MATHEMATICS 11 INTERMEDIATE

PART 1

Bilal Article

Chapter 13.

INVERSE TRIGONOMETRIC FUNCTION

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Inverse Trigonometric Functions

The functions Sin^{-1}, Cos^{-1}, Tan^{-1}, $Cosec^{-1}$, Sec^{-1} *and* Cot^{-1} *are called inverse*

- Inverse trigonometric functions are not dingle valued functions.

Principal valued functions:

Since inverse trigonometric functions are not single valued functions , but they can be made single valued if we restricted their domains.

These functions are called principal valued functions denoted by

Sin^{-1}, Cos^{-1}, Tan^{-1}, $Cosec^{-1}, \dots e.t.c$

Remember that

Sin^{-1}, Cos^{-1}, Tan^{-1}, $Cosec^{-1}, \dots e.t.c$ are single valued functions. While $\sin^{-1}$, $\cos^{-1}$, $tan^{-1}, \dots$ *are* not Single valued functions.

The inverse sine function:

the inverse sine function is denoted by Sin^{-1} and defined as

$$y = Sin^{-1}x \ \ iff \ \ x = siny$$

Where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $or -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

and $x \in [-1,1]$ $or -1 \leq x \leq 1$

The inverse cosine function:

The inverse cosine functions is denoted by Cos^{-1} *and* Defined as;

$$y = Cos^{-1} \ iff \ \ x = cosy$$

Where

$$y \in [0, \pi] \ or \ 0 \leq y \leq \pi$$

And

$$x \in [-1,1] \ or -1 \leq x \leq 1$$

The inverse tangent function:

The inverse tangent function is denoted by Tan^{-1} *and is defined as;*

$$y = Tan^{-1}x \ \ iff \ \ x = Tany$$

Where

$$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ \ or -\frac{\pi}{2} < y < \frac{\pi}{2}$$

And

$$x \in (-\infty, \infty) \ or -\infty < x < \infty \ or \ x \in R$$

Note:

It must be remember that

$$\boldsymbol{sin^{-1}x \neq (sinx)^{-1}}$$
$$\boldsymbol{cos^{-1}x \neq (cosx)^{-1}}$$
$$\boldsymbol{tan^{-1}x \neq (tanx)^{-1}}$$

The inverse cosecant function:

The inverse cosecant function is denoted by $Cosec^{-1}$ *and defined as*

$$y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ \ but \ \ y \neq o$$

And

$$x \in [-1,1] \ or -1 \leq x \leq 1$$

The inverse secant function:

The inverse secant function is denoted by Sec^{-1} *and defined as*

$$y = \sec^{-1}x \ \ iff \ \ x = secy$$

Where

$$y \in [0, \pi] \ but \ y \neq \frac{\pi}{2}$$

And

$$x \in [-1,1] \ \ or \ \ -1 \leq x \leq 1$$

The inverse cotangent functions:

the inverse cotangent function is denoted by co^{-1} *and defined as* $y = \cot^{-1}x \ \ iff \ \ x = coty$

where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $but \ y \neq o$

and

$$x \in \ (-\infty, \infty) \ or \ \ x \in R$$

Domains and Ranges of principal Trigonometric functions and inverse Trigonometric functions.

$$\boldsymbol{y = sinx}$$

$$domain; \ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$Range \ ; -1 \leq x < 1$$

$$\boldsymbol{y = \sin^{-1} x}$$

Domain; $-1 \leq x \leq 1$

Range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$\boldsymbol{y = cosx}$$

$$Domain; o \leq x \leq \pi$$

$$Range; \ -1 \leq x \leq 1$$

$$\boldsymbol{y = \cos^{-1} x}$$

$$Domain; \ -1 \leq x \leq 1$$

$$Range \ ; \ \ 0 \leq x \leq \pi$$

$$\boldsymbol{y = tanx}$$

$$Domain; \ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$Range \ ; \ \ (-\infty, \infty) \ or \ R$$

$$\boldsymbol{y = \tan^{-1} x}$$

$$Domain\ \ (-\infty,\infty)\ or\ R$$
$$Range\ -\frac{\pi}{2}<x<\frac{\pi}{2}$$
$$\boldsymbol{y=\cot^{-1}x}$$
$$Domain;\ \ (\infty,\infty)\ or\ R$$
$$Range\ \ 0<x<\pi$$
$$\boldsymbol{y=cotx}$$
$$Domain;\ \ o<x<\pi$$
$$Rnage;\ \ o<x<\pi$$
$$\boldsymbol{y=secx}$$
$$Domain;\ \ [0,\pi],x\neq\frac{\pi}{2}$$
$$Range;\ \ y\leq-1\ \ or\ \ y\geq1$$
$$\boldsymbol{y=\sec^{-1}x}$$
$$Domain;\ \ x\geq-1\ \ or\ x\leq1$$
$$Range\ \ y\leq-1\ or\ y\geq1$$
$$\boldsymbol{y=cscx}$$
$$Domain\ ;\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right],x\neq o$$
$$Range;[0,\pi],y\neq1$$
$$\boldsymbol{y=\csc^{-1}x}$$
$$Domain;\ \ x\leq-1\ \ or\ x>-1$$
$$Rnage;\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right],\qquad y\neq0$$

Exercise 13.1

Question # 1. Evaluate without using calculator

(i). $\mathbf{\sin^{-1}(1)}$

Solution.

Suppose $y=\sin^{-1}(1)$, $where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow Siny=1$

$\Rightarrow y=\frac{\pi}{2}$ $\quad ,Since\ \sin\left(\frac{\pi}{2}\right)=1$

$\Rightarrow \sin^{-1}(1)=1$

Which is required.

(ii). $\mathbf{\sin^{-1}(-1)}$

Solution.

Suppose $y=\sin^{-1}(-1)$, $where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow Siny=-1$

$\Rightarrow y=-\frac{\pi}{2}$ $\quad ,Since\ \sin\left(-\frac{\pi}{2}\right)=-1$

$\Rightarrow \sin^{-1}(-1)=-\frac{\pi}{2}$

Which is required.

(iii). $\mathbf{Cos^{-1}(\frac{\sqrt{3}}{2})}$

Solution.

Suppose $y=\text{Cos}^{-1}(\frac{\sqrt{3}}{2})$ $\quad ,where\ y\in[0,\pi]$

$\Rightarrow Cosy=\frac{\sqrt{3}}{2}$

$\Rightarrow y=\frac{\pi}{6}$ $\quad ,Since\ \text{Cos}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

$\Rightarrow \text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

Which is required.

(iii). $\mathbf{Tan^{-1}(-\frac{1}{\sqrt{3}})}$

Solution

Suppose $y=\text{Tan}^{-1}(-\frac{1}{\sqrt{3}})$ $\quad ,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow \tan y=-\frac{1}{\sqrt{3}}$

$\Rightarrow y=-\frac{\pi}{6}$ $\quad ,Since\ \tan\left(-\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}$

$\Rightarrow \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

Which is required.

(v). $\mathbf{Cos^{-1}(\frac{1}{2})}$

Solution.

Suppose $y=\text{Cos}^{-1}(\frac{1}{2})$ $\quad ,where\ y\in[0,\pi]$

$\Rightarrow Cosy=\frac{1}{2}$

$\Rightarrow y=\frac{\pi}{3}$ $\quad ,Since\ \text{Cos}\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\Rightarrow \text{Cos}^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

Which is required.

(vi). $\text{Tan}^{-1}(\frac{1}{\sqrt{3}})$

Solution.

Suppose $y = \text{Tan}^{-1}(\frac{1}{\sqrt{3}})$ $\quad$ $, where\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \tan y = \frac{1}{\sqrt{3}}$

$\Rightarrow y = \frac{\pi}{6}$ $\quad$ $, Since\ \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Which is required.

(vii). $\text{Cot}^{-1}(-1)$

Solution.

Suppose $y = \text{Cot}^{-1}(-1)$ $\quad$ $, where\ y \in [0, \pi]$

$\Rightarrow Cosy = -1$

$\Rightarrow y = \frac{3\pi}{4}$ $\quad$ $, Since\ \text{Cot}\left(\frac{3\pi}{4}\right) = -1$

$\Rightarrow \text{Cot}^{-1}(-1) = \frac{3\pi}{4}$

Which is required.

(viii). $\text{cosec}^{-1}(-\frac{2}{\sqrt{3}})$

Solution.

Suppose $y = \text{cosec}^{-1}(-\frac{2}{\sqrt{3}})$ $\quad$ $, where\ y \in$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $, y \neq 0$

$\Rightarrow \text{cosec}\, y = -\frac{2}{\sqrt{3}}$

$\Rightarrow Sin\ y = -\frac{\sqrt{3}}{2}$

$\Rightarrow y = -\frac{\pi}{3}$ $\quad$ $, Since\ \text{Sin}\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow \text{Cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$

Which is required.

(ii). $\sin^{-1}(-\frac{1}{\sqrt{2}})$

Solution

Suppose $y = \sin^{-1}(-\frac{1}{\sqrt{2}})$ $\quad$ $, where\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow Siny = -\frac{1}{\sqrt{2}}$

$\Rightarrow y = -\frac{\pi}{4}$ $\quad$ $, Since\ \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

Which is required.

Question # 2 . Without using calculator Show that

(i). $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

Solution.

Suppose $\alpha = \sin^{-1}\frac{5}{13} - -(1)\ where\ \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$Sin\alpha = \frac{5}{13}$

Now $\cos\alpha = \pm\sqrt{1 - Sin^2\alpha}$

Since $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ there cos is positive

$\cos\alpha = +\sqrt{1 - Sin^2\alpha}$

$\cos\alpha = +\sqrt{1 - \left(\frac{5}{13}\right)^2}$

$\cos\alpha = +\sqrt{1 - \frac{25}{169}}$

$\cos\alpha = +\sqrt{\frac{144}{169}}$

$\cos\alpha = \frac{12}{13}$

Now $\quad \tan\alpha = \frac{Sin\alpha}{cos\alpha}$

$\tan\alpha = \frac{\frac{5}{13}}{\frac{12}{13}}$

$\tan\alpha = \frac{5}{12}$

$\alpha = \tan^{-1}\frac{5}{12}$

$\sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$,using (1).

Hence Proved.

(ii). $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

Solution.

Suppose $\alpha = 2\cos^{-1}\frac{4}{5} - - - (1)\ Where\ \frac{\alpha}{2} \in [0, \pi]$

$\frac{\alpha}{2} = \cos^{-1}\frac{4}{5}$

$Cos\ \frac{\alpha}{2} = \frac{4}{5}$

Now $Sin\frac{\alpha}{2} = \pm\sqrt{1 - Cos^2\frac{\alpha}{2}}$

Since $\frac{\alpha}{2} \in [0, \pi]$ therefore sin is positive.

$Sin\frac{\alpha}{2} = \sqrt{1 - Cos^2\frac{\alpha}{2}}$

$Sin\frac{\alpha}{2} = \sqrt{1 - \left(\frac{4}{5}\right)^2}$

$Sin\frac{\alpha}{2} = \sqrt{1 - \frac{16}{25}}$

$Sin\frac{\alpha}{2} = \sqrt{\frac{9}{25}}$

$Sin\frac{\alpha}{2} = \frac{3}{5}$

Now $Sin\alpha = 2Sin\frac{\alpha}{2}\ Cos\ \frac{\alpha}{2}$

$Sin\alpha = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

$Sin\alpha = \frac{24}{25}$

$\alpha = \sin^{-1}\frac{24}{25}$

$2\cos^{-1}\frac{4}{5}=\sin^{-1}\frac{24}{25}$, using (1)

Hence Proved.

(iii). $\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$

Solution.

Suppose $\alpha=\cos^{-1}\frac{4}{5}---(1)\ Where\ \alpha\in[0,\pi]$

$\cos\alpha=\frac{4}{5}$

Now $Sin\alpha=\pm\sqrt{1-Cos^2\alpha}$

Since $\alpha\in[0,\pi]$ therefore sin is positive.

$Sin\alpha=\sqrt{1-Cos^2\alpha}$

$Sin\alpha=\sqrt{1-\left(\frac{4}{5}\right)^2}$

$Sin\alpha=\sqrt{1-\frac{16}{25}}$

$Sin\alpha=\sqrt{\frac{9}{25}}$

$Sin\alpha=\frac{3}{5}$

Now $\cot\alpha=\frac{\cos\alpha}{\sin\alpha}=\frac{\frac{4}{5}}{\frac{3}{4}}=\frac{4}{3}$

$\Rightarrow\alpha=\cot^{-1}\frac{4}{3}----(2)$

From (1) and (2) , we have

$\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$

Hence Proved.

Question # 3 . Find the value of each expression.

(i). $Cos(\sin^{-1}\frac{1}{\sqrt{2}})$

Solution.

Suppose $y=\sin^{-1}(\frac{1}{\sqrt{2}})$ $,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow Siny=\frac{1}{\sqrt{2}}$

$\Rightarrow y=\frac{\pi}{4}$ $,Since\ \sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$

Now $Cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=Cos(\frac{\pi}{4})=\frac{1}{\sqrt{2}}.$

Which is required.

(ii). $Sec\ (\cos^{-1}\frac{1}{2})$

Solution.

Suppose $y=Cos^{-1}(\frac{1}{2})$ $,where\ y\in[0,\pi]$

$\Rightarrow Cosy=\frac{1}{2}$

$\Rightarrow y=\frac{\pi}{3}$ $,Since\ Cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\Rightarrow Cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

Now $Sec\left(\cos^{-1}\frac{1}{2}\right)=Sec\left(\frac{\pi}{3}\right)=\frac{1}{\cos\frac{\pi}{3}}=\frac{1}{\frac{1}{2}}=2.$

Which is required.

(iii). $\tan(\cos^{-1}\frac{\sqrt{3}}{2})$

Solution.

Suppose $y=Cos^{-1}(\frac{\sqrt{3}}{2})$ $,where\ y\in[0,\pi]$

$\Rightarrow Cosy=\frac{\sqrt{3}}{2}$

$\Rightarrow y=\frac{\pi}{6}$ $,Since\ Cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

$\Rightarrow Cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

Now $\tan(\cos^{-1}\frac{\sqrt{3}}{2})=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$

Which is required.

(iv). $Csc(\tan^{-1}(-1))$

Solution.

Suppose $y=Tan^{-1}(-1)$ $,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow\tan y=-1$

$\Rightarrow y=-\frac{\pi}{4}$ $,Since\ \tan\left(-\frac{\pi}{4}\right)=-1$

$\Rightarrow\tan^{-1}(-1)=-\frac{\pi}{4}$

Now $Csc(\tan^{-1}(-1))=Csc\left(-\frac{\pi}{4}\right)=\frac{1}{Sin\left(-\frac{\pi}{4}\right)}=-\sqrt{2}$

Which is required.

(v). $Sec(\sin^{-1}\left(-\frac{1}{2}\right)$

Solution.

Suppose $y=\sin^{-1}(-\frac{1}{2})$ $,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow Siny=-\frac{1}{2}$

$\Rightarrow y=-\frac{\pi}{6}$ $,Since\ \sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

$\Rightarrow\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

Now $Sec(\sin^{-1}\left(-\frac{1}{2}\right)=Sec\left(-\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$

Which is required.

(vi). $\tan(\tan^{-1}(-1))$

Solution.

Suppose $y=Tan^{-1}(-1)$ $,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow\tan y=-1$

$\Rightarrow y=-\frac{\pi}{4}$ $,Since\ \tan\left(-\frac{\pi}{4}\right)=-1$

$\Rightarrow\tan^{-1}(-1)=-\frac{\pi}{4}$

$\tan(\tan^{-1}(-1))=\tan\left(-\frac{\pi}{4}\right)=-1.$

Which is required.

(vii). $Sin(\sin^{-1}(\frac{1}{2}))$

Solution.

Suppose $y=\sin^{-1}(\frac{1}{2})$ $,where\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow Siny=\frac{1}{2}$

$\Rightarrow y = \frac{\pi}{6} \quad , Since\ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Now $Sin(\sin^{-1}(\frac{1}{2})) = Sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Which is required.

(viii). $\mathbf{tan(sin^{-1}(-\frac{1}{2}))}$

Solution.

Suppose $y = \sin^{-1}(-\frac{1}{2}) \quad , where\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow Siny = -\frac{1}{2}$

$\Rightarrow y = -\frac{\pi}{6} \quad , Since\ \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$\Rightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Now $\tan(\sin^{-1}(-\frac{1}{2})) = \tan(-\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}$.

Which is required.

(ix). $\mathbf{Sin(tan^{-1}(-1))}$

Solution.

Suppose $y = \mathrm{Tan}^{-1}(-1) \quad , where\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \tan y = -1$

$\Rightarrow y = -\frac{\pi}{4} \quad , Since\ \tan\left(-\frac{\pi}{4}\right) = -1$

$\Rightarrow \tan^{-1}(-1) = -\frac{\pi}{4}$

Now $\mathrm{Sin}(\tan^{-1}(-1)) = \mathrm{Sin}\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Which is required.

Addition and subtraction formulas

(1) prove that

$$\sin^{1} A + \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$$

Proof:

Let $x = \sin^{-1} A, \quad y = \sin^{-1} B$

$$\Rightarrow sinx = \frac{A}{1} = A, \quad siny = \frac{B}{1} = B$$

$$a^2 + A^2 = 1 \Rightarrow a^2 = 1 - A^2$$

$$\Rightarrow a = \sqrt{1-A^2}\ \ by\ pathagoras$$

$$b^2 + B^2 = 1 \Rightarrow b^2 = 1 - B^2$$

$$\Rightarrow b = \sqrt{1-B^2}\ \ by\ pathagoras$$

So

$$cosx = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$$

$$cosy = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$$

Now

$$\sin(x+y) = sinxcosy + cosxsiny$$

$$x + y = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$$

2)

Prove that

$$\mathbf{\sin^{1} A - \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)}$$

Proof:

Let $x = \sin^{-1} A, \quad y = \sin^{-1} B$

$$\Rightarrow sinx = \frac{A}{1} = A, \quad siny = \frac{B}{1} = B$$

$$a^2 + A^2 = 1 \Rightarrow a^2 = 1 - A^2$$

$$\Rightarrow a = \sqrt{1-A^2}\ \ by\ pathagoras$$

$$b^2 + B^2 = 1 \Rightarrow b^2 = 1 - B^2$$

$$\Rightarrow b = \sqrt{1-B^2}\ \ by\ pathagoras$$

So

$$cosx = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$$

$$cosy = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$$

Now

$$\sin(x-y) = sinxcosy - cosxsiny$$

$$x - y = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$$

(3) prove that

$$\cos^{1} A + \cos^{-1} B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$$

Proof:

Let $x = \cos^{-1} A, \quad y = \cos^{-1} B$

$$\Rightarrow cosx = \frac{A}{1} = A, \quad cosy = \frac{B}{1} = B$$

$$a^2 + A^2 = 1 \Rightarrow a^2 = 1 - A^2$$

$$\Rightarrow a = \sqrt{1-A^2}\ \ by\ pathagoras$$

$$b^2 + B^2 = 1 \Rightarrow b^2 = 1 - B^2$$

$$\Rightarrow b = \sqrt{1-B^2}\ \ by\ pathagoras$$

So

$$sinx = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$$

$$siny = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$$

Now

$$\cos(x+y) = cosxcosy - sinxsiny$$

$$x + y = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$$

$$\cos^{1} A + \cos^{-1} B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$$

(4) prove that

$$\cos^{1} A - \cos^{-1} B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$$

Proof:

Let $x = \cos^{-1} A, \quad y = \cos^{-1} B$

$$\Rightarrow cosx = \frac{A}{1} = A, \quad cosy = \frac{B}{1} = B$$

$$a^2 + A^2 = 1 \Rightarrow a^2 = 1 - A^2$$

$$\Rightarrow a = \sqrt{1 - A^2} \text{ by pathagoras}$$

$$b^2 + B^2 = 1 \Rightarrow b^2 = 1 - B^2$$

$$\Rightarrow b = \sqrt{1 - B^2} \text{ by pathagoras}$$

So

$$sinx = \frac{\sqrt{1 - A^2}}{1} = \sqrt{1 - A^2}$$

$$siny = \frac{\sqrt{1 - B^2}}{1} = \sqrt{1 - B^2}$$

Now

$$\because \cos(x - y) = cosxcosy + sinxsiny$$

$$x + y = \cos^{-1}\left(AB + \sqrt{(1 - A^2)(1 - B^2)}\right)$$

$$\cos^1 A + \cos^{-1} B = \cos^{-1}\left(AB + \sqrt{(1 - A^2)(1 - B^2)}\right)$$

(5) prove that

$$Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A + B}{1 - AB}$$

Proof:

Let $x = Tan^{-1}A, \quad y = Tan^{-1}B$

$$\Rightarrow \quad Tanx = A, \quad Tany = B$$

$$\because \quad Tan(x + y) = \frac{Tanx + Tany}{1 - TanxTany}$$

$$\Rightarrow x + y = Tan^{-1}\frac{A + B}{1 - AB} \quad \text{hence proved}$$

(6) prove that

$$Tan^{-1}A - Tan^{-1}B = Tan^{-1}\frac{A - B}{1 + AB}$$

Proof:

Let $x = Tan^{-1}A, \quad y = Tan^{-1}B$

$$\Rightarrow \quad Tanx = A, \quad Tany = B$$

$$\because \quad Tan(x - y) = \frac{Tanx - Tany}{1 + TanxTany}$$

$$\Rightarrow x + y = Tan^{-1}\frac{A - B}{1 + AB} \quad \text{hence proved}$$

Hence

$$Tan^{-1}A - Tan^{-1}B = Tan^{-1}\frac{A - B}{1 + AB}$$

(7) prove that

$$2Tan^{-1}A = Tan^{-1}\frac{2A}{1 - A^2}$$

Proof:

We know that

$$Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A + B}{1 - AB}$$

Put B=A

$$Tan^{-1}A + Tan^{-1}A = Tan^{-1}\frac{A + A}{1 - AA}$$

$$\Rightarrow 2Tan^{-1}A = Tan^{-1}\frac{2A}{1 - A^2}$$

Exercise 13.2

Question # 1 . ***Prove that***

$$\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25} = \cos^{-1}\frac{253}{325}$$

Solution.

$LL.H.S = \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25}$

$$\because \sin^{-1} A + \sin^{-1} B = \sin^{-1}(A\sqrt{1 - B^2} + B\sqrt{1 - A^2})$$

$$L.H.S = \sin^{-1}\left(\frac{5}{13}\sqrt{1 - \left(\frac{7}{25}\right)^2} + \frac{7}{25}\sqrt{1 - \left(\frac{5}{13}\right)^2}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{5}{13}\sqrt{1 - \frac{49}{625}} + \frac{7}{25}\sqrt{1 - \frac{25}{169}}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{576}{625}} + \frac{7}{25}\sqrt{\frac{144}{169}}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{5}{13}\left(\frac{24}{25}\right) + \frac{7}{25}\left(\frac{12}{13}\right)\right)$$

$$L.H.S = \sin^{-1}\left(\frac{120}{325} + \frac{84}{325}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{204}{325}\right)$$

$$L.H.S = \frac{\pi}{2} - \cos^{-1}\left(\frac{204}{325}\right)$$

$$\because \sin^{-1} A = \frac{\pi}{2} - \cos^{-1} A$$

$$L.H.S = \cos^{-1}(0) - \cos^{-1}\left(\frac{204}{325}\right) \qquad \because \frac{\pi}{2}$$

$$= \cos^{-1}(0)$$

$$\because \cos^{-1} A - \cos^{-1} B$$

$$= \cos^{-1}\left(AB + \sqrt{(1 - A^2)(1 - B^2)}\right)$$

$$L.H.S = \cos^{-1}\left((0)\left(\frac{204}{325}\right) + \sqrt{(1 - (0)^2)\left(1 - \left(\frac{204}{325}\right)^2\right)}\right)$$

$$L.H.S = \cos^{-1}\left(0 + \sqrt{\frac{64009}{105625}}\right)$$

$$L.H.S = \cos^{-1}\left(\frac{253}{325}\right)$$

$$L.H.S = R.H.S$$

$$Hence\ Proved.$$

Question # 2.

Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\left(\frac{9}{19}\right)$

Solution.

$$\text{L.H.S} = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5}$$

$$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{1}{5}}{1-\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{9}{20}}{\frac{19}{20}}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{9}{19}\right)$$

$$\text{L.H.S} = \text{R.H.S}$$

$$Hence\ Proved.$$

Question # 3 .*Prove that* $2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$.

Solution. Suppose

$$\alpha = \sin^{-1}\frac{12}{13}$$

$$\sin\alpha = \frac{12}{13}$$

Now $\cos\alpha = \sqrt{1 - Sin^2\alpha}$

$$\cos\alpha = \sqrt{1 - \left(\frac{12}{13}\right)^2}$$

$$\cos\alpha = \sqrt{1 - \frac{144}{169}}$$

$$\cos\alpha = \sqrt{\frac{25}{169}}$$

$$\cos\alpha = \frac{5}{13}$$

Now $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\frac{5}{13}}{1+\frac{5}{13}}}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{\frac{8}{13}}{\frac{18}{13}}}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{8}{18}}$$

$$\tan\frac{\alpha}{2} = \frac{2}{3}$$

$$\frac{\alpha}{2} = \tan^{-1}\frac{2}{3}$$

$$\alpha = 2\tan^{-1}\frac{2}{3}$$

$$\sin^{-1}\frac{12}{13} = 2\tan^{-1}\frac{2}{3}$$

$$Hence\ Proved.$$

Question # 4 . *Prove that* $\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$

Solution. Suppose

$$\alpha = \tan^{-1}\frac{120}{119} - - - (1)$$

$$\tan\alpha = \frac{120}{119}$$

Now $Sec\,\alpha = \sqrt{1 + \tan^2\alpha}$

$$Sec\,\alpha = \sqrt{1 + \left(\frac{120}{119}\right)^2}$$

$$Sec\,\alpha = \sqrt{1 + \frac{14400}{14161}}$$

$$Sec\,\alpha = \sqrt{\frac{28561}{14161}}$$

$$Sec\,\alpha = \frac{169}{119}$$

Now $Cos\alpha = \frac{1}{Sec\alpha} = \frac{119}{169}$

Now $\cos\frac{\alpha}{2} = \sqrt{\frac{1+Cos\alpha}{2}}$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\frac{119}{169}}{2}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{\frac{288}{169}}{2}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{144}{169}}$$

$$\cos\frac{\alpha}{2} = \frac{12}{13}$$

$$\frac{\alpha}{2} = \cos^{-1}\frac{12}{13}$$

$$\alpha = 2\cos^{-1}\frac{12}{13} - - - (2)$$

From (1) and (2) , We have

$$\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$$

Hence Proved.

Question # 5. *Prove that* $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$.

Solution.

Suppose $\alpha = Sin^{-1}\frac{1}{\sqrt{5}} - - - - - - (ii)$

$$Sin\alpha = \frac{1}{\sqrt{5}}$$

Now $Cos\alpha = \sqrt{1 - Sin^2\alpha} = \sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}$

$$Cos\alpha = \sqrt{1 - \frac{1}{5}} = \sqrt{\frac{5-1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

So $tan\alpha = \frac{SIn\alpha}{Cos\alpha} = \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{1}{2}$

$$\alpha = tan^{-1}\frac{1}{2} - - - - - - (ii)$$

From (i) *and* (ii), **we have**

$$Sin^{-1}\frac{1}{\sqrt{5}} = tan^{-1}\frac{1}{2}$$

Now $Cot^{-1}3 = \tan^{-1}\frac{1}{3}$ (Using $Cot^{-1}x = Tan^{-1}\frac{1}{x}$)

$$L.H.S = \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3$$

$$L.H.S = tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

Using $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$$L.H.S = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$L.H.S = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)$$

$$L.H.S = \tan^{-1}(1)$$

$$L.H.S = \frac{\pi}{4}$$

$$L.H.S = R.H.S$$

Hence Proved.

Question # 6 . *Prove that* $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{77}{85}$

Solution.

$$L.H.S = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$$

$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

$$L.H.S = \sin^{-1}\left(\frac{3}{5}\sqrt{1 - \left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1 - \left(\frac{3}{5}\right)^2}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{3}{5}\sqrt{1 - \frac{64}{289}} + \frac{8}{17}\sqrt{1 - \frac{9}{25}}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{225}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{3}{5}\left(\frac{15}{17}\right) + \frac{8}{17}\left(\frac{4}{5}\right)\right)$$

$$L.H.S = \sin^{-1}\left(\frac{45}{85} + \frac{32}{85}\right)$$

$$L.H.S = \sin^{-1}\left(\frac{77}{85}\right)$$

$$L.H.S = R.H.S$$

Hence Proved.

Question # 7 . *Prove that*

$$\sin^{-1}\frac{77}{85} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$$

Solution.

Take $\sin^{-1}\frac{77}{85} = x,\quad \sin^{-1}\frac{3}{5} = y$

$$\Rightarrow sinx = \frac{77}{85},\quad siny = \frac{3}{5}$$

$$cos^2\alpha = 1 - \sin^2\alpha$$

$$\cos^2 x = 1 - \left(\frac{77}{85}\right)^2 \quad , \quad \cos^2 y = 1 - \sin^2 y$$

$$\cos^2 x = 1 - \frac{5929}{7225} \quad , \quad \cos^2 y = 1 - \left(\frac{3}{5}\right)^2$$

$$\cos^2 x = \frac{7225 - 5929}{7225},$$

$$\cos^2 x = \frac{1296}{7225}$$

$$\Rightarrow cosx = \frac{36}{85} (\cos is + +ve\ in\ domin\ of\ sine)$$

⇨

$$\cos^2 y = 1 - \left(\frac{3}{5}\right)^2$$

$$= 1 - \frac{9}{25}$$

$$\cos^2 y = \frac{25-9}{25} = \frac{16}{25}$$

$$\cos y = \frac{4}{5} (\because \ \cos is + ve \ in \ domain \ of \ sine)$$

$$\because \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\Rightarrow x - y = \cos^{-1}\left(\left(\frac{36}{85}\right)\left(\frac{4}{5}\right) + \left(\frac{77}{85}\right)\left(\frac{3}{5}\right)\right)$$

$$= \cos^{-1}\left(\frac{44}{425} + \frac{231}{425}\right)$$

$$x - y = \cos^{-1}\frac{375}{425} = \cos^{-1}\frac{15}{17}$$

$$\mathbf{\sin^{-1}\frac{77}{85} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}}$$

Hence Proved.

Question # 8. Prove that

$$\boldsymbol{Cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = Sin^{-1}\frac{3}{5}}$$

Solution. Suppose $\alpha = Cos^{-1}\frac{63}{65} - - - - - - (i)$

$$\Rightarrow Cos\alpha = \frac{63}{65}$$

Now $Sin\alpha = \sqrt{1 - Cos^2\alpha} = \sqrt{1 - \left(\frac{63}{65}\right)^2} =$

$\sqrt{1 - \frac{3969}{4225}} = \sqrt{\frac{256}{4225}} = \frac{16}{65}$

$$\Rightarrow \alpha = Sin^{-1}\frac{16}{65} - - - - (ii)$$

So from equations (i) and (ii), we have

$$Cos^{-1}\frac{63}{65} = Sin^{-1}\frac{16}{65}$$

Now Suppose $\beta = \tan^{-1}\frac{1}{5} - - - - (iii)$

$$tan\beta = \frac{1}{5}$$

So $Sec\beta = \sqrt{1 + \tan^2\beta} = \sqrt{1 + \left(\frac{1}{5}\right)^2} = \sqrt{1 + \frac{1}{25}} =$

$\sqrt{\frac{26}{25}} = \frac{\sqrt{26}}{5}$

Since $Cos\beta = \frac{1}{Sec\beta} = \frac{1}{\frac{\sqrt{26}}{5}} = \frac{5}{\sqrt{26}}$

As $tan\beta = \frac{Sin\beta}{Cos\beta} \Rightarrow Sin\beta = tan\beta Cos\beta$

$$\Rightarrow Sin\beta = \left(\frac{1}{5}\right)\left(\frac{5}{\sqrt{26}}\right) = \frac{1}{\sqrt{26}}$$

$$\Rightarrow \beta = Sin^{-1}\frac{1}{\sqrt{26}} - - - - (iv)$$

From (iii) and (iv), we have

$$\tan^{-1}\frac{1}{5} = Sin^{-1}\frac{1}{\sqrt{26}}$$

Now $L.H.S = Cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5}$

$$L.H.S = Sin^{-1}\frac{16}{65} + 2Sin^{-1}\frac{1}{\sqrt{26}}$$

$$L.H.S = Sin^{-1}\frac{16}{65} + \left(Sin^{-1}\frac{1}{\sqrt{26}} + Sin^{-1}\frac{1}{\sqrt{26}}\right)$$

$$L.H.S = Sin^{-1}\frac{16}{65} + Sin^{-1}\left(\frac{1}{\sqrt{26}}\sqrt{1 - \left(\frac{1}{\sqrt{26}}\right)^2} + \frac{1}{\sqrt{26}}\sqrt{1 - \left(\frac{1}{\sqrt{26}}\right)^2}\right)$$

$$L.H.S = Sin^{-1}\frac{16}{65} + Sin^{-1}\left(\frac{2}{\sqrt{26}}\sqrt{1 - \frac{1}{26}}\right)$$

$$L.H.S = Sin^{-1}\frac{16}{65} + Sin^{-1}\left(\frac{2}{\sqrt{26}}\sqrt{\frac{25}{26}}\right)$$

$$L.H.S = Sin^{-1}\frac{16}{65} + Sin^{-1}\left(\frac{10}{26}\right)$$

$$L.H.S = Sin^{-1}\frac{16}{65} + Sin^{-1}\frac{5}{13}$$

$$L.H.S = Sin^{-1}\left(\frac{16}{65}\sqrt{1 - \left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1 - \left(\frac{16}{65}\right)^2}\right)$$

$$L.H.S = Sin^{-1}\left(\frac{16}{65}\sqrt{1 - \frac{25}{169}} + \frac{5}{13}\sqrt{1 - \frac{256}{5225}}\right)$$

$$L.H.S = Sin^{-1}\left(\frac{16}{65}\sqrt{\frac{144}{169}} + \frac{5}{13}\sqrt{\frac{3969}{4225}}\right)$$

$$L.H.S = Sin^{-1}\left(\frac{16}{65}\left(\frac{12}{13}\right) + \frac{5}{13}\left(\frac{63}{65}\right)\right)$$

$$L.H.S = Sin^{-1}\left(\frac{192}{845} + \frac{315}{845}\right)$$

$$L.H.S = Sin^{-1}\left(\frac{3}{5}\right) = R.H.S$$

Hence Proved.

Question # 9. Prove that $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

Solution.

$$L.H.S = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$$

$$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \left(\frac{3}{4}\right)\left(\frac{3}{5}\right)}\right) - \tan^{-1}\frac{8}{19}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{27}{20}}{1 - \frac{9}{20}}\right) - \tan^{-1}\frac{8}{19}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{27}{20}}{\frac{11}{20}}\right) - \tan^{-1}\frac{8}{19}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\frac{8}{19}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\left(\frac{27}{11}\right)\left(\frac{8}{19}\right)}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{425}{209}}{1+\frac{216}{209}}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{425}{209}}{\frac{425}{209}}\right)$$

$$\text{L.H.S} = \tan^{-1}(1)$$

$$\text{L.H.S} = \frac{\pi}{4} = R.H.S$$

Hence Proved.

Question # 10. Prove that

$$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65} = \frac{\pi}{2}$$

Solution.

$$L.H.S = Sin^{-1}\frac{4}{5}+Sin^{-1}\frac{5}{13}$$

$$L.H.S = Sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right)+\sin^{-1}\frac{16}{65}$$

$$L.H.S = Sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right)+\sin^{-1}\frac{16}{65}$$

$$L.H.S = Sin^{-1}\left(\frac{4}{5}\sqrt{\frac{144}{169}}+\frac{5}{13}\sqrt{\frac{9}{25}}\right)+\sin^{-1}\frac{16}{65}$$

$$L.H.S = Sin^{-1}\left(\frac{4}{5}\left(\frac{12}{13}\right)+\frac{5}{13}\left(\frac{3}{5}\right)\right)+\sin^{-1}\frac{16}{65}$$

$$L.H.S = Sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)\sin^{-1}\frac{16}{65}$$

$$L.H.S = Sin^{-1}\frac{63}{65}+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{1-\left(\frac{16}{25}\right)^2}+\frac{16}{25}\sqrt{1-\left(\frac{63}{65}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{1-\frac{256}{4225}}+\frac{16}{65}\sqrt{1-\frac{3969}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{\frac{3969}{4225}}+\frac{16}{65}\sqrt{\frac{256}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\left(\frac{63}{65}\right)+\frac{16}{65}\left(\frac{16}{65}\right)\right)$$

$$\sin^{-1}\left(\frac{3969}{4225}+\frac{256}{4225}\right)$$

$$= \sin^{-1}\left(\frac{4225}{4225}\right) = \sin^{-1}(1) = \frac{\pi}{2} = R.H.S\ proved$$

Hence Proved.

Question # 11.Prove that $\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$

Solution. $L.H.S = \tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}$

$$\because \tan^{-1}A+\tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{1}{11}+\frac{5}{6}}{1-\left(\frac{1}{11}\right)\left(\frac{5}{6}\right)}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{61}{66}}{1-\frac{5}{66}}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{61}{66}}{\frac{61}{66}}\right)$$

$$\text{L.H.S} = \tan^{-1}(1)$$

$$\text{L.H.S} = \frac{\pi}{4} ------(i)$$

$$R.H.S = \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$$

$$\because \tan^{-1}A+\tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{R.H.S} = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$$

$$\text{R.H.S} = \tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right)$$

$$\text{R.H.S} = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)$$

$$\text{R.H.S} = \tan^{-1}(1)$$

$$\text{R.H.S} = \frac{\pi}{4} ----(ii)$$

From (i) and (ii), we have

$$L.H.S = R.H.S$$

Hence Proved.

Question # 12.Prove that $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution.

$$L.H.S = 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$L.H.S = \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$\because \tan^{-1}A+\tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}\right)+\tan^{-1}\frac{1}{7}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\frac{1}{7}$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\frac{1}{7}$$

$$L.H.S = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}\right)$$

$$\text{R.H.S} = \tan^{-1}\left(\frac{\frac{25}{28}}{1-\frac{3}{28}}\right)$$

$$\text{R.H.S} = \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$$

$$\text{L.H.S} = \tan^{-1}(1)$$
$$\text{L.H.S} = \frac{\pi}{4} = R.H.S$$
$$L.H.S = R.H.S$$

Hence Proved.

Question # 13. Show that $Cos(Sin^{-1}x) = \sqrt{1-x^2}$

Solution. Suppose that $y = Sin^{-1}x$

$$\Rightarrow Siny = x$$

Since $Cosy = \sqrt{1-Sin^2y} = \sqrt{1-x^2}$

Using value of y

$$Cos(Sin^{-1}x) = \sqrt{1-x^2}.$$

Hence proved.

Question # 14. Show that $Sin(2Cos^{-1}x) = 2x\sqrt{1-x^2}$

Solution. Suppose that $y = Cos^{-1}x$

$$\Rightarrow Cosy = x$$

Since $Siny = \sqrt{1-Cos^2y} = \sqrt{1-x^2}$

Now

$$Sin(2Cos^{-1}x) = Sin2y$$
$$Sin(2Cos^{-1}x) = 2SinyCosy$$
$$Sin(2Cos^{-1}x) = 2x\sqrt{1-x^2}.$$

Hence proved.

Question # 15. Show that $Cos(2Sin^{-1}x) = 1-2x^2$

Solution. Suppose that $y = Sin^{-1}x$

$$\Rightarrow Siny = x$$

Since $Cosy = \sqrt{1-Sin^2y} = \sqrt{1-x^2}$

Now $Cos(2Sin^{-1}x) = Cos2y = Cos^2y - Sin^2y$

$$Cos(2Sin^{-1}x) = Cos^2y - Sin^2y$$
$$Cos(2Sin^{-1}x) = \left(\sqrt{1-x^2}\right)^2 - x^2$$
$$Cos(2Sin^{-1}x) = 1-x^2-x^2$$
$$Cos(2Sin^{-1}x) = 1-2x^2$$

Hence proved.

Question # 16. Show that $\tan^{-1}(-x) = -\tan^{-1}x$

Solution. Suppose that $y = \tan^{-1}(-x)$

$$\Rightarrow tany = -x$$
$$\Rightarrow -tany = x$$

Since $\tan(-\theta) = -tan\theta$

$$\Rightarrow \tan(-y) = x$$
$$\Rightarrow -\text{y} = \tan^{-1}x$$
$$\Rightarrow \text{y} = -\tan^{-1}x$$

Using Value of y then

$$\Rightarrow \tan^{-1}(-x) = -\tan^{-1}x$$

Hence proved.

Question # 17. Show that $\sin^{-1}(-x) = -\sin^{-1}(x)$

Solution. Suppose that $y = \sin^{-1}(-x)$

$$\Rightarrow Siny = -x$$
$$\Rightarrow -Siny = x$$

Since $\text{Sin}(-\theta) = -Sin\theta$

$$\Rightarrow \text{Sin}(-y) = x$$
$$\Rightarrow -\text{y} = \text{Sin}^{-1}x$$
$$\Rightarrow \text{y} = -\text{Sin}^{-1}x$$

Using Value of y then

$$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}(x)$$

Hence proved.

Question # 18. show that $\cos^{-1}(-x) = \pi - \cos^{-1}x$

Solution.

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$
$$\Rightarrow \cos^{-1}(-x) + \cos^{-1}x = \pi$$
$$L.H.S = \cos^{-1}(-x) + \cos^{-1}x$$
$$= \cos^{-1}(-x(x) - \sqrt{1-(-x^2)(1-x^2)})$$
$$= \cos^{-1}(-x^2 - \sqrt{(1-x^2)(1-x^2)})$$
$$= \cos^{-1}(-x^2 - \sqrt{(1-x^2)^2}$$
$$= \cos^{-1}(-x^2 - (1-x^2))$$
$$= \cos^{-1}(-1) = \pi = R.H.S$$

Hence L.H.S=R.H.S

Question # 19. show that $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

Solution. Suppose that $y = Sin^{-1}x$

$$\Rightarrow Siny = x$$

Since $Cosy = \sqrt{1-Sin^2y} = \sqrt{1-x^2}$

Now $tany = \frac{Siny}{Cosy}$

$$\tan(Sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}.$$

Hence proved.

Question # 20. Given that $x = Sin^{-1}\frac{1}{2}$, find the value of remaining trigonometric functions $sinx, cosx, tanx, secx, cosecx, cotx$.

Solution.

Given $x = Sin^{-1}\frac{1}{2}$

$$\Rightarrow Sinx = \frac{1}{2} \Rightarrow Cosecx = \frac{1}{Sinx} = 2$$

Now $Cosx = \sqrt{1-Sin^2x} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$$Cosx = \frac{\sqrt{3}}{2} \Rightarrow Secx = \frac{2}{\sqrt{3}}.$$

Since

$$tanx = \frac{Sinx}{Cosx} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$
$$\Rightarrow Cotx = \sqrt{3}.$$

Hence required.