\frac{1-2}{8})\frac{h}{n}+\cdots$$

$$\approx \frac{1}{2}-\frac{1}{8}\frac{h}{n} \approx \frac{1}{2}-\frac{h}{8n} \rightarrow (ii)$$

By (i) and (ii)

$$L.H.S = R.H.S$$

hence proved

Question No.9 identify the flowing series as binomial expansion and find the sum in each case.

(i) $1-\frac{1}{2}(\frac{1}{4})+\frac{1.3}{2!4}(\frac{1}{4})^2-\frac{1.3.5}{3!8}(\frac{1}{4})^3+\cdots$

Solution: consider

$y = 1-\frac{1}{2}(\frac{1}{4})+\frac{1.3}{2!4}(\frac{1}{4})^2-\frac{1.3.5}{3!8}(\frac{1}{4})^3+\cdots$

Let the series be identity with

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots . \to (ii)$

By comparing (i) and (ii)

$$y=(1+x)^n \to (iii)$$

Now $nx = -\frac{1}{2}(\frac{1}{4}) \Rightarrow nx = -\frac{1}{8}$

$$\Rightarrow x = -\frac{1}{8n}$$

And $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!4}(\frac{1}{4})^2$

$$\Rightarrow \frac{n(n-1)}{2!}(-\frac{1}{8n})^2 = \frac{1.3}{2!\,4}(\frac{1}{4})^2 \quad \because x = -\frac{1}{8n}$$

$$\Rightarrow \frac{n(n-1)}{2}.\frac{1}{64n^2} = \frac{3}{128}$$

$$\Rightarrow n(n-1).\frac{1}{64n^2} = \frac{3}{64}$$

$$\Rightarrow \frac{n-1}{n} = 3$$

$$\Rightarrow n-1 = 3n \Rightarrow -1 = 3n-n$$

$$\Rightarrow -1 = 2n \Rightarrow n = -\frac{1}{2}$$

$$so\ \ x = -\frac{1}{8(-\frac{1}{2})} \Rightarrow x = \frac{1}{4}$$

Putting valves of x and n in (iii)

$$y = (1+\frac{1}{4})^{-\frac{1}{2}}$$

$$= (\frac{5}{4})^{-\frac{1}{2}} = (\frac{4}{5})^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$$

$$\Rightarrow Sum\ of\ series\ is\ \frac{2}{\sqrt{5}}$$

(ii)

$$1-\frac{1}{2}(\frac{1}{2})+\frac{1.3}{2.4}(\frac{1}{2})^2-\frac{1.3.5}{2.4.6}(\frac{1}{2})^3+\cdots$$

Solution:

Consider

$$y = 1-\frac{1}{2}(\frac{1}{2})+\frac{1.3}{2.4}(\frac{1}{2})^2-\frac{1.3.5}{2.4.6}(\frac{1}{2})^3+\cdots \to (i)$$

Let the series be identity with

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots \to (ii)$$

$$by\ comparing\ (i) and\ (ii)$$

$$y = (1+x)^n \to (iii)$$

Now

$$nx = -\frac{1}{2}(\frac{1}{2}) \Rightarrow nx = -\frac{1}{4} \Rightarrow x = -\frac{1}{4n}$$

And $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2.4}(\frac{1}{2})^2$

$$\Rightarrow \frac{n(n-1)}{2}(-\frac{1}{4n})^2 = \frac{1.3}{2.4}(\frac{1}{2})^2$$

$$\Rightarrow \frac{n(n-1)}{2}(-\frac{1}{4n})^2 = \frac{3}{8}(\frac{1}{4})$$

$$n(n-1).\frac{1}{16n^2} = \frac{3}{16}$$

$$\Rightarrow \frac{n-1}{n} = 3 \Rightarrow n-1 = 3n$$

$$\Rightarrow n-3n = 1 \Rightarrow -2n = 1$$

$$\Rightarrow n = -\frac{1}{2} \quad so\ x = \frac{-1}{4(-\frac{1}{2})} = \frac{1}{2}$$

Putting valves of x and n $in(iii)$

$$y = (1+\frac{1}{2})^{-\frac{1}{2}} = (\frac{3}{2})^{-\frac{1}{2}} = (\frac{2}{3})^{\frac{1}{2}}$$

$$y = \sqrt{\frac{2}{3}}$$

$$\Rightarrow sum\ of\ series\ is\ \sqrt{\frac{2}{3}}$$

(iii)

$$1+\frac{3}{4}+\frac{3\ 5}{4\cdot 8}+\frac{3.5.7}{4.8.12}+\cdots$$

Solution:

Consider

$$1+\frac{3}{4}+\frac{3\ 5}{4\cdot 8}+\frac{3.5.7}{4.8.12}+\cdots$$

Let the series be identical with

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots \to (ii)$$

$$by\ comparing\ (i) and\ (ii)$$

$$y = (1+x)^n$$

$$\Rightarrow nx = \frac{3}{4} \Rightarrow x = \frac{3}{4n}$$

And $\frac{n(n-1)}{2!}x^2 = \frac{3.5}{4.8}$

$$\Rightarrow \frac{n(n-1)}{2}(\frac{3}{4n})^2 = \frac{15}{32}$$

$$\Rightarrow n(n-1).\frac{9}{16n^2} = \frac{15}{16}$$

$$\Rightarrow \frac{n-1}{n} = \frac{15}{9}$$

$$or\ \ 9n-9 = 15n \Rightarrow -9 = 15n-9n$$

$$\Rightarrow -9 = 6n$$

$$\Rightarrow n = -\frac{9}{6} \Rightarrow n = -\frac{3}{2}$$

$$so \quad x = \frac{3}{4(-\frac{3}{2})} = -\frac{1}{2}$$

Putting valves of x and n in (iii)

$$y = (1-\frac{1}{2})^{-3/2}$$

$$= (\frac{1}{2})^{-\frac{3}{2}}$$

$$= (2)^{3/2} = [2^{\frac{1}{2}}]^3 = (\sqrt{2})^3$$

$$\Rightarrow y = (\sqrt{2})^3$$

$$\Rightarrow Sum\ of\ series\ is\ (\sqrt{2})^3$$

(iv) $1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3$

Solution:

Consider

$$y = 1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\cdots \to (i)$$

Let the series be identical with

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots \to (ii)$$

By comparing (i) and (ii)

$$y=(1+x)^n \to (ii)$$

Now

$$nx=-\frac{1}{2}.\frac{1}{3} \Rightarrow nx=-\frac{1}{6} \Rightarrow x=-\frac{1}{6n}$$

And $\frac{n(n-1)}{2!}x^2=\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2$

$$\Rightarrow \frac{n(n-1)}{2!}x^2\left(-\frac{1}{6n}\right)^2=\frac{3}{8}\left(\frac{1}{9}\right)$$

$$n(n-1)\left(\frac{\frac{1}{n}}{36^2}\right)=\frac{3}{36}$$

$$\Rightarrow \frac{n-1}{n}=\frac{1}{12}\times 36 \Rightarrow \frac{n-1}{n}=3$$

$$\Rightarrow n-1=3n \Rightarrow n-3n=1$$

$$\Rightarrow -2n=1 \Rightarrow n=-\frac{1}{2}$$

So $x=-\frac{1}{6\left(-\frac{1}{2}\right)}$

$$\Rightarrow x=\frac{1}{3}$$

Putting valves of x and n in (iii)

$$y=\left(1+\frac{1}{3}\right)^{-\frac{1}{2}}$$

$$=\left(\frac{4}{3}\right)^{-\frac{1}{2}}$$

$$=\left(\frac{3}{4}\right)^{\frac{1}{2}}$$

$$y=\frac{\sqrt{3}}{2}$$

$$\Rightarrow sum\ of\ series\ is\ \frac{\sqrt{3}}{2}$$

Question No.10

Uses binomial theorem to show that

$$\mathbf{1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots=\sqrt{2}}$$

Solution:

$$y=1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots=\sqrt{2}$$

Solution:

$$y=1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots \to (i)$$

Let *the series be identical with*

$$(1+x)^n+1+nx+\frac{n(n-1)}{2!}x^2+\cdots \to (iii)$$

By comparing (i) and (ii)

$$y=(1+x)^n \to (iii)$$

Now $nx=\frac{1}{4} \Rightarrow x=\frac{1}{4n}$

And $\frac{n(n-1)}{2!}x^2=\frac{1.3}{4.8}$

$$\Rightarrow \frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2=\frac{3}{32}$$

$$\Rightarrow \frac{n(n-1)}{2}.\frac{1}{16n^2}=\frac{3}{32}$$

$$\frac{n-1}{n}=3 \Rightarrow n-1=3n$$

$$-1-3n-n \Rightarrow -1=2n$$

$$\Rightarrow n=-\frac{1}{2}\quad so\quad x=\frac{1}{4\left(-\frac{1}{2}\right)}=-\frac{1}{4\left(-\frac{1}{2}\right)}=-\frac{1}{2}$$

Putting valves of x and n in (iii)

$$y=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$y=\left(\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$\Rightarrow y=(2)^{\frac{1}{2}} \Rightarrow y=\sqrt{2}$$

Hence

$$\mathbf{1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots=\sqrt{2}}$$

Question No.11

If $y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots$

Then prove that $y^2+2y-2=0$

Solution:

Here

$$\mathbf{y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots \to (i)}$$

Adding (1) on both sides

$$\mathbf{1+y=1+\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots}$$

Let the series be identical

$$(1+x)^n+1+nx+\frac{n(n-1)}{2!}x^2+\cdots \to (iii)$$

By comparing (i) and (ii)

$$y=(1+x)^n \to (iii)$$

Now $nx=\frac{1}{3} \Rightarrow x=\frac{1}{3n}$

And $\frac{n(n-1)}{2!}x^2=\frac{1.3}{2!}\left(\frac{1}{3}\right)^2$

$$\Rightarrow n(n-1)x^2=\frac{1}{3}$$

$$\Rightarrow n(n-1)\left(\frac{1}{3n}\right)^2=\frac{1}{3}\quad \because x=\frac{1}{3n}$$

$$n(n-1).\frac{1}{9n^2}=\frac{1}{3}$$

$$\Rightarrow \frac{n-1}{n} = 3 \Rightarrow n-1 = 3n$$
$$\Rightarrow n - 3n = 1 \Rightarrow -2n = 1$$
$$\Rightarrow n = -\frac{1}{2} \quad so \quad x = \frac{1}{3\left(-\frac{1}{2}\right)} = -\frac{2}{3}$$

Putting valves of x and n in (iii)

$$1 + y = \left(1 - \frac{2}{3}\right)^{-\frac{1}{2}} \Rightarrow 1 + y = \left(\frac{1}{3}\right)^{-\frac{1}{2}}$$
$$\Rightarrow 1 + y = (3)^{\frac{1}{2}} \Rightarrow 1 + y = \sqrt{3}$$

Squaring both sides

$$(1+y)^2 = (\sqrt{3})^2 \Rightarrow 1 + y^2 + 2y = 3$$
$$\Rightarrow y^2 + 2y + 1 - 3 = 0$$
$$\Rightarrow y^2 + 2y - 2 = 0 \quad hence\ proved.$$

Question No.12

$$if \quad 2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!} \cdot \frac{1}{2^6} + \cdots$$

Prove that $4y^2 + 4y - 1 = 0$

Solution:

Here

$$2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!} \cdot \frac{1}{2^6} + \cdots \rightarrow (i)$$

Let the series be identical with

$$(1+x)^n + 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \rightarrow (iii)$$

By comparing (i) and (ii)

$$y = (1+x)^n \rightarrow (iii)$$

Now $nx = \frac{1}{2^2} \Rightarrow x = \frac{1}{4n}$

And $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!} = \frac{1}{2^4}$

$$\Rightarrow \quad n(n-1)\left(\frac{1}{4n}\right)^2 = \frac{3}{16}$$
$$n(n-1).\frac{1}{16n^2} = \frac{3}{16}$$
$$\frac{n-1}{n} = 3 \quad n-1 = 3n$$
$$\Rightarrow n - 3n = 1 \Rightarrow -2n = 1$$
$$\Rightarrow n = -\frac{1}{2} \quad so \quad x = \frac{1}{4\left(-\frac{1}{2}\right)}$$
$$\Rightarrow x = -\frac{1}{2}$$

Putting valves of x and n in (iii)

$$1 + 2y = \left(1 - \frac{1}{2}\right)^{\frac{1}{2}} \Rightarrow 1 + 2y = \left(\frac{1}{2}\right)^{-\frac{1}{2}}$$
$$\Rightarrow 1 + 2y = (2)^{\frac{1}{2}}$$
$$\Rightarrow 1 + 2y = \sqrt{2} \quad (squaring)$$
$$(1+2y)^2 = \sqrt{2}^2$$
$$\Rightarrow 1 + 4y^2 + 4y = 2$$
$$\Rightarrow 4y^2 + 4y + 1 - 2 = 0$$
$$\Rightarrow 4y^2 + 4y - 1 = 0$$

Hence proved.

Question No.13

$$\textbf{If} \quad y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \cdots$$

Show that $y^2 + 2y - 4 = 0$

Solution:

Here

$$y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \cdots$$

Adding 1 on both sides

$$1 + y = 1 + \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \cdots \rightarrow (i)$$

By comparing (i) and (ii)

$$1 + y = (1+x)^n \rightarrow (iii)$$

$$nx = \frac{2}{5} \Rightarrow x = \frac{2}{5n} \quad and$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!} \cdot \left(\frac{2}{5}\right)^2 \Rightarrow n(n-1)\left(\frac{2}{5n}\right)^2 = \frac{12}{25}$$

$$\Rightarrow n(n-1).\frac{4}{25n^2} = \frac{12}{25} \Rightarrow \frac{n-1}{n} = 3$$

$$n - 1 = 3n \Rightarrow n - 1 = 3n - n = 2n$$

$$\Rightarrow n = -\frac{1}{2} \quad so\ x = \frac{2}{5\left(-\frac{1}{2}\right)} = -\frac{4}{5}$$

So (iii)

$$1 + y = \left(-\frac{4}{5}\right)^{-\frac{1}{2}} = \left(\frac{1}{5}\right)^{-\frac{1}{2}}$$

$$\Rightarrow 1 + y = \sqrt{5} \Rightarrow (1+y)^2 = 5$$
$$1 + 2y + y^2 = 5 \Rightarrow y^2 + 2y + 1 - 5 = 0$$
$$\Rightarrow y^2 + 2y - 4 = 0 \quad hence\ proved.$$