

Bilal Article

Chapter 12.

APPLICATION OF TRIGONOMETRIC FUNCTION

A project of: <https://newsongoogle.com/>

Solution of triangle:

A triangle has six important elements; three angles and three sides.

If any three elements out of these six elements, out of which at least one side, are given, the remaining three elements can be determined.

This process of finding the unknown elements is called the "solution of the triangle.

in $\triangle ABC$, angles are denoted by α, β, γ and three sides opposite to them are denoted by a, b and c .

Tables of trigonometric ratios:

in the solution of triangles, we may have to solve problems involving angles other than $0^\circ, 30^\circ, 45^\circ$ and 90° in such cases, we shall have to consult natural \sin, \cos, \tan tables

Note:

Tables / calculator will also be used for finding the measures of the angles when value of trigonometric ratios are given e.g; to find θ when $\sin\theta = x$

Important points about tables:

- In four figure tables, the interval is 6 minutes and different corresponding to 1,2,3,4,5 minutes are given in the difference column.
- In given tables we have total 16-column. First column on extreme left is of angles in degree from 0° to 90°
- Next ten column are of angles in minute with difference of 6 minute.
- Last five column are columns of difference of 1 minute.

Note:

As $\sin\theta, \sec\theta$ and $\tan\theta$ go on increasing as θ increasing from 0° to 90° . So the numbers in the columns of difference for $\sin\theta, \sec\theta$ and $\tan\theta$ are added.

since $\cos\theta, \operatorname{cosec}\theta$ and $\cot\theta$ decrease as θ increasing from 0° to 90° so for $\cos\theta, \operatorname{cosec}\theta$ and $\cot\theta$ the numbers in the column of the difference are subtracted.

Exercise 12.1**Question#1**

Find the values of:

(i). $\sin 53^\circ 40'$

Solution:

From trigonometric table or calculator we have

$$\sin 53^\circ 40' = 0.805$$

(ii). $\cos 36^\circ 20'$

Solution:

$$\cos 36^\circ 20' = 0.8055$$

(iii). $\tan 19^\circ 30'$

Solution:

$$\tan 19^\circ 30' = 0.3541$$

(iv). $\cot 33^\circ 50'$

Solution:

$$\cot 33^\circ 50' = \frac{1}{\tan 33^\circ 50'} = 1.4919$$

(v). $\cos 42^\circ 38'$

Solution:

$$\cos 42^\circ 38' = 0.7357$$

(vi). $\tan 25^\circ 34'$

Solution:

$$\tan 25^\circ 34' = 0.4785$$

(vii). $\sin 18^\circ 31'$

Solution:

$$\sin 18^\circ 31' = 0.3176$$

(viii). $\cos 52^\circ 13'$

Solution:

$$\cos 52^\circ 13' = 0.6128$$

(ix). $\cot 89^\circ 9'$

Solution:

$$\cot 89^\circ 9' = \frac{1}{\tan 89^\circ 9'} = 0.1736$$

Question#2

Find θ , if

(i). $\sin\theta = 0.5791$

Solution:

$$\sin\theta = 0.5791$$

$$\Rightarrow \theta = \sin^{-1}0.5791$$

$$\Rightarrow \theta = 35^\circ 23'$$

(ii). $\cos\theta = 0.9316$

Solution:

$$\cos\theta = 0.9316$$

$$\Rightarrow \theta = \cos^{-1} 0.9316$$

$$\Rightarrow \theta = 21^\circ 19'$$

(iii). $\cos\theta = 0.5257$

Solution:

$$\cos\theta = 0.5257$$

$$\Rightarrow \theta = \cos^{-1} 0.5257$$

$$\Rightarrow \theta = 58^\circ 17'$$

(iv). $\tan\theta = 1.705$

Solution:

$$\tan\theta = 1.705$$

$$i \Rightarrow \theta = \tan^{-1} 1.705$$

$$\Rightarrow \theta = 87^\circ 23'$$

(v). $\tan\theta = 21.943$

Solution:

$$\tan\theta = 21.943$$

$$\Rightarrow \theta = \tan^{-1} 21.943$$

$$\Rightarrow \theta = 31^\circ 14'$$

(vi). $\sin\theta = 0.5186$

Solution:

$$\sin\theta = 0.5186$$

$$\Rightarrow \theta = \sin^{-1}0.5186$$

$$\Rightarrow \theta = 31^\circ 14'$$

Solution of right Triangles:

To solve a right triangles, we have to find

- i The measures of two acute angles.
- ii The length of the tree sides.

Note:

A trigonometric ratio of an acute angle of a right triangle involves 3 quantities "length of two sides and measure of an angle." Thus if two out of these three quantities are known. We can find the third quantity.

Case -I

When measure of two sides are given;

Case -II

When measure of one side and one angle are given.

Exercise 12.2

Question#1Solve the right triangle ABC, in which $\gamma = 90^\circ$

(i)

Solution:

from figure

$$a = 4, \alpha = 45^\circ$$

$$\therefore \frac{a}{c} = \tan 45^\circ$$

$$\Rightarrow \frac{4}{c} = (1) \Rightarrow 4 = c$$

Also

$$\frac{3}{b} = \sin 45^\circ$$

$$\Rightarrow \frac{4}{b} = 0.707$$

$$\Rightarrow \frac{4}{0.707} = b$$

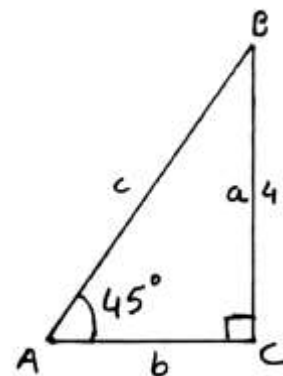
$$\Rightarrow b = 5.657$$

Now,

$$\alpha + \gamma = 90^\circ$$

$$\gamma = 90 - \alpha = 90 - 45$$

$$\Rightarrow \gamma = 45^\circ$$



(ii) here

$$\alpha = 60^\circ, \gamma = 90^\circ, c = 12$$

$$\beta = ?, a = ?, b = ?$$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 60^\circ - 90^\circ$$

$$\Rightarrow \beta = 30^\circ$$

$$\therefore \cos \alpha = \frac{b}{c} \Rightarrow b = c \cos \alpha$$

$$\Rightarrow b = (12) \cos 60^\circ \Rightarrow (12) \left(\frac{1}{2}\right) = 6$$

$$\therefore \sin \alpha = \frac{a}{c} \Rightarrow a = c \sin \alpha$$

$$\Rightarrow a = 12 \sin 60^\circ = 12 \left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}$$

$$\text{Hence } a = 6\sqrt{3}, b = 6, \beta = 30^\circ$$

iii).

Here

$$b = 5, c = 10$$

From Pythagoras theorem

$$c^2 = a^2 + b^2$$

$$\Rightarrow (10)^2 = a^2 + (5)^2$$

$$\Rightarrow 100 = a^2 + 25$$

$$\Rightarrow 100 - 25 = a^2$$

$$\Rightarrow a^2 = 75$$

$$\Rightarrow a = \sqrt{75}$$

$$\Rightarrow a = 8.66$$

Now ,

$$\tan \alpha = \frac{a}{b} = \frac{8.66}{5} = 1.732$$

$$\Rightarrow \alpha = \tan^{-1}(1.732) \Rightarrow \alpha = 59.999 \approx 60$$

$$\Rightarrow \alpha = 60^\circ$$

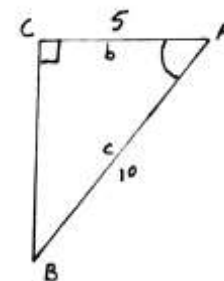
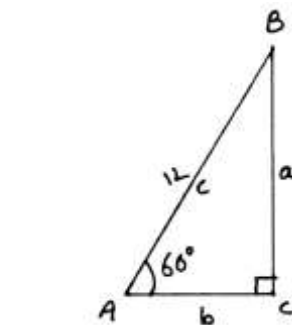
Now,

$$\alpha + \beta = 90^\circ$$

$$\Rightarrow \beta = 90 - \alpha$$

$$\Rightarrow \beta = 90 - 60$$

$$\Rightarrow \beta = 30^\circ$$



(iv) here

$$a = 8, \alpha = 40^\circ, \gamma = 90^\circ$$

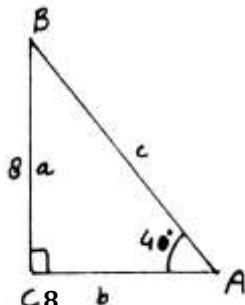
$$b = ?, c = ?, \beta = ?$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$\beta = 180^\circ - 40^\circ - 90^\circ$$

$$\Rightarrow \beta = 50^\circ$$



$$\because \sin \alpha = \frac{a}{c} \Rightarrow c = \frac{a}{\sin \alpha} = \frac{8}{\sin 40^\circ}$$

$$\Rightarrow c = \frac{8}{0.643} = 12.44$$

$$\because \sin \alpha = \frac{b}{c} \Rightarrow b = c \cos \alpha$$

$$\Rightarrow b = 12.44 \cos 40^\circ = (12.44)(0.766)$$

$$\Rightarrow b = 9.529$$

Hence $b = 9.529, c = 12.44, \beta = 50^\circ$

(v) here

$$c = 15, \alpha = 56^\circ, \gamma = 90^\circ$$

$$\alpha = ?, \beta = ?, b = ?$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 56^\circ - 90^\circ$$

$$\Rightarrow \beta = 34^\circ$$

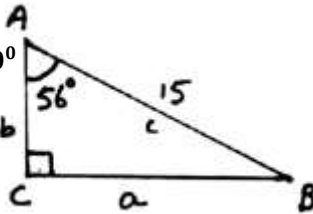
$$\because \sin \alpha = \frac{a}{c} \Rightarrow a = c \sin \alpha$$

$$\Rightarrow a = (15) \sin 56^\circ = (15)(0.829) = 12.435$$

$$\because \cos \alpha = \frac{b}{c} \Rightarrow b = c \cos \alpha$$

$$\Rightarrow b = 15 \cos 56^\circ = (15)(0.559) = 8.4^\circ$$

hence $b = 8.4, a = 12.435, \beta = 34^\circ$



(vi) here

$$a = 8, b = 8, \gamma = 90^\circ$$

$$a = 8, b = 8, \alpha = ?$$

$$\tan \alpha = \frac{b}{a} = \frac{8}{8} = 1$$

$$\Rightarrow \alpha = \tan^{-1}(1) = 45^\circ$$

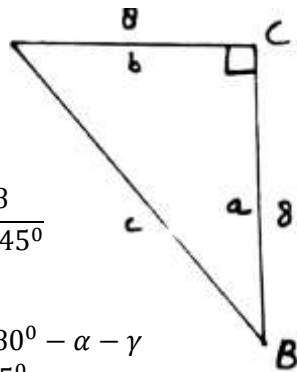
$$\because \sin \alpha = \frac{a}{c} \Rightarrow c = \frac{a}{\sin \alpha} = \frac{8}{\sin 45^\circ}$$

$$\Rightarrow c = \frac{8}{\frac{1}{\sqrt{2}}} = 8\sqrt{2}$$

$$\because \alpha + \beta + \gamma = 180^\circ \Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$\Rightarrow \beta = 180^\circ - 45^\circ - 90^\circ = 45^\circ$$

Hence $c = 8\sqrt{2}, \alpha = 45^\circ, \gamma = 45^\circ$



Question#2

$$\alpha = 37^\circ 20', a = 243$$

Solution:

$$\alpha + \beta = 90^\circ$$

$$\Rightarrow \beta = 90 - \alpha$$

$$\Rightarrow \beta = 90 - 37^\circ 20'$$

$$\Rightarrow \beta = 52^\circ 40'$$

Now,

$$\sin \alpha = \frac{a}{c} = \sin 37^\circ 20' = \frac{243}{c}$$

$$\Rightarrow 0.606 = \frac{243}{c} \Rightarrow c = \frac{243}{0.606}$$

$$\Rightarrow c = 400.692$$

Now,

$$\frac{a}{b} = \tan \alpha \Rightarrow \frac{243}{b} = \tan 37^\circ 20'$$

$$\Rightarrow \frac{243}{\tan 37^\circ 20'} = b$$

$$\Rightarrow \frac{243}{0.763} = b$$

$$\Rightarrow b = 318.598$$

Question#3

$$\alpha = 62^\circ 40', b = 796$$

Solution:

$$\alpha = 62^\circ 40', b = 796, \gamma = 90^\circ$$

$$\beta = ? \quad \alpha = ? \quad c = ?$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 62^\circ 40' - 90^\circ$$

$$\beta = 27^\circ 20'$$

$$\because \cos \alpha = \frac{b}{c} \Rightarrow c = \frac{b}{\cos \alpha}$$

$$\Rightarrow c = \frac{796}{\cos 62^\circ 40'} = \frac{796}{0.459} = 1734$$

$$\because \sin \alpha = \frac{a}{c} \Rightarrow a = c \sin \alpha$$

$$\Rightarrow a = (1734) \sin 62^\circ 40'$$

$$\Rightarrow a = (1734)(0.888) = 1540$$

Hence $a = 1540, c = 1734, \beta = 27^\circ 20'$

Question#4

$$a = 3.28, b = 5.74$$

Solution:

Here

$$a = 3.28, b = 5.74, \gamma = 90^\circ$$

$$c = ? \quad \beta = ? \quad \alpha = ?$$

$$\because \tan \alpha = \frac{a}{b} = \frac{3.28}{5.74} = 0.5644$$

$$\Rightarrow \alpha = \tan^{-1}(0.564) = 29^\circ 25'$$

$$\because \sin \alpha = \frac{a}{c} \Rightarrow c = \frac{a}{\sin \alpha}$$

$$\Rightarrow c = \frac{3.28}{\sin 29^\circ 25'} = \frac{3.28}{0.49}$$

$$\Rightarrow c = 6.69$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 29^\circ 25' - 90^\circ$$

$$\Rightarrow \beta = 60^\circ 35'$$

Hence $c = 6.69, \beta = 60^\circ 35', \alpha = 29^\circ 25'$

Question#5

$$b = 68.4, c = 96.2$$

Solution:

Here

$$b = 68.4, c = 96.2$$

From Pythagoras theorem

$$c^2 = a^2 + b^2$$

$$\Rightarrow (96.2)^2 = a^2 + (68.4)^2$$

$$\Rightarrow 9254.44 = a^2 + 4678.56$$

$$\Rightarrow 9254.44 - 4678.56 = a^2$$

$$\Rightarrow a^2 = 4575.88$$

$$\Rightarrow a = \sqrt{4575.88}$$

$\Rightarrow a = 67.645$

Now ,

$\tan \alpha = \frac{a}{b} = \frac{67.645}{68.4} = 0.98897$

$\Rightarrow \alpha = \tan^{-1}(0.98897)$

$\Rightarrow \alpha = 44.68^\circ = 44^\circ 41'$

Now,

$\alpha + \beta = 90^\circ$

$\Rightarrow \beta = 90 - \alpha$

$\Rightarrow \beta = 90 - 44^\circ 41'$

$\Rightarrow \beta = 45^\circ 19'$

Question#6

$a = 5429, c = 6294$

Solution:

Here $a = 5429, c = 6294, \gamma = 90^\circ$

$b = ?, \alpha = ?, \beta = ?$

$\therefore \sin \alpha = \frac{a}{c} = \frac{5429}{6294} = 0.863$

$\Rightarrow \alpha = \sin^{-1}(0.863) = 59^\circ 36'$

$\therefore \cos \alpha = \frac{b}{c}$

$\Rightarrow b = c \cos \alpha = 6294 \cos 59^\circ 36'$

$\Rightarrow b = (6294)(0.506) = 3184$

$\therefore \alpha + \beta + \gamma = 180^\circ$

$\Rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 59^\circ 36' - 90$

$\Rightarrow \beta = 30^\circ 24'$

Hence $b = 3184, \alpha = 59^\circ 36', \beta = 30^\circ 24'$

Question#7

$\beta = 50^\circ 10', c = 0.832$

Solution:

Here $\beta = 50^\circ 10', c = 0.832, \gamma = 90^\circ$

$a = ?, b = ?, \alpha = ?$

$\therefore \alpha + \beta + \gamma = 180^\circ$

$\Rightarrow \alpha = 180^\circ - \beta - \gamma$

$\alpha = 180^\circ - 50^\circ 10' - 90^\circ$

$\alpha = 39^\circ 50'$

$\therefore \cos \alpha = \frac{b}{c} \Rightarrow b = c \cos \alpha$

$\Rightarrow b = (0.832) \cos 39^\circ 50'$

$b = (0.832)(0.767) = 0.638$

$\therefore \sin \alpha = \frac{a}{c} \Rightarrow a = c \sin \alpha$

$\Rightarrow a = (0.832)(\sin 39^\circ 50')$

$a = (0.832)(0.64) = 0.533$

hence $a = 0.533, b = 0.638, \alpha = 39^\circ 50'$

Heights and distances

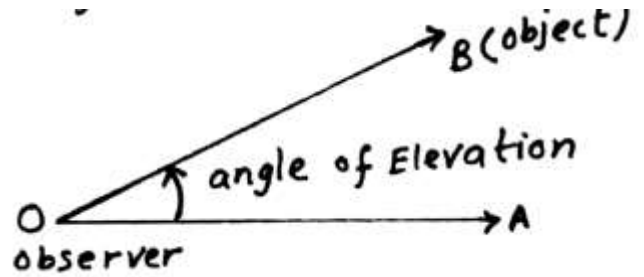
To find heights and distance the following procedures is adopted.

1. Construct a clear labelled diagram showing the known measurements.
2. Established the relationship between the quantities in the diagram to form equations containing trigonometric ratios.
3. Use tables /calculator to find the solution.

Angles Elevation and Depression:

Angles of Elevation:

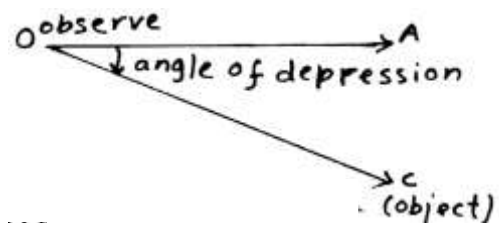
When an object is at higher level from the observers eye then the angle made by observers eye is called angle of elevation.



In fig $m\angle AOB$ is angle of elevation.

Angle of Depression:

When an object is at lower level from the observer's eye then the angle made by observers eye then the angle made by observers eye is called angle of depression.



In fig $m\angle AOC$ is angle of depression.

Exercise 12.3

Question#1

A vertical pole is 8 m high and the length of its shadow is 6 m. What is the angle of elevation of the sun at that moment?

Solution:

Let α be the required angle then

$\tan \alpha = \frac{8}{6}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^\circ = 53^\circ 8'$

Question#2

A man 18 dm tall observes that the angle of elevation of the top of a tree at a distance of 12 m from him is 32 . What is the height of the tree?

Solution:

Let h be the height of the tree and AC be the man then

$AC = BD = 18dm = 1.8m$

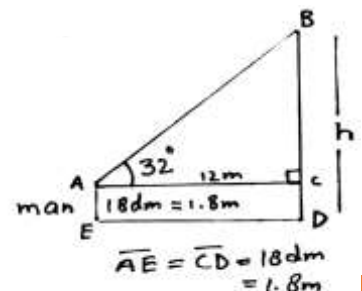
And

$AB = CD = 12m$

And

$D = h - 1.8$

Now from triangle C



$$\frac{DE}{CD} = \tan 32^\circ$$

$$\Rightarrow \frac{h-1.8}{12} = 0.6249$$

$$\Rightarrow h - 1.8 = 0.6249(12)$$

$$\Rightarrow h - 1.8 = 7.4984$$

$$\Rightarrow h = 7.4984 + 1.8$$

$$\Rightarrow h = 9.23 \text{ m}$$

Question#3

At the top of a cliff 80 m high, the angle of depression of a boat is 12° . How far is the boat from the cliff?

Solution:

Let x be the required distance then

$$\tan 12^\circ = \frac{BC}{AB}$$

$$\Rightarrow 0.2126 = \frac{80}{x}$$

$$\Rightarrow x = \frac{80}{0.2126}$$

$$\Rightarrow x = 376.37 \text{ m}$$

Question#4

A ladder leaning against a vertical wall makes an angle of 24° with the wall. Its foot is 5m from the wall. Find its length.

Solution:

Consider l be the length of the ladder then

$$\cos 24^\circ = \frac{AB}{AC}$$

$$\Rightarrow 0.9135 = \frac{5}{l}$$

$$\Rightarrow l = \frac{5}{0.9135}$$

$$\Rightarrow l = 12.29 \text{ m}$$

Question#5

A kite lying at a height of 67.2 m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.

Solution:

Let l be the length of the string then

$$\sin 55^\circ = \frac{BC}{AC}$$

$$\Rightarrow 0.819 = \frac{67.2}{l}$$

$$\Rightarrow l = \frac{67.2}{0.819}$$

$$\Rightarrow l = 82.04$$

Question#6

When the angle between the ground and the suri is 30° , lag pole casts a shadow of 40m long. Find the height of the top of the lag.

Solution:

Let h be height of flag then

$$\tan \theta = \frac{BC}{AC} = \frac{h}{40}$$

$$\Rightarrow \tan 30^\circ = \frac{h}{40}$$

$$\Rightarrow h = 40 \tan 30^\circ \\ = 40(0.577)$$

$$\Rightarrow h = 23.1 \text{ m}$$

Question#7

A plane lying directly above a post 6000 m away from an anti-aircraft gun observes the gun at an angle of depression of 27° . Find the height of the plane.

Solution:

Let height of the plane be h then

$$\tan 27^\circ = \frac{BC}{AB}$$

$$\Rightarrow 0.5095 = \frac{h}{6000}$$

$$\Rightarrow (0.5095)(6000) = h$$

$$\Rightarrow h = 3057.15 \text{ m}$$

Question#8

A man on the top of a 100 m high lighthouse is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respectively. Find the distance between the ships

Solution:

Let the ships be at A and B and a man at the light house at D

Let distance between the ships = x

By triangle BCD

$$\tan 19^\circ = \frac{DC}{BC}$$

$$\Rightarrow 0.344 = \frac{100}{BC}$$

$$\Rightarrow BC = \frac{100}{0.344}$$

$$\Rightarrow BC = 290.421$$

Now by triangle ACD

$$\tan 17^\circ = \frac{DC}{AC}$$

$$\Rightarrow 0.3057 = \frac{100}{AC}$$

$$\Rightarrow BC = \frac{100}{0.3057}$$

$$\Rightarrow AC = 327.09$$

Now

$$x = AC - BC = 327.09 - 290.421$$

$$\Rightarrow x = 36.66 \text{ m}$$

Question#9

P and Q are two points in line with a tree. If the distance between P and Q be 30 m and the angles of elevation of the top of the tree at P and Q be 12° and 15° respectively, find the height of the tree.

Solution:

Let h be the height of the tree and $QR = x$ then $PR = x + 30$

From triangle QRS

$$\tan 15^\circ = \frac{RS}{QR}$$

$$\Rightarrow 0.2679 = \frac{h}{x}$$

$$\Rightarrow 0.2679 x = h \dots \dots (i)$$

Also From triangle PRS

$$\tan 12^\circ = \frac{RS}{PR}$$

$$\Rightarrow 0.2126 = \frac{h}{x+30}$$

$$\Rightarrow 0.2679(x+30) = h$$

$$\Rightarrow 0.2126x + 6.378 = h \dots \dots (ii)$$

Comparing (i) and (ii)

$$0.2679x = 0.2126x + 6.378$$

$$\Rightarrow 0.2679x - 0.2126x = 6.378$$

$$\Rightarrow 0.0553x = 6.378$$

$$\Rightarrow x = \frac{6.378}{0.0553} = 115.335$$

Putting in (i)

$$h = 0.2679(115.335)$$

$$\Rightarrow h = 30.898 \text{ m}$$

Question#10

Two men are on the opposite sides of a 100 m high tower. If the measures of the angles of elevation of the top of the tower are 18° and 22° respectively find the distance between them.

Solution:

Let the men be at pt. A and B also $AC = x_1$ and $CB = x_2$

By triangle ACD

$$\frac{CD}{AC} = \tan 18^\circ$$

$$\Rightarrow \frac{100}{x_1} = 0.3249$$

$$\Rightarrow \frac{100}{0.3249} = x_1$$

$$\Rightarrow x_1 = 307.768$$

Also by triangle BCD

$$\frac{CD}{CB} = \tan 22^\circ$$

$$\Rightarrow \frac{100}{x_2} = 0.4040$$

$$\Rightarrow \frac{100}{0.4040} = x_2$$

$$\Rightarrow x_2 = 247.50$$

So distance between men = $x_1 + x_2 = 307.768 + 247.50 = 555.28 \text{ m}$

Question#11

A man standing 60 m away from a tower notices that the angles of elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find the length of the flag staff.

Solution:

Let height of the tower be h_1 and the height of the flag staff be h then

$$BD = h + h_1$$

From triangle ABC

$$\frac{BC}{AB} = \tan 62^\circ$$

$$\Rightarrow \frac{h_1}{60} = 1.8807$$

$$\Rightarrow h_1 = (1.8807)(60)$$

$$\Rightarrow h_1 = 112.844 \text{ m}$$

Now From triangle ABD

$$\frac{BD}{AB} = \tan 64^\circ$$

$$\Rightarrow \frac{h+h_1}{60} = 2.0503$$

$$\Rightarrow h + h_1 = (2.0503)(60)$$

$$\Rightarrow h + h_1 = 123.018$$

$$\Rightarrow h = 123.018 - h_1$$

$$\Rightarrow h = 123.018 - 112.844$$

$$\Rightarrow h = 10.174 \text{ m}$$

Question#12

The angle of elevation of the top of a 60 m high tower from a point A, on the same level as the foot of the tower, is 25° . Find the angle of elevation of the top of the tower from a point B, 20 m nearer to A from the foot of the tower.

Solution:

Let α be the required angle and $BC = x$ then $AC = x + 20$

From triangle ACD

$$\frac{DC}{AC} = \tan 25^\circ$$

$$\Rightarrow \frac{60}{x+20} = 0.4663$$

$$\Rightarrow 60 = (0.4663)(x+20)$$

$$\Rightarrow 60 = 0.4663x + 9.326$$

$$\Rightarrow 60 - 9.326 = 0.4663x$$

$$\Rightarrow 50.674 = 0.4663x$$

$$\Rightarrow x = \frac{50.674}{0.4663}$$

$$\Rightarrow x = 108.6722$$

Now from triangle BCD

$$\tan \alpha = \frac{DC}{BC} = \frac{60}{x} = \frac{60}{108.6722} = 0.552$$

$$\Rightarrow \alpha = \tan^{-1}(0.552) = 28.904^\circ = 28^\circ 54'$$

$$\Rightarrow \alpha = 28^\circ 54'$$

Question#13

Two buildings A and B are 100 m apart. The angle of elevation from the top of the building A to the top of the building B is 20° . The angle of elevation from the base of the building B to the top of the building A is 50° . Find the height of the building B.

Solution:

Let height of the building A = $CA = h_1$ and height of the building B = h then

$$EB = h - h_1$$

from triangle CDA

$$\frac{BC}{AB} = \tan 50^\circ$$

$$\Rightarrow \frac{h_1}{100} = 1.1918$$

$$\Rightarrow h_1 = (1.1918)(100)$$

$$\Rightarrow h_1 = 119.175$$

Now From triangle ABD

$$\frac{EB}{AE} = \tan 20^\circ$$

$$\begin{aligned} \Rightarrow \frac{h-h_1}{100} &= 0.36397 \\ \Rightarrow h - h_1 &= (0.36397)(100) \\ \Rightarrow h - h_1 &= 36.397 \\ \Rightarrow h &= 36.397 + h_1 \\ \Rightarrow h &= 36.397 + 119.75 \\ \Rightarrow h &= 155.572 \text{ m} \end{aligned}$$

Question#14

A window washer is working in a hotel building. An observer at a distance of 20 m from the building finds the angle of elevation of the worker to be of 30°. The worker

climbs up 12 m and the observer moves 4 m farther away from the building. Find the new angle of elevation of the worker.

Solution:

Let the required angle be α and $AB = 20$ then $DA = 4$

From triangle ACD

$$\frac{BC}{AB} = \tan 30^\circ$$

$$\Rightarrow \frac{BC}{20} = 0.577$$

$$\Rightarrow BC = (0.577)(20)$$

$$\Rightarrow BC = 11.547$$

$$\Rightarrow BE = BC + CE = 11.547 + 12 = 23.547$$

Now from triangle DBE

$$\tan \alpha = \frac{BE}{DB} = \frac{23.547}{242} = 0.981$$

$$\Rightarrow \alpha = \tan^{-1}(0.981) = 44.454^\circ = 44^\circ 27'$$

$$\Rightarrow \alpha = 44^\circ 27'$$

Question#15

A man standing on the bank of a canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is 60°. On retreating 40 meters from the bank, he finds the measure of the angle of elevation of the tree as 30°. Find the height of the tree and the width of the canal.

Solution:

Let h be the height of the tree be h and the width of the canal be x

From triangle ABC

$$\tan 60^\circ = \frac{BC}{AB}$$

$$\Rightarrow 1.732 = \frac{h}{x}$$

$$\Rightarrow h = 0.1.732x \dots \dots (i)$$

Also From triangle DBC

$$\tan 30^\circ = \frac{BC}{DB}$$

$$\Rightarrow 0.577 = \frac{h}{40+x}$$

$$\Rightarrow 0.577(40+x) = h$$

$$\Rightarrow h = 23.094 + 0.577x \dots \dots (ii)$$

Comparing (i) and (ii)

$$0.1.732x = 23.094 + 0.577x$$

$$\Rightarrow 0.1.732x - 0.577x = 23.094$$

$$\Rightarrow 1.155x = 23.094$$

$$\Rightarrow x = \frac{23.094}{1.155} = 19.995 \text{ m}$$

Putting in (i)

$$h = 0.1.732(19.995)$$

$$\Rightarrow h = 34.63 \text{ m}$$

Oblique triangles:

A triangle which is not right angled is called oblique triangle.

The law of cosine

In any triangle ABC with usual notation, prove that

$$i \quad a^2 = b^2 + c^2 - 2bccos\alpha$$

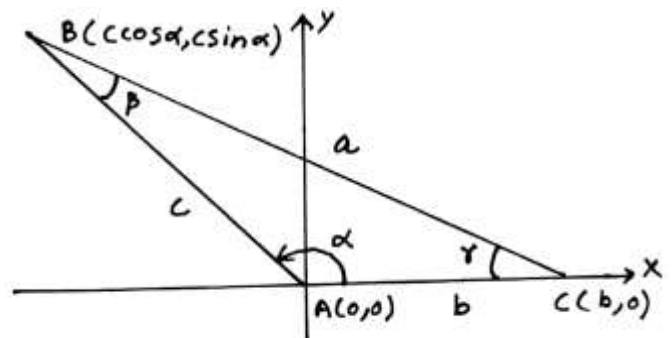
$$ii \quad b^2 = c^2 + a^2 - 2cacos\beta$$

$$iii \quad c^2 = a^2 + b^2 - 2abcos\gamma$$

Proof:

In any triangle ABC , coordinates of points are $A(0,0)$

$B(ccos\alpha, csina)$ and $C(b, 0)$



By distance formula

$$\begin{aligned} |BC|^2 &= (ccos\alpha - b)^2 + (csina - 0)^2 \\ &= c^2 \cos^2 \alpha + b^2 - 2bccos\alpha + c^2 \sin^2 \alpha \\ &= c^2(\cos^2 \alpha + \sin^2 \alpha) + b^2 - 2bccos\alpha \\ &\Rightarrow a^2 = c^2(1) + b^2 - 2bccos\alpha \\ &\Rightarrow a^2 = b^2 + c^2 - 2bccos\alpha \end{aligned}$$

Similarly,

$$b^2 = c^2 + a^2 - 2cacos\beta$$

$$c^2 = a^2 + b^2 - 2abcos\gamma$$

They can also be expressed as

$$cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Note:

If $\triangle ABC$ is right then,

Law of cosine reduce to Pythagoras theorem.

$$\text{if } \alpha = 90^\circ \text{ then } b^2 + c^2 = a^2$$

$$\text{or if } \beta = 90^\circ \text{ then } c^2 + a^2 = b^2$$

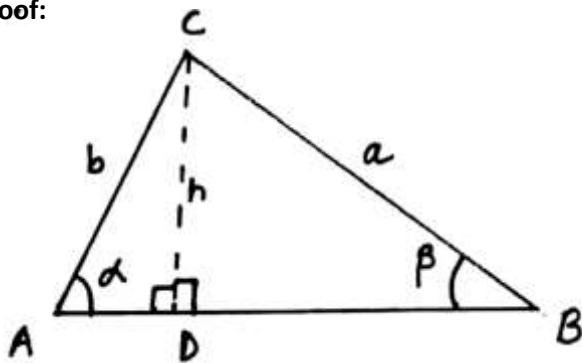
$$\text{if } \gamma = 90^\circ \text{ then } a^2 + b^2 = c^2$$

The law of Sines:

In any triangle ABC with usual notation, prove that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof:



In any triangle draw a \perp ar C to \overline{AB} at D , in right triangle CAD

$$\sin\alpha = \frac{\overline{CD}}{\overline{AC}} \Rightarrow \sin\alpha = \frac{h}{b}$$

$$\Rightarrow h = b\sin\alpha \rightarrow (i)$$

in right triangle CAD

$$\sin\beta = \frac{\overline{CD}}{\overline{BC}} \Rightarrow \sin\beta = \frac{h}{a}$$

$$\Rightarrow h = a\sin\beta \rightarrow (ii)$$

From (i) $a\sin\beta = b\sin\alpha$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \rightarrow (iii)$$

Similarly if we draw a \perp ar from A to \overline{BC} than

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Combining (iii) and (iv) we get

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

The law of tangents

In any triangle ABC with usual notation, prove that

$$(i) \quad \frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$$

$$(ii) \quad \frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}}$$

$$(iii) \quad \frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$$

Proof:

We have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D(\text{say})$$

Then

$$a = D\sin\alpha, \quad b = D\sin\beta, \quad c = D\sin\gamma$$

$$\frac{a-b}{a+b} = \frac{D\sin\alpha - D\sin\beta}{D\sin\alpha + D\sin\beta} = \frac{D(\sin\alpha - \sin\beta)}{D(\sin\alpha + \sin\beta)}$$

$$= \frac{2\cos\frac{\alpha+\beta}{2} \sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}}$$

$$= \frac{\cos\frac{\alpha+\beta}{2} \tan\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$

$$\Rightarrow \frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} \quad \text{if } a > b$$

Similarly

$$\frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \quad \text{if } b > c$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} \quad \text{if } c > a$$

Half Angle formulas

(a) the sine Half the angle in terms of the sides in any triangle ABC , prove that

$$(i) \quad \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$(ii) \quad \sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$(iii) \quad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Where $2s = a + b + c$

Proof:

law of cosines

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Also

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\Rightarrow 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$$

$$= 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$2\sin^2\frac{\alpha}{2} = \frac{2bc - b^2 + a^2}{2bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{a^2 - (b^2 + c^2 - 2bc)}{4bc}$$

$$= \frac{a^2 - (b-c)^2}{4bc}$$

$$\sin^2 \frac{\alpha}{2} = \frac{(a+b-c)(a-(b-c))}{4bc}$$

$$= \frac{(a+b-c)(a-b+c)}{4bc}$$

$$\sin^2 \frac{\alpha}{2} = \frac{(a+b-c)(a+c-b)}{4bc}$$

$$\therefore s = \frac{a+b+c}{2} \Rightarrow a+b+c = 2s$$

$$a+b = 2s - c$$

$$\Rightarrow a+b-c = 2s - c - c$$

$$\Rightarrow a+b-c = 2s - 2c = 2(s-c)$$

Also

$$a+b+c = 2s$$

$$\Rightarrow a+b = 2s - b$$

$$\Rightarrow a+b-c = 2s - b - b$$

$$a+c-b = 2s - 2b = 2(s-b)$$

So,

$$\sin^2 \frac{\alpha}{2} = \frac{2(s-c) - 2(s-b)}{4bc}$$

$$= \frac{4(s-c)(s-b)}{4bc}$$

$$\sin^2 \frac{\alpha}{2} = \frac{(s-c)(s-b)}{bc}$$

$$\Rightarrow \sin \frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}$$

$\therefore \alpha$ is measure of angle of ΔABC

$$\therefore \frac{\alpha}{2} < 90^\circ \Rightarrow \sin \frac{\alpha}{2} = +ve$$

Similarly,

$$\sin \frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \quad \sin \frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

The cosine of Half the angle in term of the sides.

in any ΔABC with usual notation, prove that

$$(i) \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$(ii) \cos \frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$(iii) \cos \frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Where $2s = a + b + c$

Proof:

\therefore law of cosines

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

also

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1$$

$$\Rightarrow 2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$2 \cos^2 \frac{\alpha}{2} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$2 \cos^2 \frac{\alpha}{2} = \frac{(b+c)^2 - a^2}{4bc}$$

$$\cos^2 \frac{\alpha}{2} = \frac{(a+b+c)(b+c-a)}{4bc}$$

$$\therefore a+b+c = 2s$$

$$\Rightarrow b+c = 2s - a$$

$$b+c-a = 2s - a - a = 2s - 2a - 2(s-a)$$

So

$$\cos^2 \frac{\alpha}{2} = \frac{2s \cdot 2(s-a)}{4bc}$$

$$\cos^2 \frac{\alpha}{2} = \frac{4s(s-a)}{4bc}$$

$$\Rightarrow \cos^2 \frac{\alpha}{2} = \frac{s(s-a)}{bc}$$

$\therefore \alpha$ is measure of angle of ΔABC

$$\therefore \frac{\alpha}{2} \text{ is acute} \Rightarrow \cos \frac{\alpha}{2} = +ve$$

Similarly,

$$\cos \frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ca}}, \quad \cos \frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

© The Tangent Of Half The Angle In Terms Of The Sides.

In any triangle ABC with usual notation, prove that

$$(i) \tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan \frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan \frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Where $2s = a + b + c$

Proof:

$$\therefore \sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \text{ and}$$

$$\cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$\Rightarrow \tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly,

$$\tan \frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan \frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Solution of Oblique triangle

We know that

A triangle can be constructed

- One side and two angles are given.
- Two sides and their include angle are given
- Three sides are given

In the same way

An oblique triangle can be sides if

- One side and two angles are given
- Two sides and their include angle are given
- Three sides are given.

Case -1

When measure of one side and two angles are given

Exercise 12.4

Solve the triangle ABC, if

Question#1

$$\beta = 60^\circ, \gamma = 15^\circ, b = \sqrt{6}$$

Solution:

We know that:

$$\alpha + \beta + \gamma = 180^\circ$$

Now,

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 60^\circ - 15^\circ$$

$$\Rightarrow \alpha = 105^\circ$$

By law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\Rightarrow a = b \cdot \frac{\sin \alpha}{\sin \beta}$$

$$\Rightarrow a = \sqrt{6} \cdot \frac{\sin 105^\circ}{\sin 60^\circ} = \sqrt{6} \cdot \frac{0.9659}{0.8660} = \sqrt{6} \times 1.1153 =$$

$$2.731$$

Again By law of sines

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\Rightarrow c = b \cdot \frac{\sin \gamma}{\sin \beta}$$

$$\Rightarrow c = \sqrt{6} \cdot \frac{\sin 15^\circ}{\sin 60^\circ} = \sqrt{6} \cdot \frac{0.2588}{0.8660} =$$

$$\sqrt{6} \times 0.2989 = 0.732$$

$$\text{Hence } \alpha = 105^\circ, a = 2.731, c = 0.732$$

Question#2

$$\beta = 52^\circ, \gamma = 89^\circ 35', a = 89.35$$

Solution:

$$\beta = 52^\circ, \gamma = 89^\circ 35', a = 89.35$$

We know that:

$$\alpha + \beta + \gamma = 180^\circ$$

Now,

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 52^\circ - 89^\circ 35'$$

$$\Rightarrow \alpha = 38^\circ 25'$$

By law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\Rightarrow b = a \cdot \frac{\sin \beta}{\sin \alpha}$$

$$\Rightarrow b = 89.35 \cdot \frac{\sin 52^\circ}{\sin 38^\circ 25'} = 89.35 \cdot \frac{0.7880}{0.6213} = 89.35 \times$$

$$1.268 = 113.31$$

Again By law of sines

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\Rightarrow c = b \cdot \frac{\sin \gamma}{\sin \beta}$$

$$\Rightarrow c = 113.31 \cdot \frac{\sin 89^\circ 35'}{\sin 52^\circ} = 113.31 \cdot \frac{0.999}{0.6213} =$$

$$113.31 \times 1.268 = 143.79$$

$$\text{Hence } \alpha = 38^\circ 25', a = 113.31, c = 143.79$$

Question#3

$$b = 125, \gamma = 53^\circ, \alpha = 47^\circ$$

Solution:

We know that:

$$\alpha + \beta + \gamma = 180^\circ$$

Now,

$$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 47^\circ - 53^\circ$$

$$\Rightarrow \beta = 80^\circ$$

By law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\Rightarrow a = b \cdot \frac{\sin \alpha}{\sin \beta}$$

$$\Rightarrow a = 125 \cdot \frac{\sin 47^\circ}{\sin 80^\circ} = 125 \cdot \frac{0.731}{0.985} = 125 \times 0.742 =$$

$$93$$

Again By law of sines

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\Rightarrow c = b \cdot \frac{\sin \gamma}{\sin \beta}$$

$$\Rightarrow c = 125 \cdot \frac{\sin 53^\circ}{\sin 80^\circ} = 125 \cdot \frac{0.7986}{0.985} = 125 \times 0.811 =$$

$$101$$

$$\text{Hence } \beta = 80^\circ, a = 93, c = 101.7$$

Question#4

$$c = 16.1, \alpha = 42^\circ 45', \gamma = 74^\circ 32'$$

Solution:

We know that:

$$\alpha + \beta + \gamma = 180^\circ$$

Now,

$$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 42^\circ 45' - 74^\circ 32'$$

$$\Rightarrow \beta = 62^\circ 43'$$

By law of sines

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow b = c \cdot \frac{\sin\beta}{\sin\gamma}$$

$$\Rightarrow b = 113.31 \cdot \frac{\sin 62^\circ 43'}{\sin 42^\circ 45'} = 16.1 \cdot \frac{0.899}{0.674} = 16.1 \times$$

$$0.992 = 14.85$$

Again By law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow a = b \cdot \frac{\sin\alpha}{\sin\beta}$$

$$\Rightarrow a = 14.85 \cdot \frac{\sin 42^\circ 45'}{\sin 62^\circ 43'} = 14.85 \cdot \frac{0.678}{0.899} = 14.85 \times$$

$$0.7635 = 11.34$$

$$\text{Hence } \beta = 62^\circ 43', b = 14.85, a = 11.33$$

Question#5

$$a = 53, \beta = 88^\circ 36', \gamma = 31^\circ 54'$$

Solution:

We know that:

$$\alpha + \beta + \gamma = 180^\circ$$

Now,

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 88^\circ 36' - 31^\circ 54'$$

$$\Rightarrow \alpha = 59^\circ 30'$$

By law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow b = a \cdot \frac{\sin\beta}{\sin\alpha}$$

$$\Rightarrow b = 53 \cdot \frac{\sin 88^\circ 36'}{\sin 59^\circ 30'} = 53 \cdot \frac{0.9997}{0.8616}$$

$$= 53 \times 1.16 = 61.49$$

Again By law of sines

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$\Rightarrow c = b \cdot \frac{\sin\gamma}{\sin\beta}$$

$$\Rightarrow c = 61.49 \cdot \frac{\sin 31^\circ 54'}{\sin 88^\circ 36'}$$

$$= 61.49 \cdot \frac{0.528}{0.8616}$$

$$= 61.49 \times 0.5286 = 32.51$$

$$\text{Hence } \alpha = 59^\circ 30', b = 61.49, c = 32.5$$

Case -II

When measures of two sides and their include angles are given.

In this case we can use

- First law of cosine and then law of sines,
- Or first law of tangent and then law of sines.

Exercise 12.5

Solve the triangle ABC in which:

Question#1

$$b = 95, c = 34, \text{ and } \alpha = 52^\circ$$

Solution:

$$b = 95, c = 34, \text{ and } \alpha = 52^\circ$$

By law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos\alpha$$

$$= (95)^2 + (34)^2 - 2(95)(34)\cos 52^\circ$$

$$= 9025 + 1156 - 6460(0.6157) = 6203.578$$

$$\Rightarrow a = \sqrt{6203.578}$$

$$\Rightarrow a = 78.76$$

Again by law of cosine

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(34)^2 + (78.76)^2 - (95)^2}{2(34)(78.76)}$$

$$= \frac{1156 + 6203.138 - 9025}{5355.68} = -\frac{1665.862}{5355.68} = -0.311$$

$$\beta = \cos^{-1}(-0.311)$$

$$\Rightarrow \beta = 108^\circ 7' 20''$$

Since in any triangle

$$\alpha + \beta + \gamma = 180$$

$$\gamma = 180 - \alpha - \beta = 180^\circ - 52^\circ - 108^\circ 7' 20''$$

$$\Rightarrow \gamma = 19^\circ 52' 40''$$

Question#2

$$b = 12.5, c = 23, \text{ and } \alpha = 38^\circ 20'$$

Solution:

$$b = 12.5, c = 23, \text{ and } \alpha = 38^\circ 20'$$

By law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos\alpha$$

$$= (12.5)^2 + (23)^2 - 2(12.5)(23)\cos 38^\circ 20'$$

$$= 156.25 + 529 - 575(0.7844) = 234.21$$

$$\Rightarrow a = \sqrt{234.21}$$

$$\Rightarrow a = 15.304$$

Again by law of cosine

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(23)^2 + (15.304)^2 - (12.5)^2}{2(23)(15.304)}$$

$$= \frac{529 + 234.21 - 156.25}{703.984} = \frac{606.96}{703.984} = 0.8622$$

$$\beta = \cos^{-1}(0.8622)$$

$$\Rightarrow \beta = 30^\circ 26'$$

Since in any triangle

$$\alpha + \beta + \gamma = 180$$

$$\gamma = 180 - \alpha - \beta = 180^\circ - 38^\circ 20' - 30^\circ 26'$$

$$\Rightarrow \gamma = 111^\circ 14'$$

Question#3

$$a = \sqrt{3} - 1, b = \sqrt{3} + 1, \text{ and } \gamma = 60^\circ$$

Solution:

$$a = \sqrt{3} - 1 = 0.732, b = \sqrt{3} + 1 = 2.732, \gamma = 60^\circ$$

By law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos\gamma$$

$$= (0.732)^2 + (2.732)^2 - 2(0.732)(2.732)\cos 60^\circ$$

$$= 0.5358 + 7.4638 - 1.9998 = 5.9998 \approx 6$$

$$\Rightarrow c = \sqrt{6} = 2.449$$

$$\Rightarrow c = 2.449$$

Again by law of cosine

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2.732)^2 + (2.449)^2 - (0.732)^2}{2(2.732)(2.449)}$$

$$= \frac{7.4638 + 5.9976 - 0.5358}{13.3813} = \frac{12.9256}{13.3813} = 0.9659$$

$$\alpha = \cos^{-1}(0.9659)$$

$$\Rightarrow \alpha = 15^\circ$$

Since in any triangle

$$\alpha + \beta + \gamma = 180$$

$$\beta = 180 - \alpha - \gamma = 180 - 15 - 60 \Rightarrow \beta = 105^\circ$$

Question#4

$$a = 3, c = 6, \text{ and } \gamma = 36^\circ 20'$$

solution:

law of cosine:

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$= (3)^2 + (6)^2 - 2(3)(6) \cos 36^\circ 20'$$

$$= 9 + 36 - 36(0.8055)$$

$$b^2 = 45 - 29.0010 = 15.99$$

$$\Rightarrow b = 3.998 \text{ or } b = 4$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(4)^2 + (6)^2 - (3)^2}{2(4)(6)}$$

$$\cos \alpha = \frac{16 + 36 - 9}{48} = \frac{43}{48}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{43}{48}\right) = \cos^{-1}(0.8958)$$

$$\alpha = 26^\circ 23' 4''$$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 26^\circ 23' 4'' - 36^\circ 20'$$

$$\gamma = 117^\circ 16' 56''$$

Here $\alpha = 26^\circ 23' 4''$, $\gamma = 117^\circ 16' 56''$, $b = 4$

Question#5

$$a = 7, b = 3, \text{ and } \alpha = 38^\circ 13'$$

solution:

law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \alpha$$

$$c^2 = (7)^2 + (3)^2 - 2(7)(3) \cos 38^\circ 13'$$

$$c^2 = 49 + 9 - 42(0.7856)$$

$$c^2 = 58 - 32.9984 = 25.0016 \approx 25$$

$$\Rightarrow c = 5$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(3)^2 + (5)^2 - (7)^2}{2(3)(5)} = \frac{9 + 25 - 49}{30}$$

$$\cos \alpha = -\frac{1}{2} \Rightarrow \alpha = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \alpha = 120^\circ$$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - 120^\circ - 38^\circ 13' = 21^\circ 47'$$

$$\text{Hence } \alpha = 120^\circ, \beta = 21^\circ 47', c = 5$$

Solve the following triangles, using first

Law of tangents and then Law of sines:

Question#6

$$a = 36.21, b = 42.09, \text{ and } \gamma = 44^\circ 29'$$

Solution:

Since,

$$\alpha + \beta + \gamma = 180$$

$$\Rightarrow \alpha + \beta = 180 - \gamma$$

$$= 180 - 44^\circ 29'$$

$$\Rightarrow \alpha + \beta = 135^\circ 31' \dots \dots (i)$$

By law of tangent

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow \frac{36.21-42.09}{36.21+42.09} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{135^\circ 31'}{2}\right)}$$

$$\Rightarrow \frac{-5.88}{78.3} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan(67^\circ 45')}$$

$$\Rightarrow -0.0751 = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{2.4443}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = -0.0751(2.4443)$$

$$= -0.1836$$

$$\Rightarrow \frac{\alpha-\beta}{2} = \tan^{-1}(-0.1836)$$

$$\Rightarrow \frac{\alpha-\beta}{2} = -10^\circ 24'$$

$$\Rightarrow \alpha - \beta = -20^\circ 48' \dots \dots (ii)$$

Adding (i) & (ii)

$$\alpha + \beta = 135^\circ 31'$$

$$\alpha - \beta = -20^\circ 48'$$

$$2\alpha = 114^\circ 43'$$

$$\Rightarrow \alpha = 57^\circ 22'$$

Putting value of α in eq. (i)

$$57^\circ 22' + \beta = 135^\circ 22''$$

$$\Rightarrow \beta = 135^\circ 22'' - 57^\circ 22'$$

$$\Rightarrow \beta = 78^\circ 9'$$

Now by law of sine

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha}$$

$$\Rightarrow \frac{c}{\sin 44^\circ 29'} = \frac{36.21}{\sin 57^\circ 22'}$$

$$\Rightarrow c = \frac{36.21}{\sin 57^\circ 22'} \cdot \sin 44^\circ 29'$$

$$= \frac{36.21}{0.8421} \cdot 0.7007$$

$$\Rightarrow c = 30.13$$

Question#7

$$a = 93, b = 101, \text{ and } \gamma = 80^\circ$$

solution:

$$a = 93, c = 101, \beta = 80^\circ$$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \alpha + \gamma = 180^\circ - \beta = 180^\circ - 80^\circ = 100^\circ \rightarrow (i)$$

$$\frac{\alpha + \gamma}{2} = \frac{100^\circ}{2} = 50^\circ$$

Law of tangents

$$\frac{\tan \frac{\gamma - \alpha}{2}}{\tan \frac{\gamma + \alpha}{2}} = \frac{c - a}{c + a}$$

$$\Rightarrow \tan \frac{\gamma - \alpha}{2} = \frac{101 - 93}{101 + 93} \tan 50^\circ$$

$$\tan \frac{\gamma - \alpha}{2} = \frac{8}{194} (1.19) = 0.0476$$

$$\Rightarrow \frac{\gamma - \alpha}{2} = \tan^{-1}(0.0476)$$

$$\Rightarrow \frac{\gamma - \alpha}{2} = \tan^{-1}(0.0476)$$

$$\frac{\gamma - \alpha}{2} = 2.72 \Rightarrow \gamma - \alpha = 5.45$$

$$\Rightarrow \gamma - \alpha = 5^{\circ}27' \rightarrow (ii)$$

$$\text{By (i) + (ii)} \Rightarrow 2r = 105^{\circ}27'$$

$$\Rightarrow \gamma = \frac{105^{\circ}27'}{2} = 52^{\circ}43' \text{ put in (1)}$$

$$(i) \Rightarrow \alpha + 52^{\circ}43' = 100^{\circ}$$

$$\alpha = 100^{\circ} - 52^{\circ}43' = 47^{\circ}16'$$

By law of sine

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow b = \frac{c \sin\beta}{\sin\gamma} = \frac{(101)\sin 80^{\circ}}{\sin 52^{\circ}43'}$$

$$b = \frac{(101)(0.98)}{0.795} = 124.5$$

Hence $\alpha = 47^{\circ}16'$, $\gamma = 52^{\circ}43'$, $b = 124.5$

Question#8

$a = 14.8$, $c = 16.1$, and $\gamma = 42^{\circ}45'$

solution:

$$\because \alpha + \beta + \gamma = 180^{\circ}$$

$$\Rightarrow \beta + \gamma = 180^{\circ} - \alpha = \beta + \gamma = 180^{\circ} - 47^{\circ}16'$$

$$\Rightarrow \beta + \gamma = 137^{\circ}15' \rightarrow (i)$$

$$\frac{\beta + \gamma}{2} = 68^{\circ}37'15''$$

by law of tangent

$$\frac{\tan \frac{\beta - \gamma}{2}}{\tan \frac{\beta + \gamma}{2}} = \frac{b - c}{b + c}$$

$$\tan \frac{\beta - \gamma}{2} = \frac{14.8 - 16.1}{14.8 + 16.1} \tan 68^{\circ}37'15''$$

$$\tan \frac{\beta - \gamma}{2} = \frac{-1.3}{30.9} (2.55) = -0.1072$$

$$\Rightarrow \frac{\beta - \gamma}{2} = \tan^{-1}(-0.1072) = -6^{\circ}7'7''$$

$$\beta - \gamma = -12^{\circ}14'14''$$

$$\text{By (i) + (ii)} \Rightarrow 2\beta = 125^{\circ}$$

$$\Rightarrow \beta = 62^{\circ}30'22'' \text{ put in (i)}$$

$$62^{\circ}30'22'' + \gamma = 137^{\circ}15'$$

$$\gamma = 74^{\circ}44'37''$$

By law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow a = \frac{b \sin\alpha}{\sin\beta} = \frac{14.8 \sin 42^{\circ}45'}{\sin 62^{\circ}30'}$$

$$\Rightarrow a = \frac{14.8(0.678)}{0.887} = 11.31$$

$$\text{hence } a = 11.31, \gamma = 74^{\circ}44'37''$$

$$\beta = 62^{\circ}30'22''$$

Question#9

$a = 319$, $c = 168$, and $\gamma = 110^{\circ}22'$

solution:

$$\because \alpha + \beta + \gamma = 180^{\circ}$$

$$\Rightarrow \beta + \alpha = 180^{\circ} - \gamma = \beta + \alpha = 180^{\circ} - 110^{\circ}22'$$

$$\Rightarrow \beta + \alpha = 69^{\circ}38' \rightarrow (i)$$

$$\frac{\alpha + \beta}{2} = 34^{\circ}49'$$

by law of tangent

$$\frac{\tan \frac{\alpha - \beta}{2}}{\tan \frac{\alpha + \beta}{2}} = \frac{a - b}{a + b}$$

$$\tan \frac{\alpha - \beta}{2} = \frac{319 - 168}{319 + 168} \tan 34^{\circ}49'$$

$$\tan \frac{\alpha - \beta}{2} = \frac{151}{487} (0.695) = 0.215$$

$$\Rightarrow \frac{\alpha - \beta}{2} = \tan^{-1}(0.215) = -6^{\circ}7'7''$$

$$\alpha - \beta = 24^{\circ}16'3'' \rightarrow (iii)$$

$$\text{By (i) + (ii)} \Rightarrow 2\alpha = 93^{\circ}54'3''$$

$$\Rightarrow \alpha = 46^{\circ}57'1'' \text{ put in (i)}$$

$$46^{\circ}57'1'' + \beta = 137^{\circ}15'$$

$$\beta = 22^{\circ}40'58''$$

By law of sines

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$\Rightarrow c = \frac{a \sin\gamma}{\sin\alpha} = \frac{(319)\sin 110^{\circ}22'}{\sin 46^{\circ}57'1''}$$

$$\Rightarrow c = \frac{(319)(0.937)}{0.73} = 409.12$$

Hence

$$c = 409.12, \beta = 22^{\circ}40'58'', \alpha = 59^{\circ}30'$$

Question#10

$a = 61$, $b = 32$, and $\gamma = 59^{\circ}30'$

Solution:

Since,

$$\alpha + \beta + \gamma = 180$$

$$\Rightarrow \beta + \gamma = 180 - \alpha$$

$$= 180 - 59^{\circ}30'$$

$$\Rightarrow \beta + \gamma = 120^{\circ}30' \dots \dots (i)$$

By law of tangent

$$\frac{b - c}{b + c} = \frac{\tan \left(\frac{\beta - \gamma}{2} \right)}{\tan \left(\frac{\beta + \gamma}{2} \right)} \Rightarrow \frac{61 - 32}{61 + 32} = \frac{\tan \left(\frac{\beta - \gamma}{2} \right)}{\tan \left(\frac{120^{\circ}30'}{2} \right)}$$

$$\Rightarrow \frac{29}{93} = \frac{\tan \left(\frac{\beta - \gamma}{2} \right)}{\tan(60^{\circ}15')}$$

$$\Rightarrow 0.3118 = \frac{\tan \left(\frac{\beta - \gamma}{2} \right)}{1.7496}$$

$$\Rightarrow \tan \left(\frac{\beta - \gamma}{2} \right) = 0.3118(1.7496)$$

$$= 0.5455$$

$$\Rightarrow \frac{\beta - \gamma}{2} = \tan^{-1}(0.5455)$$

$$\Rightarrow \frac{\beta - \gamma}{2} = 28^{\circ}37'$$

$$\Rightarrow \beta - \gamma = 57^{\circ}14' \dots \dots (ii)$$

Adding (i) & (ii)

$$\beta + \gamma = 120^{\circ}30'$$

$$\beta - \gamma = 57^{\circ}14'$$

$$2\beta = 177^{\circ}44'$$

$$\Rightarrow \beta = 88^\circ 52'$$

Putting value of α in eq. (i)

$$88^\circ 52' + \gamma = 120^\circ 30'$$

$$\Rightarrow \gamma = 120^\circ 30' - 88^\circ 52'$$

$$\Rightarrow \gamma = 31^\circ 38'$$

Now by law of sine

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha}$$

$$\Rightarrow \frac{32}{\sin 31^\circ 38'} = \frac{36.21}{\sin 59^\circ 30'}$$

$$\Rightarrow c = \frac{32}{\sin 31^\circ 38'} \cdot \sin 59^\circ 30'$$

$$= \frac{36.21}{0.5244} \cdot 0.8616$$

$$\Rightarrow c = 52.57$$

Question#11

Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure 57° . Find the remaining two angles.

Solution:

Let $a : b = 3 : 2$

$$\text{i.e. } \frac{a}{b} = \frac{3}{2}$$

$$a = \frac{3}{2} b$$

$$\gamma = 57^\circ$$

Since,

$$\alpha + \beta + \gamma = 180$$

$$\Rightarrow \alpha + \beta = 180 - \gamma$$

$$= 180^\circ - 57^\circ$$

$$\Rightarrow \alpha + \beta = 123^\circ \dots \dots (i)$$

By law of tangent

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow \frac{\frac{3}{2}b-b}{\frac{3}{2}b+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{123^\circ}{2}\right)}$$

$$\Rightarrow \frac{\frac{1}{2}b}{\frac{5}{2}b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan(61^\circ 30')}$$

$$\Rightarrow \frac{1}{5} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{1.8418}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{5}(1.8418)$$

$$= 0.3684$$

$$\Rightarrow \frac{\alpha-\beta}{2} = \tan^{-1}(0.3684)$$

$$\Rightarrow \frac{\alpha-\beta}{2} = 20^\circ 13'$$

$$\Rightarrow \alpha - \beta = 40^\circ 26' \dots \dots (ii)$$

Adding (i) & (ii)

$$\alpha + \beta = 123^\circ$$

$$\alpha - \beta = 40^\circ 26'$$

$$2\alpha = 163^\circ 27'$$

$$\Rightarrow \alpha = 81^\circ 44'$$

Putting value of α in eq. (i)

$$81^\circ 44' + \beta = 123^\circ$$

$$\Rightarrow \beta = 123^\circ - 81^\circ 44'$$

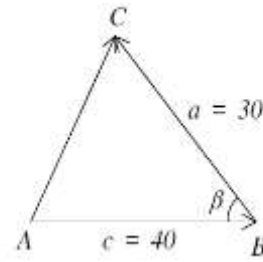
$$\beta = 41^\circ 16'$$

Question#12

Two forces of 40 N and 30 N are represented by \overline{AB} and \overline{BC} which are inclined at an angle of $147^\circ 25''$. Find \overline{AC} , the resultant of \overline{AB} and \overline{BC}

Solution:

Since



$$\overline{AB} = c = 40N$$

$$\overline{BC} = a = 30N$$

$$m\angle B = \beta = 147^\circ 25''$$

$$\overline{AC} = b = ?$$

By law of cosine

$$b^2 = c^2 + a^2 - 2ca \cos \alpha$$

$$= (40)^2 + (30)^2 - 2(40)(30)\cos 147^\circ 25''$$

$$= 1600 + 900 - 2400(-0.8426)$$

$$= 4522.26$$

$$\Rightarrow b = \sqrt{4522.26} = 67.248$$

$$\overline{AC} = 67.248N$$

Case III

When measure of three sides are given

In this case we can use

- The law of cosine
- The half angle formulas

Exercise 12.6

Solve the following triangles, in which

Question#1

$$a = 7, b = 7, \text{ and } c = 9$$

Solution:

$$s = \frac{a+b+c}{2} = \frac{7+7+9}{2} = \frac{23}{2} = 11.5$$

Now,

$$\cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(11.5)(11.5-7)}{(7)(9)}} = \sqrt{\frac{(11.5)(11.5-7)}{63}} = \sqrt{0.821} = 0.906$$

$$\frac{\alpha}{2} = \cos^{-1}(0.906) = 24.99 \approx 25$$

$$\Rightarrow \alpha = 2(25)$$

$$\Rightarrow \alpha = 50^\circ$$

Now,

$$\cos \frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$= \sqrt{\frac{(11.5)(11.5-7)}{(9)(7)}} = \sqrt{\frac{(11.5)(11.5-7)}{63}} = \sqrt{0.821} = 0.906$$

$$\frac{\beta}{2} = \cos^{-1}(0.906) = 24.99 \approx 25$$

$$\Rightarrow \beta = 2(25)$$

$$\Rightarrow \beta = 50^\circ$$

Now,

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 50^\circ - 50^\circ$$

$$\Rightarrow \gamma = 80^\circ$$

Question#2

$$a = 32, b = 40, \text{ and } c = 66$$

Solution:

$$s = \frac{a+b+c}{2} = \frac{32+40+66}{2} = \frac{138}{2} = 69$$

Now,

$$\cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(69)(69-32)}{(40)(66)}} = \sqrt{\frac{(69)(37)}{2640}} = \sqrt{0.9670} = 0.9833$$

$$\frac{\alpha}{2} = \cos^{-1}(0.9833) = 10.48 \approx 11$$

$$\Rightarrow \alpha = 2(11)$$

$$\Rightarrow \alpha = 22^\circ$$

Now,

$$\cos \frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$= \sqrt{\frac{(69)(69-40)}{(32)(66)}} = \sqrt{\frac{(69)(29)}{2001}} = \sqrt{0.9474} = 0.9733$$

$$\frac{\beta}{2} = \cos^{-1}(0.9733) = 13.26 \approx 14$$

$$\Rightarrow \beta = 2(14)$$

$$\Rightarrow \beta = 28^\circ$$

Now,

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 22^\circ - 28^\circ$$

$$\Rightarrow \gamma = 130^\circ$$

Question#3

$$a = 28.3, b = 31.7, \text{ and } c = 42.8$$

Solution:

$$s = \frac{a+b+c}{2} = \frac{28.3+31.7+42.8}{2} = \frac{102.8}{2} = 51.4$$

Now,

$$\cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(51.4)(51.4-28.3)}{(31.7)(42.8)}} = \sqrt{\frac{(51.4)(23.1)}{2640}} = \sqrt{0.8770} =$$

$$0.9364$$

$$\frac{\alpha}{2} = \cos^{-1}(0.9364) = 20.54 \approx 21$$

$$\Rightarrow \alpha = 2(21)$$

$$\Rightarrow \alpha = 42^\circ$$

Now,

$$\cos \frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$= \sqrt{\frac{(51.4)(51.4-31.7)}{(28.3)(42.8)}} = \sqrt{\frac{(51.4)(19.7)}{1211.24}}$$

$$= \sqrt{0.8359} = 0.9142$$

$$\frac{\beta}{2} = \cos^{-1}(0.9142) = 23.90 \approx 24$$

$$\Rightarrow \beta = 2(24)$$

$$\Rightarrow \beta = 48^\circ$$

Now,

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 42^\circ - 48^\circ$$

$$\Rightarrow \gamma = 90^\circ$$

Question#4

$$a = 31.9, b = 56.31, \text{ and } c = 40.27$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(56.31)^2 + (42.8)^2 - (31.9)^2}{2(56.31)(42.8)}$$

$$= \frac{3170.81 + 1621.67 - 1017.61}{4}$$

$$\cos \alpha = \frac{3447.88}{4535.21} = 0.8323$$

$$\cos \alpha = \frac{3447.88}{4535.21} = 0.8323$$

$$\Rightarrow \alpha = \cos^{-1}(0.8323) = 33^\circ 39' 51''$$

$$\therefore \cos \beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{(42.8)^2 + (31.9)^2 - (56.31)^2}{2(42.8)(31.9)}$$

$$= \frac{1621.67 + 1017.61 - 3170.81}{2569.2}$$

$$\cos \beta = -0.207$$

$$\Rightarrow \cos \beta = -0.207$$

$$\Rightarrow \beta = \cos^{-1}(-0.207) = 101^\circ 56' 47''$$

$$\therefore \gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 33^\circ 39' 51'' - 101^\circ 56' 47''$$

$$\gamma = 44^\circ 23' 21''$$

$$\text{Hence } \alpha = 33^\circ 39' 51'', \quad \beta = 101^\circ 56' 47''$$

$$\gamma = 44^\circ 23' 21''$$

Question#5

$$a = 4584, b = 5140, \text{ and } c = 3624$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(5140)^2 + (3624)^2 - (4584)^2}{2(5140)(3624)}$$

$$= \frac{26419600 + 13133376 - 21013056}{37254720}$$

$$\cos \alpha = 0.4977$$

$$\Rightarrow \alpha = \cos^{-1}(0.4977) \approx 60^\circ 9' 7''$$

$$\therefore \cos \beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{(3624)^2 + (4584)^2 - (5140)^2}{2(3624)(4584)}$$

$$= \frac{13133376 + 21013056 - 26419600}{33224832}$$

$$\Rightarrow \cos \beta = 0.233$$

$$\begin{aligned}\Rightarrow \beta &= \cos^{-1}(0.233) = 76^{\circ}30'33'' \\ \therefore \gamma &= 180^{\circ} - 60^{\circ}9'7'' - 76^{\circ}30'33'' \\ \therefore \gamma &= 180^{\circ} - 60^{\circ}9'7'' - 76^{\circ}30'33'' \\ \gamma &= 43^{\circ}20'22''\end{aligned}$$

Hence

$$\alpha = 60^{\circ}9'7'', \beta = 76^{\circ}30'33'', \text{ and } \gamma = 43^{\circ}20'22''$$

Question#6

Find the smallest angle of the triangle ABC, when $a = 37.34$, $b = 3.24$, $c = 35.06$.

Solution:

Since angle opposite to the smallest side is smallest angle So,

$$\begin{aligned}\cos\beta &= \frac{c^2 + a^2 - b^2}{2ca} \\ &= \frac{(35.06)^2 + (37.4)^2 - (3.24)^2}{2(35.06)(37.34)} \\ &= \frac{2612.98}{2618.28} = 0.9979\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}(0.9979) = 3^{\circ}37'27'' \\ \Rightarrow \beta &= 3^{\circ}37'27''\end{aligned}$$

Question#7

Find the measure of the greatest angle, if sides of the triangle are 16, 20, 33.

Solution:

$$\text{Let } a = 16, b = 20, c = 33$$

Since angle opposite to the largest side is largest.

Here

$$c =$$

33 is the largest side therefore γ is the greatest angle

$$\begin{aligned}\cos\gamma &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{(16)^2 + (20)^2 - (33)^2}{2(16)(20)} \\ &= \frac{256 + 400 - 1089}{640} \\ &= -\frac{433}{640}\end{aligned}$$

$$\begin{aligned}\cos\gamma &= -0.68 \Rightarrow \gamma = \cos^{-1}(-0.68) \\ \Rightarrow \gamma &= 132^{\circ}50'37''\end{aligned}$$

Question#8

The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Prove that the greatest angle of the triangle is 120° .

Solution:

Let

$$a = x^2 + x + 1, \quad b = 2x + 1 \quad c = x^2 - 1$$

Since $a = x^2 + x + 1$ is the largest side therefore α is the greatest angle

Now,

$$\begin{aligned}\cos\alpha &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x-1)} \\ &= \frac{4x^2+4x+1+x^4-2x^2+1-(x^4+x^2+1+2x^3+2x+2x^2)}{2(2x^3+x^2-2x-1)} \\ &= \frac{x^4+2x^2+4x+2-x^4-x^2-1-2x^3-2x-2x^2}{2(2x^3+x^2-2x-1)}\end{aligned}$$

$$\begin{aligned}&= \frac{-2x^3-x^2+2x+1}{2(2x^3+x^2-2x-1)} \\ &= \frac{-(2x^3+x^2-2x-1)}{2(2x^3+x^2-2x-1)} \\ &\Rightarrow \cos\alpha = -\frac{1}{2}\end{aligned}$$

$$\Rightarrow \alpha = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \alpha = 120^{\circ}$$

Question#9

The measures of side of a triangular plot are 413, 214 and 375 meters. Find the measures of the corner angles of the plot

Solution:

$$\text{Let } a = 413, b = 214, c = 375$$

Since,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(214)^2 + (375)^2 - (413)^2}{2(214)(375)}$$

$$= \frac{45796 + 140625 - 170569}{160500}$$

$$\cos\alpha = \frac{15852}{160500} = 0.987$$

$$\Rightarrow \alpha = \cos^{-1}(0.987)$$

$$\Rightarrow \alpha = 84^{\circ}19'54''$$

$$\therefore \cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{(375)^2 + (413)^2 - (214)^2}{2(375)(413)}$$

$$\cos\beta = \frac{265398}{309750} = 0.856$$

$$\beta = \cos^{-1}(0.856) = 31^{\circ}2'21''$$

$$\therefore \gamma = 64^{\circ}37'44''$$

Hence

$$\alpha = 84^{\circ}19'54'', \quad \beta = 31^{\circ}2'21'', \quad \gamma = 64^{\circ}37'44''$$

Question#10

Three villages A, B and C are connected by straight roads 6 km, 9 km and 13 km. What angles these roads make with each other?

Solution:

$$\text{Let } a = 6, b = 9, c = 13$$

Now,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(9)^2 + (13)^2 - (6)^2}{2(9)(13)}$$

$$= \frac{214}{234} = 0.9145$$

$$\Rightarrow \cos\alpha = 0.9145$$

$$\Rightarrow \alpha = \cos^{-1}(0.9145)$$

$$\Rightarrow \alpha = 23^{\circ}52'$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{(13)^2 + (6)^2 - (9)^2}{2(13)(6)} = \frac{124}{156} = 0.7948$$

$$\beta = \cos^{-1}(0.7948) = 37^\circ 21'$$

$$\Rightarrow \beta = 37^\circ 21'$$

Now, $\alpha + \beta + \gamma = 180^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 23^\circ 52' - 37^\circ 21'$$

$$\Rightarrow \gamma = 118^\circ 47'$$

Thus road of villages make angle $23^\circ 52'$, $37^\circ 21'$ and $118^\circ 47'$ with each other

Area of triangles

Case I

Area of triangle on terms of the measures of two sides and their included angle

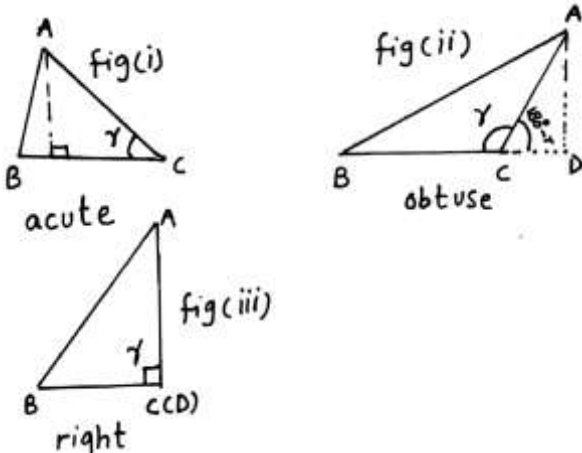
“with usual notations, prove that

Area of triangle

$$\begin{aligned} \Delta ABC &= \frac{1}{2} bcsin\alpha = \frac{1}{2} casin\beta \\ &= \frac{1}{2} absiny \end{aligned}$$

Proof:

Consider three different kinds of ΔABC with $m\angle C = \gamma$ as



In fig(i) $\frac{AD}{AC} = siny \rightarrow (i)$

in fig(ii) $\frac{AD}{AC} = \sin(180^\circ - \gamma) \rightarrow (ii)$

from (i), (ii) and (iii) it is clear that

$$\Rightarrow \frac{AD}{AC} = siny$$

$$\Rightarrow AD = bcsiny$$

Now

$$\text{area of } \Delta ABC = \frac{1}{2} (\text{base})(\text{altitude})$$

$$= \frac{1}{2} (BC)(AD)$$

$$\Delta = \frac{1}{2} absiny$$

Similarly

$$\Delta = \frac{1}{2} bcsin\alpha, \quad \frac{1}{2} casin\beta$$

Case-II

Area of triangle in terms of one side two angles

“in any triangle ΔABC , with usual notation,

Prove that

$$\begin{aligned} \text{Area of triangle} &= \frac{a^2 \sin\beta \sin\gamma}{2 \sin\alpha} \\ &= \frac{b^2 \sin\gamma \sin\alpha}{2 \sin\beta} = \frac{c^2 \sin\alpha \sin\beta}{2 \sin\gamma} \end{aligned}$$

Proof:

$$\begin{aligned} \frac{a}{\sin\alpha} &= \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \\ \Rightarrow a &= \frac{c \sin\alpha}{\sin\gamma} \quad \text{and} \quad b = \frac{c \sin\beta}{\sin\gamma} \end{aligned}$$

Now

$$\begin{aligned} \Delta &= \frac{1}{2} absiny \\ &= \frac{1}{2} \frac{c \sin\alpha}{\sin\gamma} \cdot \frac{c \sin\beta}{\sin\gamma} \cdot \sin\gamma \\ &= \frac{1}{2} \frac{c^2 \sin\alpha \sin\beta}{\sin\gamma} \end{aligned}$$

Similarly

$$\Delta = \frac{1}{2} \frac{a^2 \sin\beta \sin\gamma}{\sin\alpha}, \quad \Delta = \frac{1}{2} \frac{b^2 \sin\alpha \sin\gamma}{\sin\beta}$$

Case III

Area of triangle in terms of the measures of its sides.

Hero's formula

In a ΔABC with usual notation

Prove that

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Where } s = \frac{a+b+c}{2}$$

Proof:

$$\Delta = \frac{1}{2} bcsin\alpha$$

$$\therefore \sin 2\left(\frac{\alpha}{2}\right) = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

So

$$\begin{aligned}\Delta &= \frac{1}{2}bc \left(2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right) \\ &= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}} \\ \Delta &= bc \sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}} \\ \Delta &= \frac{bc \sqrt{s(s-a)(s-b)(s-c)}}{bc} \\ \Delta &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$

Which is called Hero's formula.

Exercise 12.7

Question#1

Find the area of the triangle ABC, given two sides and their included angle:

(i).

$a = 200$, $b = 120$, and $\gamma = 150^\circ$

Solution:

$$\text{Area of triangle } ABC = \Delta = \frac{1}{2} a b \sin \gamma$$

$$= \frac{1}{2} (200)(120) \sin 150^\circ$$

$$= \frac{1}{2} (200)(120)(0.5)$$

$$= 6000 \text{ sq. unit}$$

(ii). $b = 37$, $c = 45$, and $\alpha = 30^\circ 50'$

Solution:

$$\begin{aligned}\because \Delta &= \frac{1}{2} b c \sin \alpha \\ &= \frac{1}{2} (37)(45) (\sin 30^\circ 50') \\ \Delta &= 426.69\end{aligned}$$

(iii). $b = 4.33$, $c = 9.25$, and $\gamma = 56^\circ 44'$

Solution:

$$\begin{aligned}\because \Delta &= \frac{1}{2} a b \sin \gamma \\ &= \frac{1}{2} (4.33)(9.25) (\sin 56^\circ 44') \\ &= \frac{1}{2} (4.33)(9.25)(0.8361)\end{aligned}$$

$$\Delta = 16.74 \text{ sq. units}$$

Question#2

(i).

$b = 25.4$, $\gamma = 36^\circ 41'$, and $\alpha = 45^\circ 17'$

Solution:

Since in any triangle

$$\alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 45^\circ 17' - 36^\circ 41' \Rightarrow$$

$$\beta = 98^\circ 2'$$

$$\text{Now Area of triangle} = \frac{b^2 \sin \gamma \sin \alpha}{2 \sin \beta}$$

$$= \frac{(25.4)^2 \sin 36^\circ 41' \cdot \sin 45^\circ 17'}{2 \sin 98^\circ 2'}$$

$$= \frac{(645.16)(0.5974)(0.2924)}{2(0.9902)}$$

$$= 138.293 \text{ sq. unit}$$

(ii). $c = 32$, $\alpha = 47^\circ 24'$, and $\beta = 70^\circ 16'$

Solution:

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \gamma = 180^\circ - \beta - \alpha$$

$$180^\circ - 70^\circ 16' - 47^\circ 24'$$

$$\gamma = 62^\circ 20'$$

$$\text{Area of } \Delta ABC = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}$$

$$= \frac{(32)^2 \sin 47^\circ 24' \sin 70^\circ 16'}{2 \sin \gamma}$$

$$= \frac{(1024)(0.7361)(0.9413)}{2(0.8857)}$$

$$\Delta ABC = \frac{709.52}{1.7714} = 400.54 \text{ sq. units}$$

(iii). $b = 8.2$, $\alpha = 83^\circ 42'$, and $\gamma = 37^\circ 12'$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 83^\circ 42' - 37^\circ 12'$$

$$\beta = 59^\circ 6'$$

$$\because \text{Area of } \Delta ABC = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha}$$

$$= \frac{(4.8)^2 \sin 59^\circ 6' \sin 27^\circ 12'}{2 \sin 83^\circ 42'}$$

$$= \frac{(23.04)(0.8581)(0.6046)}{2(0.99)}$$

$$= \frac{11.953}{1.986} = 6.02 \text{ sq. units}$$

Question#3

Find the area of the triangle ABC, given three sides:

(i).

$a = 18$, $b = 24$, and $c = 30$

Solution:

$$s = \frac{a+b+c}{2} = \frac{18+24+30}{2} = \frac{72}{2} = 36$$

$$s - a = 36 - 18 = 18$$

$$s - b = 36 - 24 = 12$$

$$s - c = 36 - 30 = 6$$

So,

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{36(18)(12)(6)}$$

$$= \sqrt{46656}$$

$$= 216 \text{ sq. unit}$$

(ii).

$$a = 524, b = 276, \text{ and } c = 315$$

solution:

$$\therefore s = \frac{a+b+c}{2} = \frac{524+276+315}{2}$$

$$s = \frac{1165}{2} = 557.25$$

$$s - a = 557.25 - 524 = 33.5$$

$$s - b = 557.25 - 276 = 281.5$$

$$s - c = 557.25 - 315 = 242.5$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(557.25)(33.5)(281.5)(242.5)}$$

$$= \sqrt{127491.09}$$

$$\Delta = \sqrt{35705.89} \text{ sq. units.}$$

(iii).

$$a = 32.65, b = 42.81, \text{ and } c = 64.92$$

solution:

$$\therefore s = \frac{a+b+c}{2} = \frac{32.65+42.81+64.92}{2}$$

$$s = \frac{140.38}{2} = 70.19$$

Now

$$s - a = 70.19 - 32.65 = 37.54$$

$$s - b = 70.19 - 42.81 = 27.38$$

$$s - c = 70.19 - 64.92 = 5.27$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(70.19)(37.54)(27.38)(5.27)}$$

$$\Delta = \sqrt{380201.28} = 616.6 \text{ sq. units.}$$

Question#4

The area of triangle is 2437. If $a = 79$, and $c = 97$, then find angle β .

Solution:

$$\text{Area of triangle} = 2437$$

$$a = 79, c = 97, \beta = ?$$

$$\text{Area of triangle} = \frac{1}{2} a c \sin \beta$$

$$\Rightarrow 2437 = \frac{1}{2} (79)(97) \sin \beta$$

$$\Rightarrow 2437 = \frac{1}{2} (79)(97) \sin \beta$$

$$\Rightarrow 2437 = 3831.5 \sin \beta$$

$$\Rightarrow \frac{2437}{3831.5} = \sin \beta$$

$$\Rightarrow 0.636 = \sin \beta$$

$$\Rightarrow \beta = \sin^{-1}(0.636)$$

$$\Rightarrow \beta = 39^\circ 30'$$

Question#5

The area of triangle is 121.34. If $\alpha = 32^\circ 15'$, $\beta = 65^\circ 37'$ then find c and angle γ .

Solution:**Solution:**

$$\text{Area of triangle} = 121.34$$

$$\alpha = 32^\circ 15', \beta = 65^\circ 37' \quad c = ? \quad \gamma = ?$$

Since in any triangle

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 32^\circ 15' - 65^\circ 37' \Rightarrow$$

$$\gamma = 82^\circ 8'$$

$$\text{Now Area of triangle} = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}$$

$$\Rightarrow 121.34 =$$

$$\Rightarrow 121.34 = \frac{c^2 \sin 32^\circ 15' \cdot \sin 65^\circ 37'}{2 \sin 82^\circ 8'}$$

$$\Rightarrow 121.34 = \frac{c^2 (0.5336)(0.9108)}{2(0.9906)}$$

$$= c^2 (0.2453)$$

$$\Rightarrow \frac{121.34}{0.2453} = c^2$$

$$\Rightarrow c^2 = 494.66$$

$$\Rightarrow c = \sqrt{494.66}$$

$$\Rightarrow c = 22.24$$

Question#6

One side of a triangular garden is 30 m. If its two corner angles are $22 \frac{1}{2}^\circ$, and $112 \frac{1}{2}^\circ$ find the cost of planting the grass at the rate of Rs. 5 per square meter.

Solution:

Suppose that ABC be a triangular garden such that

$$a = 30 \text{ m}, \beta = 22^\circ \frac{1}{2}, \gamma = 112^\circ \frac{1}{2}$$

Since in any triangle

$$\alpha + \beta + \gamma = 180^\circ$$

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 22^\circ \frac{1}{2} - 112^\circ \frac{1}{2} \Rightarrow$$

$$\alpha = 45^\circ$$

$$\text{Now Area of triangle} = \frac{c^2 \sin \beta \sin \gamma}{2 \sin \alpha}$$

$$= \frac{a^2 \sin 22^\circ \frac{1}{2} \cdot \sin 112^\circ \frac{1}{2}}{2 \sin 45^\circ}$$

$$(30)^2 (0.3827)(0.9329)$$

$$2(0.7072)$$

$$= 224.99 \text{ m}^2 \approx 225 \text{ m}^2$$

Since the cost of planting the grass per square meter = 5 Rs.

$$\text{Therefore the cost of planting } 225 \text{ m}^2 = 5 \times 225 \text{ m}^2 = 1125 \text{ m}^2$$

Circles connected with triangle:

We have three kinds of circles related to a triangle.

(i) Circum-circle

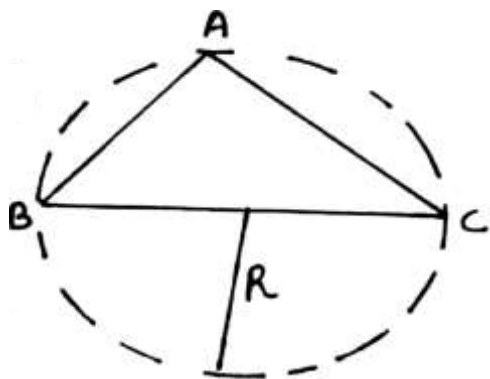
(ii) in-circle

(iii) Ex-circle

(i) **Circum Circle:**

The circle passing through the vertices of a triangle is called a circum-circle.

Its centre is called circum-centre and its radius is denoted by R.

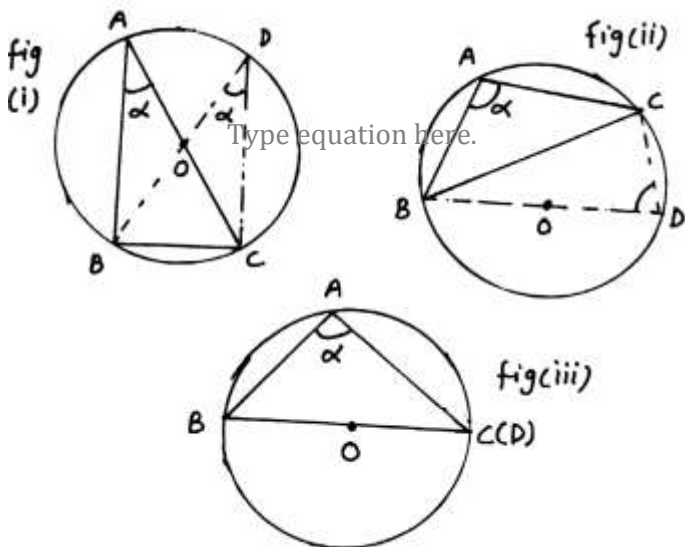


Prove that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

With usual notations.

Proof:



In fig (i) in right triangle ΔBCD

$$\frac{m\overline{Bc}}{m\overline{BD}} = \sin\alpha \rightarrow (I)$$

In fig(ii) $m\angle BDC + m\angle BAC = 180^\circ$
 (\because sum of opposite angle of cyclical quadrilateral)
 $= 180^\circ$

$$\Rightarrow m\angle BDC = 180^\circ - m\angle A = 180^\circ - \alpha$$

In right triangle BCD

$$\frac{m\overline{Bc}}{m\overline{BD}} = \sin m\angle BDC$$

$$\Rightarrow \frac{m\overline{Bc}}{m\overline{BD}} = \sin(180^\circ - \alpha) = \sin\alpha \rightarrow (II)$$

In fig (iii) clearly

$$\frac{m\overline{Bc}}{m\overline{BD}} = 1 = \sin 90^\circ = \sin\alpha \rightarrow (III)$$

From (I), (II), (III)

$$\frac{m\overline{Bc}}{m\overline{BD}} = \sin\alpha \Rightarrow \sin\alpha = \frac{a}{2R}$$

Where $\overline{Bc} = a, \overline{BD} = 2R$

$$\Rightarrow R = \frac{a}{2\sin\alpha}$$

Similarly

$$R = \frac{b}{2\sin\beta} \text{ and } R = \frac{c}{2\sin\gamma}$$

Deduction of law of sines:

We know that

$$R = \frac{a}{2\sin\alpha}, R = \frac{b}{2\sin\beta} \text{ and } R = \frac{c}{2\sin\gamma}$$

$$2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \text{ which is the law of sines.}$$

(b) prove that

$$R = \frac{abc}{4\Delta}$$

Proof:

We know that

$$\begin{aligned} R &= \frac{a}{2\sin\alpha} \\ \Rightarrow R &= \frac{a}{2 \cdot 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}} \\ (\because \sin\alpha &= 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}) \\ &= \frac{a}{4 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}} \\ &= \frac{a}{4 \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-c)}{bc}}} \\ R &= \frac{a}{4 \sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}}} \\ R &= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \\ R &= \frac{abc}{4\Delta} \text{ hence proved.} \end{aligned}$$

Encircle:

a circle drawn inside the triangle and touching its three sides is called an incircle.

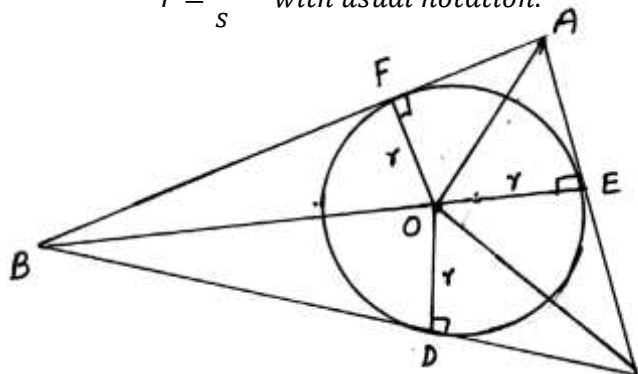
- Its centre is called in-centre and its radius is called in-radius. Denoted by r.

Important Note:

- Circum –center is the point of intersection of the perpendicular bisectors of sides of a triangle
- In centre is the point of intersection of the angle bisection of a triangle.

Prove that:

$$r = \frac{\Delta}{s} \text{ with usual notation.}$$



In triangle ABC, OD, OE and OF are \perp ar to BC, AC And AB respectively.

Then from fig

$$\text{area of } \Delta ABC = \text{area of } \Delta OBC + \text{area of } \Delta OCA + \text{area of } \Delta OAB$$

$$\Rightarrow \Delta = \frac{1}{2}BC \times OD + \frac{1}{2}CA \times OE + \frac{1}{2}AB \times OF$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

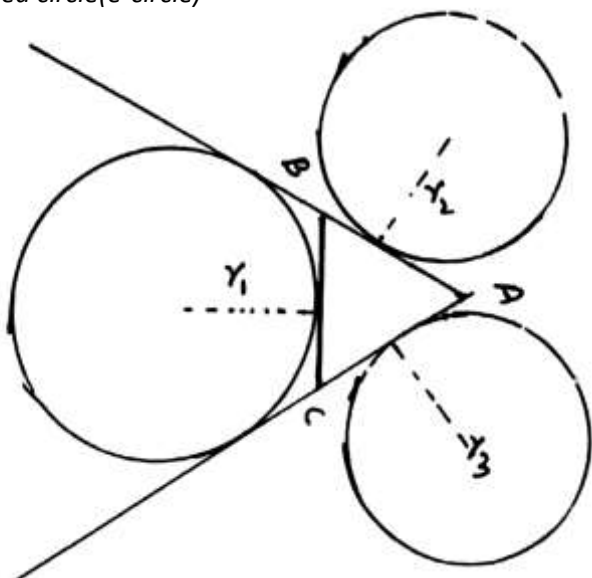
$$= \frac{1}{2}r(a + b + c)$$

$$= \frac{1}{2}r(2s)$$

$$\Delta = rs \Rightarrow r = \frac{\Delta}{s}$$

Escribed circles

A circle which touches one side of a triangle externally and the other two produced sides internally is called escribed circle(e-circle)

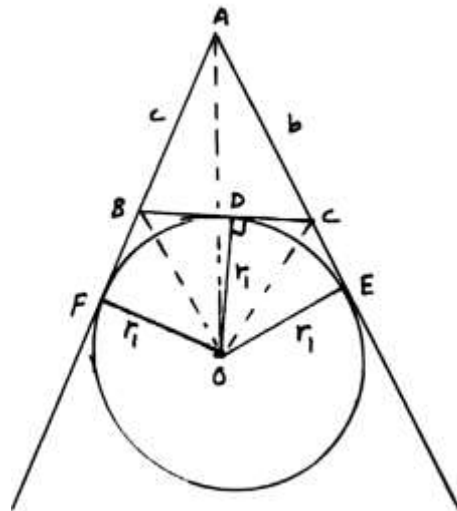


Important note:

- The radius of escribed circle is called escribed radius or e- radius.
- The centre of escribed circle is called escribed centre or e-centre.
- Escribed circles are of three kinds.
- Circle drawn opposite to vertex A has radius r_1
- Circle drawn opposite to vertex B has radius r_2
- Circle drawn opposite to vertex C has radius r_3

Prove that

$$r_1 = \frac{\Delta}{s - a}, r_2 = \frac{\Delta}{s - b}, r_3 = \frac{\Delta}{s - c}$$



Let O be the centre of established circle. Draw \perp ar D, E, F then

$$\Delta ABC = \Delta OAB + \Delta AOC - \Delta OBC$$

$$= \frac{1}{2}(AB)(OF) + \frac{1}{2}(AC)(OE) - \frac{1}{2}(BC)(OD)$$

$$\Delta = \frac{1}{2}Cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$

$$= \frac{1}{2}r_1(c + b - a)$$

$$= \frac{1}{2}r_1(b + c - a)$$

$$= \frac{1}{2}r_1(2s - a - a)$$

$$= \frac{1}{2}r_1(2s - 2a)$$

$$\frac{1}{2}2r_1(s - a)$$

$$\Delta = r_1(s - a)$$

$$\Rightarrow r_1 = \frac{\Delta}{s - a}, \text{ hence proved.}$$

Similarly

$$r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

Exercise 12.8

Question#1

(i).

$$r = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

Solution:

$$R.H.S = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$= 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= 4R \sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{(bc)(ac)(ab)}}$$

$$= 4R \sqrt{\frac{(s-a)^2 (s-b)^2 (s-c)^2}{a^2 b^2 c^2}}$$

$$= 4R \frac{(s-a)(s-b)(s-c)}{abc}$$

$$= 4R \left(\frac{abc}{4\Delta} \right) \frac{(s-a)(s-b)(s-c)}{abc} \quad \because R = \frac{abc}{4\Delta}$$

$$= \frac{(s-a)(s-b)(s-c)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{s\Delta}$$

$$= \frac{\Delta^2}{s\Delta} \quad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta}{s}$$

$$r = L.H.S$$

(ii).

$$s = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$R.H.S = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$= 4R \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}}$$

$$= 4R \sqrt{\frac{s \cdot s^2 (s-a)(s-b)(s-c)}{(bc)(ac)(ab)}}$$

$$= 4R \sqrt{\frac{s^2 \Delta^2}{a^2 b^2 c^2}}$$

$$= 4R \frac{s\Delta}{abc}$$

$$= 4R \left(\frac{abc}{4\Delta} \right) \frac{s\Delta}{abc} \quad \because R = \frac{abc}{4\Delta}$$

$$= s$$

Question#2

Show that:

$$r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} =$$

$$c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$$

Solution:**We take**

$$a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2}$$

$$= a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \frac{1}{\cos \frac{\alpha}{2}}$$

$$= a \sqrt{\frac{(s-b)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \frac{1}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{bc}{s(s-a)}}$$

$$= a \sqrt{\frac{(s-a)(s-c)(s-a)(s-b)(bc)}{(ac)(ab)s(s-a)}}$$

$$= a \sqrt{\frac{(s-a)(s-b)(s-c)}{(ac)(ab)s(s-a)}}$$

$$= a \sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2 s^2}}$$

$$= a \sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2 s^2}}$$

$$= a \frac{\sqrt{s(s-a)(s-b)(s-c)}}{as}$$

$$= \frac{\Delta}{s} = r$$

$$\Rightarrow r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} \dots \dots (i)$$

Also

$$R.H.S = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2}$$

$$= b \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{ac}{s(s-b)}}$$

$$= b \sqrt{\frac{(s-a)(s-b)^2 (s-c)}{s}}$$

$$= \frac{b}{b} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}}$$

$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

$$b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} \dots \dots (ii)$$

$$\text{Also } L.H.S = c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$$

$$c \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{ab}{s(s-c)}}$$

$$= c \sqrt{\frac{(s-a)(s-c)^2 (s-b)}{s}}$$

$$= \frac{c}{c} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}}$$

$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

$$c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2} \dots \dots (iii)$$

From (i), (ii), and (iii)

$$r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} = c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$$

Question#3

(i).

$$r_1 = 4R \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution:

$$\begin{aligned} R.H.S &= 4R \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\ &= 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}} \\ &= 4R \sqrt{\frac{(s-b)(s-c)s(s-b)s(s-c)}{(bc)(ac)(ab)}} \\ &= 4R \sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}} \\ &= 4R \frac{s(s-b)(s-c)}{abc} \\ &= 4R \left(\frac{abc}{4\Delta} \right) \frac{s(s-b)(s-c)}{abc} \cdot \frac{(s-a)}{(s-a)} \quad \because R = \frac{abc}{4\Delta} \\ &= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)} \\ &= \frac{\Delta^2}{\Delta(s-a)} \quad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{\Delta}{(s-a)} \end{aligned}$$

$$r_1 = L.H.S$$

(ii).

$$r_2 = 4R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution:

$$\begin{aligned} R.H.S &= 4R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2} \\ &= 4R \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-c)}{ab}} \\ &= 4R \sqrt{\frac{(s-a)(s-c)s(s-a)s(s-c)}{(bc)(ac)(ab)}} \\ &= 4R \sqrt{\frac{s^2(s-a)^2(s-c)^2}{a^2b^2c^2}} \\ &= 4R \frac{s(s-a)(s-c)}{abc} \\ &= \frac{abc}{\Delta} \cdot \frac{s(s-a)(s-c)}{abc} = \frac{s(s-a)(s-c)}{\Delta} \\ &= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-b)} = \frac{\Delta^2}{\Delta(s-b)} \\ &= \frac{\Delta}{s-b} = r_2 = L.H.S \end{aligned}$$

Hence proved.

(iii).

$$r_3 = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$$

Solution:

$$\begin{aligned} R.H.S &= 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \\ &= 4R \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \\ &= 4R \sqrt{\frac{(s-a)(s-b)s(s-a)s(s-b)}{(bc)(ac)(ab)}} \end{aligned}$$

$$\begin{aligned} &= 4R \sqrt{\frac{s^2(s-a)^2(s-b)^2}{a^2b^2c^2}} \\ &= 4R \frac{s(s-a)(s-b)}{abc} \\ &= \frac{abc}{\Delta} \cdot \frac{s(s-a)(s-b)}{abc} = \frac{s(s-a)(s-b)}{\Delta} \\ &= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-c)} = \frac{\Delta^2}{\Delta(s-c)} \\ &= \frac{\Delta}{s-c} = r_3 = L.H.S \end{aligned}$$

Question#4

$$(i). r_1 = s \tan \frac{\alpha}{2}$$

Solution:

$$\begin{aligned} R.H.S &= s \tan \frac{\alpha}{2} \\ &= s \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ &= s \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \cdot \frac{s(s-a)}{s(s-a)} \\ &= s \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}} \\ &= s \sqrt{\frac{\Delta^2}{s^2(s-a)^2}} \\ &= s \frac{\Delta}{s(s-a)} = \frac{\Delta}{(s-a)} = r_1 = L.H.S \end{aligned}$$

(ii).

$$r_2 = s \tan \frac{\beta}{2}$$

$$R.H.S = s \tan \frac{\beta}{2}$$

$$\begin{aligned} &= s \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \\ &= s \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \cdot \frac{s(s-b)}{s(s-b)} \\ &= s \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-b)^2}} \\ &= s \sqrt{\frac{\Delta^2}{s^2(s-b)^2}} \\ &= s \frac{\Delta}{s(s-b)} = \frac{\Delta}{(s-b)} = r_2 = L.H.S \end{aligned}$$

(iii).

$$r_3 = s \tan \frac{\gamma}{2}$$

Solution:

$$\begin{aligned} R.H.S &= s \tan \frac{\gamma}{2} \\ &= s \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= s \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \cdot \frac{s(s-c)}{s(s-c)} \\ &= s \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}} \\ &= s \sqrt{\frac{\Delta^2}{s^2(s-c)^2}} \\ &= s \frac{\Delta}{s(s-c)} = \frac{\Delta}{(s-c)} = r_3 = L.H.S \end{aligned}$$

Question#5

$$(i). r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$$

Solution:

$$\begin{aligned}
 L.H.S &= r_1 r_2 + r_2 r_3 + r_3 r_1 \\
 &= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) + \left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right) \\
 &= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)} \\
 &= \Delta^2 \left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} \right) \\
 &= \Delta^2 \left(\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)} \right) \\
 &= \Delta^2 \left(\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)} \right) \\
 &= \Delta^2 \left(\frac{3s-2s}{(s-a)(s-b)(s-c)} \right) \quad \because s = \frac{a+b+c}{2} \\
 &= \Delta^2 \left(\frac{s}{(s-a)(s-b)(s-c)} \cdot \frac{s}{s} \right) \\
 &= \Delta^2 \left(\frac{s^2}{s(s-a)(s-b)(s-c)} \right) \\
 &= \Delta^2 \left(\frac{s^2}{\Delta^2} \right) \\
 &= s^2 = R.H.S
 \end{aligned}$$

$$(ii). r r_1 r_2 r_3 = \Delta^2$$

Solution:

$$\begin{aligned}
 L.H.S &= r r_1 r_2 r_3 \\
 &= \left(\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) \\
 &= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \frac{\Delta^4}{\Delta^2} = \Delta^2 = R.H.S
 \end{aligned}$$

$$(iii). r_1 + r_2 + r_3 - r = 4R$$

Solution:

$$\begin{aligned}
 L.H.S &= r_1 + r_2 + r_3 - r \\
 &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} \\
 &= \Delta \left(\frac{1}{(s-a)} + \frac{1}{(s-b)} + \frac{1}{(s-c)} - \frac{1}{s} \right) \\
 &= \Delta \left(\frac{(s-b)+(s-a)}{(s-a)(s-b)} + \frac{s-(s-c)}{s(s-c)} \right) \\
 &= \Delta \left(\frac{2s-b-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)} \right) \\
 &= \Delta \left(\frac{a+b+c-b-a}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right) \quad \because 2s = a + b + c \\
 &= \Delta \left(\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right) \\
 &= c\Delta \left(\frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)} \right) \\
 &= c\Delta \left(\frac{s(s-c)+(s-a)(s-b)}{s(s-c)(s-a)(s-b)} \right)
 \end{aligned}$$

$$= c\Delta \left(\frac{s^2-sc+s^2-as-bs+ab}{\Delta^2} \right)$$

$$= c \left(\frac{2s^2-s(a+b+c)+ab}{\Delta} \right)$$

$$= c \left(\frac{2s^2-s(2s)+ab}{\Delta} \right)$$

$$= c \left(\frac{2s^2-2s^2+ab}{\Delta} \right)$$

$$= \frac{abc}{\Delta} = 4R = R.H.S$$

$$(iv). r r_1 r_2 r_3 = rs^2$$

Solution:

$$\begin{aligned}
 L.H.S &= r r_1 r_2 r_3 \\
 &= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) \\
 &= \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{s\Delta^3}{s(s-a)(s-b)(s-c)} = \frac{s\Delta^3}{\Delta^2} = s\Delta \\
 s^2 \frac{\Delta}{s} &= s^2 r = rs^2 = R.H.S
 \end{aligned}$$

Question#6

Find r , r_1 , r_2 , r_3 . if measures of the sides of triangle ABC are

$$(i). a = 13, b = 14, \text{ and } c = 15$$

Solution:

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$$s - a = 21 - 13 = 8$$

$$s - b = 21 - 14 = 7$$

$$s - c = 21 - 15 = 6$$

So,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{21(8)(7)(6)}$$

$$\sqrt{7056} = 84$$

Now,

$$R = \frac{abc}{4\Delta}$$

$$= \frac{(13)(14)(15)}{4(84)} = 8.125$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{7} = 12$$

$$r_1 = \frac{\Delta}{s-c} = \frac{84}{6} = 14$$

(ii). $a = 34$, $b = 20$, and $c = 42$

Solution:

$$\therefore R = \frac{abc}{4\Delta}, \quad r = \frac{\Delta}{s}, \quad r_1 = \frac{\Delta}{s-a}$$

$$r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

$$\therefore s = \frac{a+b+c}{2} = \frac{34+20+42}{2} = \frac{96}{2} = 48$$

$$s-a = 48-34, \quad s-b = 48-20 = 28$$

$$s-c = 48-42 = 6$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \sqrt{48(14)(28)(6)} = \sqrt{11896} = 336$$

$$R = \frac{abc}{4\Delta} = \frac{(34)(20)(42)}{4(336)} = 21.25$$

Question#7

Prove that in an equilateral triangle,

(i). $r : R : r_1 = 1 : 2 : 3$

In an equilateral triangle, all the sides of a triangle are equal so $a = b = c$

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$s-a = \frac{3a}{2} - a = \left(\frac{3}{2} - 1\right)a = \frac{1}{2}a$$

Now,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-a)(s-a)}$$

$$= \sqrt{s(s-a)^3}$$

$$= \sqrt{\frac{3a}{2} \left(\frac{1}{2}a\right)^3}$$

$$= \sqrt{\frac{3a}{2} \left(\frac{a^3}{8}\right)}$$

$$= \sqrt{\frac{3a^4}{16}}$$

$$= \frac{\sqrt{3}a^2}{4}$$

Now,

$$r = \frac{\Delta}{s} = \frac{\sqrt{3}a^2/4}{3a/2} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{3a}$$

$$R = \frac{abc}{4\Delta}$$

$$= \frac{a.a.a}{4 \cdot \frac{\sqrt{3}a^2}{4}} = \frac{a}{\sqrt{3}} = \frac{a}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}a}{3}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}a^2/4}{\frac{1}{2}a} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

$$r_1 = \frac{\Delta}{s-c} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

Now,

i) $r : R : r_1 = 1 : 2 : 3$

$$L.H.S = r : R : r_1$$

$$= \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}a}{2}$$

$$" \times ' \text{ by } \frac{2\sqrt{3}}{a}$$

$$= \frac{a}{2\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{a}{\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{\sqrt{3}a}{2} \times \frac{2\sqrt{3}}{a}$$

$$= 1 : 2 : 3 = R.H.S$$

(ii). $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

Solution:

In an equilateral triangle, all the sides of a triangle are equal so $a = b = c$

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$s-a = \frac{3a}{2} - a = \left(\frac{3}{2} - 1\right)a = \frac{1}{2}a$$

Now,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-a)(s-a)}$$

$$= \sqrt{s(s-a)^3}$$

$$= \sqrt{\frac{3a}{2} \left(\frac{1}{2}a\right)^3}$$

$$= \sqrt{\frac{3a}{2} \left(\frac{a^3}{8}\right)}$$

$$= \sqrt{\frac{3a^4}{16}}$$

$$= \frac{\sqrt{3}a^2}{4}$$

Now,

$$r = \frac{\Delta}{s} = \frac{\sqrt{3}a^2/4}{3a/2} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{3a}$$

$$R = \frac{abc}{4\Delta}$$

$$= \frac{a.a.a}{4 \cdot \frac{\sqrt{3}a^2}{4}} = \frac{a}{\sqrt{3}} = \frac{a}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}a}{3}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}a^2/4}{\frac{1}{2}a} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

Now,

$$r : R : r_1 : r_2 : r_3 = \frac{\sqrt{3}a}{6} : \frac{\sqrt{3}a}{3} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2} :$$

$$\frac{\sqrt{3}a}{2}$$

÷ ing by $\sqrt{3}a$

$$r : R : r_1 : r_2 : r_3 = \frac{1}{6} : \frac{1}{3} : \frac{1}{2} : \frac{1}{2} : \frac{1}{2}$$

× ing by 6

$$r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$$

Question#8

$$(i). \Delta = r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

$$= r^2 \frac{1}{\tan \frac{\alpha}{2}} \cdot \frac{1}{\tan \frac{\beta}{2}} \cdot \frac{1}{\tan \frac{\gamma}{2}}$$

$$= r^2 \cdot \frac{1}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$= r^2 \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= r^2 \sqrt{\frac{s^3 (s-a)(s-b)(s-c)}{(s-a)^2 (s-b)^2 (s-c)^2}}$$

$$= r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$= r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}} \cdot \frac{s}{s}$$

$$= r^2 \sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}}$$

$$= r^2 \sqrt{\frac{s^4}{\Delta^2}}$$

$$= r^2 \cdot \frac{s^2}{\Delta}$$

$$= \left(\frac{\Delta}{s}\right)^2 \frac{s^2}{\Delta} \quad \because r = \frac{\Delta}{s}$$

$$\frac{\Delta^2}{s^2} \frac{s^2}{\Delta} = \Delta = L.H.S$$

$$(ii). r = s \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$$

$$R.H.S = s \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$$

$$= s \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= s \sqrt{\frac{(s-a)^2 (s-b)^2 (s-c)^2}{s^3 (s-a)(s-a)(s-b)(s-c)}}$$

$$= s \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}}$$

$$= s \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}} \times \frac{s}{s}$$

$$= s \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}}$$

$$= \frac{s \sqrt{s(s-a)(s-b)(s-c)}}{s^4}$$

$$= \frac{\Delta}{s} = r = L.H.S$$

$$(iii). \Delta = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution:

$$R.H.S = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$= 4Rr \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}}$$

$$= 4Rr \sqrt{\frac{s(s-a) \cdot s(s-b) \cdot s(s-c)}{(bc)(ac)(ab)}}$$

$$= 4Rr \sqrt{\frac{s \cdot s^2 (s-a)(s-b)(s-c)}{(bc)(ac)(ab)}}$$

$$= 4Rr \sqrt{\frac{s^2 \Delta^2}{a^2 b^2 c^2}}$$

$$= 4Rr \frac{s \Delta}{abc}$$

$$= 4Rr \left(\frac{abc}{4\Delta}\right) \left(\frac{\Delta}{s}\right) \frac{s \Delta}{abc} = \Delta$$

L.H.S**Question#9**

(i).

$$\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

Solution:

$$L.H.S = \frac{1}{2rR}$$

$$= \frac{1}{\frac{1}{2} \left(\frac{\Delta}{s}\right) \left(\frac{abc}{4\Delta}\right)} = \frac{4s\Delta}{2\Delta abc} = \frac{2s}{abc} = \frac{a+b+c}{abc} \quad \because 2s =$$

a + b + c

$$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc}$$

$$= \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$$

R.H.S

(ii).

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Solution:

$$R.H.S = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{1}{\frac{\Delta}{s-a}} + \frac{1}{\frac{\Delta}{s-b}} + \frac{1}{\frac{\Delta}{s-c}}$$

$$= \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{s-a+s-b+s-c}{\Delta}$$

$$= \frac{3s-(a+b+c)}{\Delta}$$

$$= \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{\frac{\Delta}{s}} = \frac{1}{r}$$

L.H.S**Question#10**

$$r = \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} = \frac{b \sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}{\cos \frac{\beta}{2}} = \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}}$$

Solution:**We take**

$$\frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} = \frac{a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{bc}{s(s-a)}}$$

$$= \frac{a}{a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

Hence proved.

Also

$$\begin{aligned}
 R.H.S &= \frac{b \sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}{\cos \frac{\beta}{2}} \\
 &= \frac{b \sqrt{\frac{(s-b)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-b)}{bc}}} \\
 &= b \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{ac}{s(s-b)}} \\
 &= \frac{b}{b} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\
 &= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}} \\
 &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S
 \end{aligned}$$

Hence proved.

Also

$$\begin{aligned}
 R.H.S &= \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}} \\
 &= \frac{c \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-c)}{ab}}} \\
 &= c \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{ab}{s(s-c)}} \\
 &= \frac{c}{c} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\
 &= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}} \\
 &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S
 \end{aligned}$$

Hence proved.

Question#11

Prove that: $abc(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta s$

Solution:

$$L.H.S = abc(\sin\alpha + \sin\beta + \sin\gamma)$$

Since,

$$\Delta = \frac{1}{2} abs \sin\gamma = \frac{1}{2} bcs \sin\alpha = \frac{1}{2} cas \sin\beta$$

$$\therefore \sin\gamma = \frac{2\Delta}{ab}, \quad \sin\alpha = \frac{2\Delta}{bc}, \quad \sin\beta = \frac{2\Delta}{ca}$$

$$\text{Thus, } L.H.S = abc \left(\frac{2\Delta}{ab} + \frac{2\Delta}{bc} + \frac{2\Delta}{ca} \right)$$

$$= abc \left(\frac{2\Delta a + 2\Delta b + 2\Delta c}{abc} \right)$$

$$= 2\Delta(a + b + c) = 2\Delta(2s)$$

$$b + c$$

$$\therefore 2s = a +$$

$$= 4\Delta s = L.H.S$$

Question#12

(i).

$$(r_1 + r_2) \tan \frac{\gamma}{2} = c$$

Solution:

$$L.H.S = (r_1 + r_2) \tan \frac{\gamma}{2} = c$$

$$= \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b} \right) \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \left(\frac{\Delta(s-a) + \Delta(s-b)}{(s-a)(s-b)} \right) \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \cdot \frac{s(s-c)}{s(s-c)}$$

$$= \Delta \left(\frac{s-b+s-a}{(s-a)(s-b)} \right) \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}}$$

$$= \Delta \left(\frac{2s-a-b}{(s-a)(s-b)} \right) \sqrt{\frac{\Delta^2}{s^2(s-c)^2}}$$

$$= \Delta \left(\frac{a+b+c-a-b}{(s-a)(s-b)} \right) \frac{\Delta}{s(s-c)}$$

$$= \frac{\Delta^2 c}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2 c}{\Delta^2} = c = R.H.S$$

(ii).

$$(r_3 - r) \cot \frac{\gamma}{2} = c$$

Solution:

$$L.H.S = (r_3 - r) \cot \frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{s-c} + \frac{\Delta}{s} \right) \cdot \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \Delta \left(\frac{1}{s-c} + \frac{1}{s} \right) \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta \left(\frac{s-(s-c)}{s(s-c)} \right) \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta \left(\frac{c}{s(s-c)} \right) \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \cdot \frac{(s-c)}{(s-c)}$$

$$= \Delta \left(\frac{c}{s(s-c)} \right) \sqrt{\frac{s^2(s-c)^2}{s(s-a)(s-b)(s-c)}}$$

$$= \Delta \left(\frac{c}{s(s-c)} \right) \sqrt{\frac{s^2(s-c)^2}{\Delta^2}}$$

$$= \Delta \left(\frac{c}{s(s-c)} \right) \frac{s(s-c)}{\Delta} = c = R.H.S$$