MATHEMATICS 11 INTERMEDIATE PART 1

Bilal Article

Chapter 6.

SEQUENCE AND SERIES

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Contents

Sequence:
A sequence is a function whose domain is subset of natural numbers.
$\boldsymbol{e.g\ 2,4,6,8,\ldots}$

$$\boldsymbol{1,-1,1,-1,1,-1 \quad and}$$

$\boldsymbol{a,(a+d),(a+2d),\ldots etc}$

Real Sequence:
If all member of a sequence are real numbers, then it is called a real sequence.

Finite Sequence:
If the domain of a sequence is a finite set, then the sequence is called infinite sequence.

Note:
The number $a_1, a_2, a_3, \ldots, a_n$ are called terms of a sequence. And here $a_1 = first\ term$
a_2 =2nd term ,…., $a_n = nth$ term

- The last term of the sequence is called its nth term or general term denoted by a_n
- Sequences are also called progressions.
- An infinite sequence has no last term.
- We can find a sequence by putting $n = 1,2,3,\ldots$ in the nth or general term.

Exercise 6.1

Question#1

Write the first four terms of the following sequences, if

$\boldsymbol{(i).\ a_n = 2n-3}$

Solution:

Put $\boldsymbol{n=1}$

$a_1 = 2(1)-3 = 2-3 = -1$

Put $\boldsymbol{n=2}$

$a_2 = 2(2)-3 = 4-3 = 1$

Put $\boldsymbol{n=3}$

$a_3 = 2(3)-3 = 6-3 = 3$

Put $\boldsymbol{n=4}$

$a_4 = 2(4)-3 = 8-3 = 5$

Hence $\boldsymbol{-1,\ 1,3,5}$ ***are the first four term of the sequence.***

$\boldsymbol{(ii).\ a_n = (-1)^n n^2}$

Solution:

Put $\boldsymbol{n=1}$

$a_1 = (-1)^1(1)^2 = (-1)(1) = -1$

Put $\boldsymbol{n=2}$

$a_2 = (-1)^2(2)^2 = (1)(4) = 4$

Put $\boldsymbol{n=3}$

$a_3 = (-1)^3(3)^2 = (-1)(9) = -9$

Put $\boldsymbol{n=4}$

$a_4 = (-1)^4(4)^2 = (-1)(16) = -16$

Hence $\boldsymbol{-1,4,-9,16}$ ***are the first four terms of the sequence.***

$\boldsymbol{(iii).\ a_n = (-1)^n(2n-3)}$

Solution:

$$a_n = (-1)^n(2n-3) \to (i)$$

Put n=1,2,3,4 in (i)

$$a_1 = (-1)^1(2(1)-3) = (-1)(-1) = 1$$
$$a_2 = (-1)^2(2(2)-3) = (1)(4-3) = (1)(1) = 1$$
$$a_3 = (-1)^3(2(3)-3) = (-1)(6-3)$$
$$= (-1)(3) = -3$$
$$a_4 = (-1)^4(2(4)-3) = (1)(8-3) = (1)(5) = 5$$

First four term $1,1,-3,5$

$\boldsymbol{(iv).\ a_n = 3n-1}$

Solution:

$a_n = 3n-5 \to (i)$

Put $n = 1,2,3,4, in\ (i)$

$$a_1 = 3(1)-5 = 3-5 = -2$$
$$a_2 = 3(2)-5 = 6-5 = -1$$
$$a_3 = 3(3)-5 = 9-5 = 4$$
$$a_4 = 3(4)-5 = 12-5 = 7$$

First four terms $-2,1,4,7$

$\boldsymbol{(v).\ a_n = \frac{n}{2n+1}}$

Solution:

$$a_n = \frac{n}{2n+1}$$

Put $n = 1,2,3,4,\ in(i)$

$$a_1 = \frac{1}{2(1)+1} = \frac{1}{2+1} = \frac{1}{3}$$
$$a_2 = \frac{2}{2(2)+1} = \frac{2}{4+1} = \frac{2}{5}$$
$$a_3 = \frac{3}{2(3)+1} = \frac{3}{6+1} = \frac{3}{7}$$
$$a_4 = \frac{4}{2(4)+1} = \frac{4}{8+1} = \frac{4}{9}$$

First four term are $\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9}$

$\boldsymbol{(vi).\ a_n = \frac{1}{2^n}}$

Solution:

$$a_n = \frac{1}{2^n} \to (i)$$

$$put\ n = 1,2,3,4\ \ in\ (i)$$

$$\boldsymbol{a_1 = \frac{1}{2^1} = \frac{1}{2}}$$
$$\boldsymbol{a_2 = \frac{1}{2^2} = \frac{1}{4}}$$
$$\boldsymbol{a_3 = \frac{1}{2^3} = \frac{1}{8}}$$
$$\boldsymbol{a_4 = \frac{1}{2^4} = \frac{1}{16}}$$

First four term are $\boldsymbol{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16}}$

$\boldsymbol{(vii).\ a_n - a_{n-1} = n+2,\ a_1 = 2}$

Solution:

Put $\boldsymbol{n=2}$

$a_1 - a_{2-1} = 2+2$

$\Rightarrow a_2 - a_1 = 4$

$\Rightarrow a_2 = 4 + a_1 = 4+2 = 6 \quad \because\ a_1 = 2$

Put $\boldsymbol{n=3}$

$a_3 - a_{3-1} = 3+2 \Rightarrow 4$

$\Rightarrow a_3 - a_2 = 5$
$\Rightarrow a_3 = 5 + a_2 = 5 + 6 = 11 \quad \because a_2 = 6$
Put $n = 4$
$a_4 - a_{4-1} = 4 + 2 \Rightarrow 6$
$\Rightarrow a_4 - a_3 = 6$
$\Rightarrow a_4 = 6 + a_3 = 6 + 11 = 17 \quad \because a_3 = 11$
Hence $2, 6, 11, 17$ are the first four terms of the sequence.

$(viii).\ a_n = na_{n-1}\ ,\ a_1 = 1$

Solution:
Put $n = 2$
$a_2 = (2)a_{2-1}$
$\Rightarrow a_2 = 2a_1$
$= 2(1) \quad \because a_1 = 1$
$= 2$
Put $n = 3$
$a_3 = (3)a_{3-1}$
$\Rightarrow a_3 = 3a_2$
$= 3(2) \quad \because a_2 = 2$
$= 6$
Put $n = 4$
$a_4 = (4)a_{4-1}$
$\Rightarrow a_4 = 4a_3$
$= 4(6) \quad \because a_3 = 6$
$= 24$
Hence $1, 2, 6, 24$ are the first four terms of the sequence.

$(ix).\ a_n + 1 = (n + 1)a_{n-1}\ ,\ a_1 = 1$

Solution:
Put $n = 2,3,4$
$a_2 = (2 + 1)a_1 = 3a_1 = 3(1) = 3 \quad \because a_1 = 1$
$a_3 = (3 + 1)a_2 = 4a_2 = 4(3) = 12 \quad a_2 = 3$
$a_4 = (4 + 1)a_3 = 5a_3 = 5(12) = 60 \quad a_3 = 12$
First four term are 1,3,12,60

$(x).\ a_n = \frac{1}{a+(n-1)d}$

Solution:
Put $n = 1$
$a_1 = \frac{1}{a+(1-1)d} \Rightarrow a_1 = \frac{1}{a+(0)d} = \frac{1}{a}$
Put $n = 2$
$a_2 = \frac{1}{a+(2-1)d} \Rightarrow a_1 = \frac{1}{a+(1)d} = \frac{1}{a+d}$
Put $n = 3$
$a_3 = \frac{1}{a+(3-1)d} \Rightarrow a_1 = \frac{1}{a+(2)d} = \frac{1}{a+2d}$
Put $n = 4$
$a_4 = \frac{1}{a+(4-1)d} \Rightarrow a_1 = \frac{1}{a+(3)d} = \frac{1}{a+3d}$
Hence, $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$ are the first four terms of the sequence.

Question#2

Find the indicated terms of the following sequences;

$(i).\ 2\ , 6\ , 11\ , 17\ , \dots a_7$

Solution:
We see that the successive difference of the given terms are 4, 5, 6 and conclude that sequence of the differences is $4, 5, 6, 7, 8, 9..$
So
$a_5 = 17 + 7 = 24$,
$a_6 = 24 + 8 = 32$
and $a_7 = 32 + 9 = 41$
Thus, the required term is $a_7 = 41$

$(ii).\ 1\ , 3\ , 12\ , 60\ , \dots a_6$

Solution:
We see that the successive multiplying factor are $3, 4, 5$ and conclude that the sequence of multiplying factors is $3, 4, 5, 6, 7, 8, 9$
So,
$a_5 = 60 \times 6 = 360$
$a_6 = 360 \times 7 = 2520$
Thus, the required term is $a_6 = 2520$

$(iii).\ 1\ , \frac{3}{2}\ , \frac{5}{4}\ , \frac{7}{8}\ , \dots a_7$

Solution:
The successive terms in numerator are $1, 3, 5, 7, \dots$, which are the consecutive odd numbers and next terms are $9, 11, 13$.
And the successive terms in denominators are $1, 2, 4, 8, \dots$ With common ratio 2, so the next terms are $16, 32, 64$.
Thus, the required term is $a_7 = \frac{14}{64}$

$(iv).\ 1, 1, -3, 5, -7, 9, .. , \dots a_8$

Solution:
We see that the common difference of odd terms is -4, so $a_7 = -7 + (-4) = -11$
And the common difference of even terms is 4, so $a_8 = 9 + 4 = 13$
Thus, the required term is $a_8 = 13$

$(v).\ 1, -3, 5, -7, 9, -11, \dots a_8$

Solution:
We see that the common difference of odd terms is 4, so $a_7 = 9 + 4 = 13$.
And the common difference of the even terms is -4, so $a_8 = -11 + (-4) = -15$
Thus, the required term is $a_8 = -15$

Question#3

Find the next two terms of the following sequences;

$(i).\ 7, 9, 12, 16, \dots$

Solution:
We see that the sequence of the successive difference is $2, 3, 4, \dots$ so the next two differences are 5 and 6.
Thus, the next two terms are $16 + 5 = 21$ *and* $21 + 6 = 27$.

$(ii).\ 1, 3, 7, 15, 31, \dots$

Solution:
We see that the sequence of the successive difference is $2, 4, 8, 16, \dots$ so the next two differences are 32 and 64.

Thus, the next two terms of the sequence are $\mathbf{31+32=\ 63}$ *and* $\mathbf{63+64=127}$

(iii). $\mathbf{-1, 2, 12, 40, \ldots}$

Solution:

The sequence of the above terms can be written as $-1\times 1, 1\times\ 2, 3\times\ 4, 5\times 8, \ldots$

So, the next two terms are $\mathbf{7\times 16\ =112}$ *and* $\mathbf{9\times 32\ =\ 288}$.

(iv). $\mathbf{1, -3, 5, -7, 9, -11\ldots}$

Solution:

We see that the common difference of odd terms is 4, so $\boldsymbol{a_7}=9+4=13$.

And the common difference of the even terms is -4*, so* $\boldsymbol{a_8}=-11+(-4)=-15$

Thus, the next two terms are 13 *and* -15.

Arithmetic progression (A.P)

A sequence $\{a_n\}$ *is* an Arithmetic sequence or Arithmetic progression $(A.P)$ *if* $a_n-a_{(n-1)}$ is same for all $n\in N$ and $n>1$.

The difference a_n-a_{n-1} *i.e* the difference of two consecutive terms of an $A.P$ is called the common difference and is usually denoted by d.

Important Note:

When the common difference (d) of any two consecutive terms of a sequence is same, then this sequence will be called an Arithmetic sequence"

Rule for the nth term of an A.P

$$a_n=a_1+(n-1)d$$

Proof:

We know that

$$a_n-a_{n-1}=d \quad n>1$$

$$or \quad a_n=a_{n-1}+d\rightarrow(i)$$

Put $n=2,3,4,\ldots.$ *in* (i)*we get*

$$a_2=a_1+d=a_1+2(2-1)d\rightarrow(ii)$$
$$a_3=a_2+d=a_1+d+d=a_1+2d$$
$$\Rightarrow a_3=a_1+(3-1)d\rightarrow(iii)$$
$$a_4=a_3+d$$
$$\Rightarrow\ a_4=a_1+2d+d=a_1+3d$$
$$\Rightarrow\ a^4=a_1+(4-1)d\rightarrow(iv)$$

From (ii) *and* (iii) *and* (iv) we observe That

$$a_n=a_1+(n-1)d$$

Hence proved.

Exercise 6.2

Question#1

Write the first four terms of the following arithmetic sequences, if

(i). $\boldsymbol{a_1=5}$ ***and other three consecutive terms are*** $\mathbf{23, 26, 29}$

Solution:

Consecutive terms are $23, 26, 29$

Since,

$a_1=5$

$d=26-23=3$

Now,

$a_2=a_1+d=5+3=8$

$a_3=a_2+d=8+3=11$

$a_4=a_3+d=11+3=14$

Hence $5, 8, 11, 14$ *are the first four terms of A.P*

(ii). $\boldsymbol{a_5=17}$ ***and*** $\boldsymbol{a_9=37}$

Solution:

Consider a_1 *be the first term and d be the common difference*

Since,

$a_5=17$

$\Rightarrow a_1+(5-1)d=17$

$\Rightarrow a_1+4d=17\ \ldots(i)$

Also,

$a_9=37$

$\Rightarrow a_1+(9-1)d=37$

$\Rightarrow a_1+8d=37\ \ldots(ii)$

Subtracting (i) *from* (ii)

$$\begin{array}{r} a_1+4d=17 \\ a_1+8d=37 \\ -\quad -\quad\ \ - \\ \hline -4d=-20 \end{array}$$

$\Rightarrow d=5$

Putting values of d in (iv)

$a_1+4(5)=17$

$\Rightarrow a_1+20=17$

$\Rightarrow a_1=17-20$

$\Rightarrow a_1=-3$

So,

$a_2=a_1+d=-3+5=2$

$a_3=a_2+d=2+5=7$

$a_4=a_3+d=7+5=12$

Hence $\mathbf{-3, 2, 7, 12}$ ***are the first four terms of A.P***

(iii). $\mathbf{3}\boldsymbol{a_7}=\boldsymbol{a_4}$ ***and*** $\boldsymbol{a_{10}}=\mathbf{33}$

Solution:

Suppose a_1 *be the first term and d be the common difference*

Since,

$3a_7=7a_4$

$\Rightarrow 3(a_1+6d)=7(a_1+3d)$
$\Rightarrow 3a_1+18d=7a_1+21d$
$\Rightarrow -4a_1-3d=0$
$\Rightarrow 4a_1+3d=0 \ldots(i)$

Also,

$a_{10}=33$
$\Rightarrow a_1+9d=33 \ldots(ii)$

Multiplying eq. (i) *by 4 Subtracting from* (i)

$$\begin{array}{r} 4a_1+3d=0 \\ 4a_1+36d=132 \\ -\quad - \qquad - \\ \hline -33d=-132 \end{array}$$

$\Rightarrow d=4$

Putting values of d in (ii)

$a_1+9(4)=33$
$\Rightarrow a_1+36=33$
$\Rightarrow a_1=33-36$
$\Rightarrow a_1=-3$

So,

$a_2=a_1+d=-3+4=-1$
$a_3=a_2+d=1+4=5$
$a_4=a_3+d=5+4=9$

Hence $-3,-1,5,9$ ***are the first four terms of A.P***

Question#2

If $a_n-3=2n-5$***, find the nth term of the sequence.***

Solution:

$a_n-3=2n-5$
$\Rightarrow a_n-3=2n-6+1$
$=2(n-3)+1$

Replacing $n-3$ *by* n

$a_n-3=2n+1$

Question#3

If the 5th term of an A.P. is 16 and the 20th term is 46, what is its 12th term?

Solution:

Consider a_1 *be the first term and d be the common difference*

Since,

$a_5=16$
$\Rightarrow a_1+(5-1)d=16$
$\Rightarrow a_1+4d=16 \ldots(i)$

Also,

$a_{20}=46$
$\Rightarrow a_1+(20-1)d=46$
$\Rightarrow a_1+19d=46 \ldots(ii)$

Subtracting (i) *from* (ii)

$$\begin{array}{r} a_1+4d=16 \\ a_1+19d=46 \\ -\quad - \qquad - \\ \hline -15d=-30 \end{array}$$

$\Rightarrow d=2$

Putting values of d in (i)

$a_1+4(2)=16$
$\Rightarrow a_1+8=16-8$
$\Rightarrow a_1=8$

So,

$a_{12}=a_1+11d$
$=8+11(2)$
$=8+2=30$

Question#4

Find the 13th term of the sequence $x,1,2-x,3-2x,..$

Solution:

Here,

$a_1=1$
$d=a_2-a_1=1-x$

Since,

$a_{13}=a_1+12d$
$=x+12(1-x)$
$=x+12-12x$
$\Rightarrow a_{13}=12-11x$

Question#5

Find the 18th term of the A.P. if its 6th term is 19 and the 9th term is 31.

Solution:

Here $a_{18}=?$

$\mathbf{a_6=19}$, $\mathbf{a_9=31}$

$\mathbf{a_6=19} \quad \Rightarrow \mathbf{a_1+5d=19\rightarrow(i)}$
$\mathbf{a_9=31} \quad \Rightarrow \mathbf{a_1+8d=31\rightarrow(ii)}$

By $(ii)-(i) \Rightarrow$

$$\begin{array}{r} \mathbf{a_1+8d=31} \\ \mathbf{\pm a_1\pm 5d=\pm 19} \\ \hline \mathbf{3d=12} \end{array}$$

$\Rightarrow \mathbf{d=4\ put\ in\ (i)}$

$\mathbf{a_1+5(4)=19 \Rightarrow a_1+20=19}$
$\mathbf{a_1=19-20 \Rightarrow a_1=-1}$

Noe $\mathbf{a_{18}=a_1+17d=-1+17(4)}$
$\mathbf{a_{18}=-1+68=67}$
$\mathbf{a_{18}=67}$

Question#6

Which term of the A.P. $5,2,-1,\ldots$ ***is*** -85***?***

Solution:

Here,

$a_1=5$
$d=a_2-a_1=2-5$
$d=-3$

Since,

$a_n=a_1+(n-1)d$
$\Rightarrow -85=5+(n-1)(-3)$
$\Rightarrow -85=5-3n+3$
$\Rightarrow 3n=5+3+85$
$\Rightarrow 3n=93$
$\Rightarrow n=31$

Question#7

Which term of the A.P. $-2, 4, 10, \ldots$ is 148?

Solution:

Given that

$$-2, 4, 10, \ldots, 148$$

Here $a_1 = -2$, $d = 4 - (-2) = 6$, $a_n = 148$

$$\because a_n = a_1 + (n-1)d$$

$$148 + 2 = (n-1)6$$

$$\frac{150}{6} = n - 1 \Rightarrow n - 1 = 25$$

⇨ $n = 25 + 1 \Rightarrow n = 26$

Hence 26^{th} term is 148

Question#8

How many terms are there in the A.P. in which $a_1 = 11$, $a_n = 68$, $d = 3$?

Solution:

$a_1 = 11$

$a_n = 68$

$d = 3$

Since,

$a_n = a_1 + (n-1)d$

⇨ $68 = 11 + (n-1)(3)$

$68 - 11 = (n-1)3$

⇨ $57 = (n-1)3$

⇨ $\frac{57}{3} = n - 1 \Rightarrow n - 1 = 19$

$$n = 20$$

Hence there are 20 term in given A.P

Question#9

If the nth term of the A.P. is $3n - 1$, find the A.P.

Solution:

Since,

$a_n = 3n - 1$

Put $n = 1$

$a_1 = 3(1) - 1 = 3 - 1 = 2$

Put $n = 2$

$a_2 = 3(2) - 1 = 6 - 1 = 5$

Put $n = 3$

$a_3 = 3(3) - 1 = 9 - 1 = 8$

Put $n = 4$

$a_4 = 3(4) - 1 = 12 - 1 = 11$

Thus, $2, 5, 8, 11 \ldots$ ***is the required A.P.***

Question#10

Determine whether $(i)\ -19$, $(ii)\ 2$ ***are the terms of the A.P.*** $17, 13, 9, \ldots$ ***or not.***

Solution:

(i).

Here,

$a_1 = 17$

$d = a_2 - a_1 = 13 - 17$

$d = -4$

Suppose -19 ***be the nth term of A.P i.e.***

Since,

$a_n = a_1 + (n-1)d$

$\Rightarrow -19 = 17 + (n-1)(-4)$

$\Rightarrow -19 = 17 - 4n + 4$

$\Rightarrow 4n = 17 + 4 + 19$

$\Rightarrow 4n = 40$

$\Rightarrow n = 10$

Thus -19 ***is the*** 10^{th} ***term of A.P***

(ii). *Here,*

$a_1 = 17$

$d = a_2 - a_1 = 13 - 17$

$d = -4$

$a_n = 2$

Suppose -19 ***be the nth term of A.P i.e.***

Since,

$a_n = a_1 + (n-1)d$

$\Rightarrow 2 = 17 + (n-1)(-4)$

$\Rightarrow 2 = 17 - 4n + 4$

$\Rightarrow 4n = 17 + 4 - 2$

$\Rightarrow 4n = 19$

$\Rightarrow n = \frac{19}{4}$

Which is a rational. Thus 2 ***is not the term of A.P.***

Question#11

If l, m, n are the pth, at and Roth terms of an A.P., show that

Solution:

Consider a_1 *be the first term and d be the common difference*

Since,

$a_p = l$

$\Rightarrow a_1 + (p-1)d = l$

$a_q = m$

$\Rightarrow a_1 + (q-1)d = m$

$a_r = n$

$\Rightarrow a_1 + (r-1)d = n$

(i). $l(q-r) + m(r-p) + n(p-q) = 0$

Solution:

L.H.S $= l(q-r) + m(r-p) + n(p-q)$

$= [a_1 + (p-1)d](q-r) + [a_1 + (q-1)d](r - p) + [a_1 + (r-1)d](p-q)$

$= (a_1 + pd - d)(q-r) + (a_1 + qd - d)(r-p) + (a_1 + rd - d)(p-q)$

$= a_1q + pqd - qd - a_1r - prd + rd + a_1r + qrd - qr - a_1p - pqd + pq + a_1p + prd - pd + -a_1q - qrd + qd$

$= 0 =$ ***R.H.S***

(ii). $p(m-n) + q(n-l) + r(l-m) = 0$

Solution:

L.H.S $= p(m-n) + q(n-l) + r(l-m)$

$= p[a_1 + (p-1)d - a_1 - (r-1)d] + q[a_1 + (r-1)d - a_1 - (q-1)d]$
$+r[a_1 + (p-1)d - a_1 - (q-1)d]$
$= p[qd - d - rd + d] + q[rd - d - pd + d] + r[pd - d - qd + d]$
$= pqd - prd + qrd - pqd + prd - qrd$
$= 0 =$ *R.H.S*

Question#12

Find the nth term of the sequence,
$\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$

Solution:

We first find the nth term of $4, 7, 10, \ldots$
$a_1 = 4$
$d = a_2 - a_1 = 7 - 4$
$d = 3$
So,
$a_n = a_1 + (n-1)d$
$= 4 + (n-1)(3)$
$= 4 + 3n - 3 = 3n + 1$
Hence nth term of the given sequence is $\left(\frac{3n+1}{3}\right)^2$

Question#13

If $\frac{1}{a}, \frac{1}{b}$ *and* $\frac{1}{c}$ *are in A.P., show that* $b = \frac{2ac}{a+b}$

Solution:

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in A.P therefore,*
$d = \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$
$\Rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{a} + \frac{1}{c}$
$\Rightarrow \frac{1+1}{b} = \frac{a+c}{ac}$
$\Rightarrow \frac{2}{b} = \frac{a+c}{ac}$
$\Rightarrow \frac{b}{2} = \frac{ac}{a+c}$
$\Rightarrow b = \frac{2ac}{a+b}$
Proved.

Question#14

If $\frac{1}{a}, \frac{1}{b}$ *and* $\frac{1}{c}$ *are in A.P., show that the common difference* $\frac{a-c}{2ac}$

Solution:

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in A.P therefore,*
$d = \frac{1}{b} - \frac{1}{a} \quad \ldots(i)$
also
$d = \frac{1}{c} - \frac{1}{b} \quad \ldots(ii)$
Comparing (i) *and* (ii)
$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$
$\Rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{a} + \frac{1}{c}$
$\Rightarrow \frac{1+1}{b} = \frac{a+c}{ac}$
$\Rightarrow \frac{2}{b} = \frac{a+c}{ac}$
$\Rightarrow \frac{b}{2} = \frac{ac}{a+c}$
$\Rightarrow b = \frac{2ac}{a+b}$

Putting the values of b in eq. (i)
$d = \frac{1}{\frac{2ac}{a+b}} - \frac{1}{a}$
$= \frac{a+c}{2ac} - \frac{1}{a}$
$= \frac{a+c-2c}{2ac}$
$= \frac{a-c}{2ac}$
Hence the common difference is $\frac{a-c}{2ac}$

Arithmetic Mean A.M

A number A is said to be arithmetic mean between two numbers a and b if a, A, b are in A.P.

Thus $d = A - a$ and $d = b - A$
⇨ $A - a = b - A$
⇨ $A + A = a + b$
⇨ $2A = a + b \Rightarrow A = \frac{a+b}{2}$

Thus A.M $= \frac{a+b}{2}$

Note:
Middle term of three consecutive terms in A.P is the A.M between the extreme terms.

- The numbers $A_1, A_2, A_3, \ldots, A_n$ are said to be in A.M between a and b if $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P
- If $A_1, A_2, A_3, \ldots, A_{n-1}, A_n$ are n A.M's between a and b and d be common difference then
$A_1 = a + d \quad, A_2 = A_1 + d = a_1 + 2d$
$a_3 = A_2 + d = a_1 + 3d, \quad A_4 = A_3 + d = a_1 + 4d$
$\ldots\ldots\ldots A_n = A_{n-1} + d = a_1 + nd$

Exercise 6.3

Question#1

Find A.M. between
(i). $3\sqrt{5}$ *and* $5\sqrt{5}$

Solution:

Here
$a = 3\sqrt{5}$ *and* $b = 5\sqrt{5}$
A.M.$= \frac{a+b}{2}$
$= \frac{3\sqrt{5}+5\sqrt{5}}{2}$
$= \frac{8\sqrt{5}}{2}$
$= 4\sqrt{5}$

(ii). $x - 3$ *and* $x + 5$

Solution:

$a = x - 3$ *and* $b = x + 5$

$A.M. = \frac{a+b}{2}$
$= \frac{x-3+x+5}{2}$
$= \frac{2x+2}{2}$
$= x+1$

(iii). $1-x+x^2$ and $1+x+x^2$

Solution:

$a = 1-x+x^2$ and $b = 1+x+x^2$
$A.M. = \frac{a+b}{2}$
$= \frac{1-x+x^2+1+x+x^2}{2}$
$= \frac{2+2x^2}{2}$
$= 1+x^2$

Question#2

If 5, 8 are two A.Ms between a and b, find a and b.

Solution:

Since 5, 8 are two A.Ms between a and b.
Therefor $a, 5, 8, b$ are in A.P.
Here
$a_1 = a$
$d = 8-5 = 3$

Now,

$a_2 = a_1 + d$
$\Rightarrow 5 = a+3$
$\Rightarrow 5-3 = a$
$\Rightarrow a = 2$
$a_4 = a_1 + 3d$
$\Rightarrow b = 2+3(3)$
$\Rightarrow b = 2+9$
$\Rightarrow b = 11$

Question#3

Find 6 A.Ms. between 2 and 5.

Solution:

Let A_1, A_2, A_3, A_4, A_5 and A_6 are the six A.Ms. between 2 and 5.
Then $2, A_1, A_2, A_3, A_4, A_5, A_6, 5$ are in A.P.
Here
$a_1 = 2$
$a_8 = 5$
$\Rightarrow a_1 + 7d = 5$
$\Rightarrow 2+7d = 5$
$\Rightarrow 7d = 5-2$
$\Rightarrow 7d = 3$
$\Rightarrow d = \frac{3}{7}$

So,

$A_1 = a_2 = a_1 + d = 2 + \frac{3}{7} = \frac{17}{7}$
$A_2 = a_3 = a_1 + 2d = 2 + 2\left(\frac{3}{7}\right) = 2 + \frac{6}{7} = \frac{20}{7}$
$A_3 = a_4 = a_1 + 3d = 2 + 3\left(\frac{3}{7}\right) = 2 + \frac{9}{7} = \frac{23}{7}$
$A_4 = a_5 = a_1 + 4d = 2 + 4\left(\frac{3}{7}\right) = 2 + \frac{12}{7} = \frac{26}{7}$
$A_5 = a_6 = a_1 + 5d = 2 + 5\left(\frac{3}{7}\right) = 2 + \frac{15}{7} = \frac{29}{7}$
$A_6 = a_7 = a_1 + 6d = 2 + 6\left(\frac{3}{7}\right) = 2 + \frac{18}{7} = \frac{32}{7}$

Hence, $\frac{17}{7}, \frac{20}{7}, \frac{23}{7}, \frac{26}{7}, \frac{29}{7}, \frac{32}{7}$ are the six A.Ms. between 2 and 5.

Question#4

Find four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

Solution:

Let A_1, A_2, A_3, and A_4 are the four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.
Then $\sqrt{2}$ $A_1, A_2, A_3, A_4, \frac{12}{\sqrt{2}}$ are in A.P.
Here
$a_1 = \sqrt{2}$
$a_6 = \frac{12}{\sqrt{2}}$
$\Rightarrow a_1 + 5d = \frac{12}{\sqrt{2}}$
$\Rightarrow \sqrt{2} + 5d = \frac{12}{\sqrt{2}}$
$\Rightarrow 5d = \frac{12}{\sqrt{2}} - \sqrt{2}$
$\Rightarrow 5d = \frac{12-2}{\sqrt{2}}$
$\Rightarrow 5d = \frac{10}{\sqrt{2}}$
$\Rightarrow d = \frac{2}{\sqrt{2}}$
$\Rightarrow d = \frac{(\sqrt{2})^2}{\sqrt{2}}$
$\Rightarrow d = \sqrt{2}$

So,

$A_1 = a_2 = a_1 + d = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$
$A_2 = a_3 = a_1 + 2d = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$
$A_3 = a_4 = a_1 + 3d = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$
$A_4 = a_5 = a_1 + 4d = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$

Hence $2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$ are the four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

Question#5

Insert 7 A.Ms. between 4 and 8.

Solution:

Let
$A_1, A_2, A_3, A_4, A_5, A_6, A_7$ be 7 A.Ms between 4 and 8
Then $4, A_1, A_2, A_3, A_4, A_5, A_6, A_7, 8$ are in A.P
Here
$a_1 = 4,\ n = 9$ and $a_9 = 8$
$\Rightarrow a_1 + 8d = 8 \quad \because a_n = a_1 + (n-1)d$
$\Rightarrow 4 + 8d = 8$
$8d = 8-4 \Rightarrow 8d = 4$
Or $d = \frac{1}{2}$ Now
$A_1 = a_1 + d = 4 + \frac{1}{2} = \frac{8+1}{2} = \frac{9}{2}$
$A_2 = a_1 + 2d = 4 + 2\left(\frac{1}{2}\right) = 4+1 = 5$

$A_3 = a_1 + 3d = 4 + 3\left(\frac{1}{2}\right) = 4 + \frac{3}{2} = \frac{8+3}{2}$

⇨ $A_5 = \frac{13}{2}$

$A_6 = a_1 + 6d = 4 + 6\left(\frac{1}{2}\right) = 4 + 3 = 7$

$A_7 = a_1 + 7d = 4 + 7\left(\frac{1}{2}\right) = 4 + \frac{7}{2} = \frac{8+7}{2} = \frac{15}{2}$

Hence 7 $A.Ms$ are $\frac{9}{2}, \frac{5,11}{2}, \frac{6,13}{2}, \frac{15}{2}$ between 4 and 8.

Question#6

Find three A.Ms between 3 and 11.

Solution:

Let A_1, A_2, A_3 *be* three A.Ms between 3 and 11 then 3, $A_1, A_2, A_3, 11$ *are in* $A.P$

Here $a_1 = 3,\ n = 5\ \ so\ a_5 = 11$

⇨ $a_1 + 4d = 11 \quad \because\ a_n = a_1 + (n-1)d$

⇨ $3 + 4d = 11$

⇨ $4d = 11 - 3 \Rightarrow 4d = 8 \Rightarrow d = 2$

Now

$A_1 = a_1 + d = 3 + 2 = 5$

$A_2 = a_1 + 2d = 3 + 2(2) = 3 + 4 = 7$

$A_3 = a_1 + 3d = 3 + 3(2) = 3 + 6 = 9$

Thus three $A.Ms$ between 3 and 11 are 5,7,9.

Question#7

Find n so that $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ may be the A.M. between a and b.

Solution:

Since we know that $A.M. = \frac{a+b}{2}\quad (i)$

But we have given $A.M. = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}\quad \dots (ii)$

Comparing (i) *and* (ii)

$\frac{a^n+b^n}{a^{n-1}+b^{n-1}} = \frac{a+b}{2}$

By cross multiplying

$\Rightarrow 2(a^n + b^n) = (a+b)(a^{n-1} + b^{n-1})$

$\Rightarrow 2a^n + 2b^n = a^n + a^{n-1}b + ab^{n-1} + b^n$

$\Rightarrow 2a^n + 2b^n - a^n - b^n = a^{n-1}b + ab^{n-1}$

$\Rightarrow a^n + b^n = a^{n-1}b + ab^{n-1}$

$\Rightarrow a^n - a^{n-1}b = ab^{n-1} - b^n$

$\Rightarrow a^{n-1+1} - a^{n-1}b = ab^{n-1} - b^{n-1+1}$

$\Rightarrow a^{n-1}(a-b) = b^{n-1}(a-b)$

$\Rightarrow a^{n-1} = b^{n-1}$

$\Rightarrow \frac{a^{n-1}}{b^{n-1}} = 1$

$\Rightarrow \left(\frac{a}{b}\right)^{n-1} = \left(\frac{a}{b}\right)^0 \qquad \because \left(\frac{a}{b}\right)^0 = 1$

$\Rightarrow n - 1 = 0$

$\Rightarrow n = 1$

Question#8

Show that the sum of n A.Ms. between a and b is equal to n times their A.M.

Solution:

Let $A_1,\ A_2, A_3, A_4, A_5 \dots\ A_n$ *are the* n *A.Ms. between* a *and* b.

Then $a, A_1, A_2, A_3, \dots A_n, b$ *are in A.P.*

Here

$a_1 = a$

$a_{n+2} = b$

$\Rightarrow a_1 + (n+2-1)d = b$

$\Rightarrow a_1 + (n+1)d = b$

$\Rightarrow (n+1)d = b - a$

$\Rightarrow d = \frac{b-a}{n+1}$

Now,

$\boldsymbol{A_1} = a_2 = a_1 + d = a + \frac{b-a}{n+1}$

$\boldsymbol{A_2} = a_3 = a_1 + 2d = a + 2\left(\frac{b-a}{n+1}\right)$

$\boldsymbol{A_3} = a_4 = a_1 + 3d = a + 3\left(\frac{b-a}{n+1}\right)$

. . . .

. . . .

. . . .

$\boldsymbol{A_n} = a_{n+1} = a_1 + nd = a + n\left(\frac{b-a}{n+1}\right)$

Now ,

Sum of A.M.s

$= A_1, A_2, A_3, A_4, A_5 \dots\ A_n$

$= a + \frac{b-a}{n+1} + a + 2\left(\frac{b-a}{n+1}\right) + a + 3\left(\frac{b-a}{n+1}\right) + \dots + a + n\left(\frac{b-a}{n+1}\right)$

$= (\boldsymbol{a} + \boldsymbol{a} + \boldsymbol{a} + \dots + \boldsymbol{a}) + \frac{b-a}{n+1} + 2\left(\frac{b-a}{n+1}\right) + 3\left(\frac{b-a}{n+1}\right) + \dots + n\left(\frac{b-a}{n+1}\right)$

$= \boldsymbol{na} + \frac{b-a}{n+1}(1 + 2 + 3, + \dots + n)$

$a_1 = 1$

$d = 2 - 1 = 1$

$\boldsymbol{n} = \boldsymbol{n}$

$= \boldsymbol{na} + \frac{b-a}{n+1}\left[\frac{n}{2}\ 2(1) + (n-1)(1)\right]$

$= \boldsymbol{na} + \frac{b-a}{n+1}\left[\frac{n}{2}\ (2 + n - 1)\right]$

$= \boldsymbol{na} + \frac{b-a}{n+1}\left[\frac{n}{2}\ (n+1)\right]$

$= \boldsymbol{na} + (b-a)\left(\frac{n}{2}\right)$

$= \boldsymbol{n}\left[\boldsymbol{a} + \frac{b-a}{2}\right]$

$= \boldsymbol{n}\left[\frac{2a+b-a}{2}\right]$

$= \boldsymbol{n}\left[\frac{a+b}{2}\right]$

$= \boldsymbol{n}(\boldsymbol{A.M`s\ between\ a\ \&b})$

Hence sum of n A.Ms between a & b is n times their A.Ms proved

Series

The sum of terms of a sequence is called series "

We know that $a_1, (a_1+d), (a_1+2d), \ldots, a_n$ is an arithmetic sequence.

So

$$a_1 + (a_1+d) + (a_1+2d) + \cdots + (a_n)$$

Is an arithmetic series.

Finite series:

If the number of terms in a series is finite , then the series is called series.

Infinite series:

If the number of terms in a series is infinite, then the series is infinite series.

Prove that $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Proof:

Let $a_1, (a_1+d), (a_1+2d), \ldots (a_n-2d), (a_n-d), a_n$

Are n-terms then

$$S_n = a_1 + (a_1+d) + (a_1+2d) + \cdots + (a_n-2d) + (a_n-d) + a_n \rightarrow (i)$$

Re-writing (i) in reverse order

$$S_n = a_n + (a_n-d) + (a_n-2d) + \cdots + (a_1+2d) + (a_1+d) + a_1 \rightarrow (ii)$$

By $(i) + (ii)$ *we get*

$$S_n + S_n = (a_1+a_n) + (a_1+d+a_n-d) + (a_1+2d + a_n-2d) + \cdots + (a_n-2d+a_1+2d) + (a_n-d+a_1+d) + (a_n+a_1)$$

$$\Rightarrow 2S_n = (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + \cdots + (a_1+a_n) + (a_1+a_n) + (a_1+a_n)$$

$$2Sn = (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + \cdots nterms$$

$$2S_n = n[a_1+a_n]$$

$$\Rightarrow S_n = \frac{n}{2}[a_1+a_n]$$

Put $a_n = a_1 + (n-1)d$

$$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d)$$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d$$

Hence proved.

Exercise 6.4

Question#1

Find the sum of all the integral multiples of 3 between 4 and 96 .

Solution:

The series of integral multiples of 3 between 4 and 96.

$6 + 9 + 12 + 15 \ldots\ldots + 96$

Here,

$a_1 = 6$

$d = a_2 - a_1 = 9 - 6 = 3$

$a_n = 96$

Since,

$a_n = a_1 + (n-1)d$

$\Rightarrow 96 = 6 + (n-1)(3)$

$\Rightarrow 96 = 6 + 3n - 3$

$\Rightarrow 96 - 6 + 3 = 3n$

$\Rightarrow 93 = 3n$

$\Rightarrow n = 31$

$S_n = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow S_{31} = \frac{31}{2}[6 + 96]$

$= \frac{31}{2}[102]$

$= (31)(51)$

$= 1581$

Question#2

Sum the series

$(i).\ -3 + (-1) + 1 + 3 + 5 + \ldots. + a_{16}$

Solution:

Here,

$a_1 = -3$

$d = a_2 - a_1 = -1 - (-3) = -1 + 3 = 2$

$n = 16$

Since,

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$\Rightarrow S_{16} = \frac{16}{2}(2(-3) + (16-1)(2))$

$= 8(-6 + 30)$

$= 8(24)$

$= 192$

$(ii).\ \frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \ldots. + a_{11}$

Solution:

Here,

$a_1 = \frac{3}{\sqrt{2}}$

$d = a_2 - a_1 = 2\sqrt{2} - \frac{3}{\sqrt{2}} = \frac{4-3}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$n = 13$

Since,

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$\Rightarrow S_{13} = \frac{13}{2}\left(2\left(\frac{3}{\sqrt{2}}\right) + (13-1)\left(\frac{1}{\sqrt{2}}\right)\right)$

$= \frac{13}{2}\left(\frac{6}{\sqrt{2}} + \frac{12}{\sqrt{2}}\right)$
$= \frac{13}{2}\left(\frac{18}{\sqrt{2}}\right)$
$= \frac{117}{\sqrt{2}}$

(*iii*). $1.11 + 1.41 + 1.71 + \ldots + a_{10}$

Solution:

Here,

$a_1 = 1.11$
$d = a_2 - a_1 = 1.41 - 1.11 = 0.31$
$n = 10$
$\because\ S_n = \frac{n}{2}(2a_1 + (n-1)d)$
$S_{10} = \frac{10}{2}\left(2(1.11) + (10-1)(0.30)\right)$
$s_{10} = 5\left(2.22 + (9)\right)(0.30)$
$S_{10} = 5(2.22 + 2.7) = 5(4.92)$
$\Rightarrow\ S_{10} = 25.60$

(*iv*). $-8 - 3\frac{1}{2} + 1 \ldots + a_{11}$

Solution:

$-8 - \frac{7}{2} + 1 \ldots \ldots \ldots + a_{11}$

Here,

$a_1 = -8$
$d = a_2 - a_1 = -\frac{7}{2} - (-8) = -\frac{7}{2} + 8 = \frac{9}{2}$
$n = 11$
$\because\ S_n = \frac{n}{2}(2a_1 + (n-1)d)$
$\Rightarrow S_{11} = \frac{11}{2}\left(2(-8) + (11-1)\left(\frac{9}{2}\right)\right)$
$S_{11} = \frac{11}{2}\left(-16 + (10)\left(\frac{9}{2}\right)\right)$
$S_{11} = \frac{11}{2}\left(-16 + 5(9)\right)$
$S_{11} = \frac{11}{2}(-16 + 45) = \frac{11}{2}(29)$
$S_{11} = \frac{319}{2} = 159.5$

(*v*). $(x - a) + (x + a) + (x + 3a) + \ldots$ *to n terms.*

Solution:

Here,

$a_1 = x - a$
$d = a_2 - a_1 = (x + a) - (x - a) = 2a$
$n = n$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow S_n = \frac{n}{2}[2(x-a) + (n-1)(2a)]$
$= \frac{n}{2}[2x + 2an - 2a]$
$= \frac{n}{2}.2[x + an - 2a]$
$= n[x + an - 2a]$
$= n[x + (n-2)a]$

(*vi*). $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \ldots$ *to n terms.*

Solution:

Here,

$a_1 = \frac{1}{1-\sqrt{x}}$
$d = a_2 - a_1 = \frac{1}{1-x} - \frac{1}{1-\sqrt{x}}$
$= \frac{1}{(1-\sqrt{x})(1+\sqrt{x})} - \frac{1}{1-\sqrt{x}}$
$= \frac{1-(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})}$
$= \frac{1-1-\sqrt{x}}{1-x}$
$= \frac{-\sqrt{x}}{1-x}$
$\&\ n = n$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow S_n = \frac{n}{2}\left[2\left(\frac{1}{1-\sqrt{x}}\right) + (n-1)\left(\frac{-\sqrt{x}}{1-x}\right)\right]$
$= \frac{n}{2}\left[\frac{2}{1-\sqrt{x}} - \frac{\sqrt{x}(n-1)}{1-x}\right]$
$= \frac{n}{2}\left[\frac{2}{1-\sqrt{x}} - \frac{\sqrt{x}(n-1)}{(1-\sqrt{x})(1+\sqrt{x})}\right]$
$= \frac{n}{2}\left[\frac{2(1+\sqrt{x}) - \sqrt{x}(n-1)}{(1-\sqrt{x})(1+\sqrt{x})}\right]$
$= \frac{n}{2}\left[\frac{2+2\sqrt{x}-\sqrt{x}(n-1)}{1-x}\right]$
$= \frac{n}{2}\left[\frac{2+(2-n+1)\sqrt{x}}{1-x}\right]$
$= \frac{n}{2}\left[\frac{2+(3-n)\sqrt{x}}{1-x}\right]$

(*vii*). $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \ldots$ *to n terms.*

Solution:

Here $a_1 = \frac{1}{1+\sqrt{x}}$
$d = \frac{1}{1-x} - \frac{1}{1+\sqrt{x}}\quad , n = n$
$= \frac{1}{(1-\sqrt{x})(1+\sqrt{x})} - \frac{1}{1+\sqrt{x}}$
$d = \frac{1-(1-\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} = \frac{1-1+\sqrt{x}}{1-x}$
$\Rightarrow\ d = \frac{\sqrt{x}}{1-x}$
$\because\ S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$= \frac{n}{2}\left[2\left(\frac{1}{1+\sqrt{x}}\right) + (n-1).\frac{\sqrt{x}}{1-x}\right]$
$= \frac{n}{2}\left[\frac{2}{1+\sqrt{x}} + \frac{(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$
$= \frac{n}{2}\left[\frac{2(1-\sqrt{x})+(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$
$= \frac{n}{2}\left[\frac{2-2\sqrt{x}+n\sqrt{x}-\sqrt{x}}{1-x}\right]$
$= \frac{n}{2}\left[\frac{2+n\sqrt{x}-3\sqrt{x}}{1-x}\right]$
$\Rightarrow\ S_n = \frac{n}{2}\left[\frac{2+(n-3)\sqrt{x}}{1-x}\right]$

Question#3

How many terms of the series?

(*i*). $-7 + (-5) + (-3) + \ldots$ *amount to* 65?

Solution:

Here,

$a_1 = -7$
$d = a_2 - a_1 = (-5) - (-7) = -5 + 7 = 2$
$S_n = 65$
$n = ?$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 65 = \frac{n}{2}[2(-7) + (n-1)(2)]$

$\Rightarrow 130 = n[-14 + 2n - 2]$

$\Rightarrow 130 = n(2n - 16)$

$\Rightarrow 130 = 2n^2 - 16n$

$\Rightarrow 2n^2 - 16n - 130 = 0$

$\Rightarrow n^2 - 8n - 65 = 0$

$\Rightarrow n^2 - 13n + 5n - 65 = 0$

$\Rightarrow n(n-13) + 5(n-13) = 0$

$\Rightarrow (n-13)(n+5) = 0$

$\Rightarrow n - 13 = 0$ *or* $n + 5 = 0$

$\Rightarrow n = 13$ *or* $n = -5$

As n can't be negative so, $n = 13$.

$(ii).$ $-7 + (-4) + (-1) + \ldots$ ***amount to*** $\mathbf{114}$?

Solution:

Here $a_1 = -7$ $d = -4 - (-7)$

$d = -4 + 7 = 3,$ $n = ?$ $, S_n = 114$

$\because$ $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$\Rightarrow 114 = \frac{n}{2}(2(-7) + (n-1)d)$

$114 = \frac{n}{2}(-14 + 3n - 3)$

$114 = \frac{n}{2}(-14 + 3n - 3)$

$228 = 3n^2 - 17n$

Or $3n^2 - 17n - 228 = 0$

$\Rightarrow 3n(n-12) + 19(n-12) = 0$

$(n-12) + 19(n-12) = 0$

$(n-12)(3n+19) = 0$

$(n-12) = 0$ $, 3n + 19 = 0$

$n = 12$ $, n = -\frac{19}{3}(\text{not possible})$

Hence n=12

Question#4

Sum the series

$(i).$ $3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 + \ldots$ ***to 3n terms.***

Solution:

$(3 + 5 - 7) + (9 + 11 - 13) + (15 + 17 - 19) + \ldots$ *to n terms*

$1 + 7 + 13 \ldots\ldots\ldots$ *to n terms*

Here,

$a_1 = 1$

$d = a_2 - a_1 = 7 - 1 = 6$

$n = n$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$= \frac{n}{2}[2(1) + (n-1)(6)]$

$= \frac{n}{2}[2 + 6n - 6]$

$= \frac{n}{2}[6n - 4]$

$= \frac{n}{2}.2[3n - 2]$

$= n(3n - 2)$

$(ii).$ $1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 + \ldots$ ***to 3n terms.***

Solution:

By adding three terms, we get

$-2 + 7 + 16 + \cdots$ *to terms*

Here $a_1 = -2$ $d = 7 - 2(-2) = 7 + 2 = 9$

$n = n$

$\because$ $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_n = \frac{n}{2}(2(-2) + (n-1)(9))$

$S_n = \frac{n}{2}(-4 + 9n - 9)$

$\Rightarrow$ $S_n = \frac{n}{2}(9n - 13)$

Question#5

Find the sum of 20 terms of the series whose rth term is $3r + 1$.

Solution:

Since,

$a_r = 3r + 1.$

Put $r = 1$

$a_1 = 3(1) + 1 = 3 + 1 = 4$

Put $n = 2$

$a_2 = 3(2) + 1 = 6 + 1 = 7$

So,

$d = a_2 - a_1 = 7 - 4 = 3$

$n = 20$

$S_n = ?$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow S_{20} = \frac{20}{2}[2(4) + (20-1)(3)]$

$= \frac{20}{2}[8 + 57]$

$= 10[65]$

$= 650$

Question#6

If $S_n = 2(2n - 1)$ ***,then find the series***

Solution:

Put $n = 1, 2, 3, 4$

$S_1 = 1(2(1) - 1) = 1(2 - 1) = 1$

$S_2 = 2(2(2) - 1) = 2(4 - 1) = 6$

$S_3 = 3(2(3) - 1) = 3(6 - 1) = 15$

$S_4 = 4(2(4) - 1) = 4(8 - 1) = 28$

Now,

$a_1 = S_1 = 1$

$a_2 = S_2 - S_1 = 6 - 1 = 5$

$a_3 = S_3 - S_2 = 15 - 6 = 9$

$a_4 = S_4 - S_3 = 28 - 15 = 13$

Hence , required series is

$1 + 5 + 9 + 13 + , \ldots$

Question#7

The ratio of the sums of n terms of two series in A.P. is 3n + 2 : n + 1. Find the ratio of their 8th terms

Solution:

Consider a_1 and a_1' are the first terms and d, d' are the common differences of two series in A.P.

Now,

$S_n : S_n' = 3n + 2 : n + 1$

$\Rightarrow \frac{S_n}{S_n'} = \frac{3n+2}{n+1}$

$\Rightarrow \frac{\frac{n}{2}[2a_1+(n-1)d]}{\frac{n}{2}[2a_1'+(n-1)d']} = \frac{3n+2}{n+1}$

$\Rightarrow \frac{2a_1+(n-1)d}{2a_1'+(n-1)d'} = \frac{3n+2}{n+1}$

$\Rightarrow \frac{2\left[a_1+\frac{(n-1)}{2}d\right]}{2\left[a_1'+\frac{(n-1)}{2}d'\right]} = \frac{3n+2}{n+1}$

$\Rightarrow \frac{a_1+\frac{(n-1)}{2}d}{a_1'+\frac{(n-1)}{2}d'} = \frac{3n+2}{n+1} \quad ....(i)$

For 8^{th} term

Consider $\frac{(n-1)}{2} = 7$

$\Rightarrow n - 1 = 14$

$\Rightarrow n = 14 + 1 = 15$

Putting in eq. (i)

$\Rightarrow \frac{a_1+7d}{a_1'+7d'} = \frac{3(15)+2}{15+1}$

$\Rightarrow \frac{a_8}{a_8'} = \frac{47}{16}$

$a_8 : a_8' = 47 : 16$ or $\frac{47}{16}$

Question#8

If S_2, S_3, S_5 are the sums of 2n, 3n, 5n terms of an A.P., show that $S_5 = 5(S_3 - S_2)$

Solution:

Since,

$S_2 = \frac{2n}{2}[2a_1 + (2n-1)d] \quad ...(i)$

$S_3 = \frac{3n}{2}[2a_1 + (3n-1)d] \quad ...(ii)$

$S_5 = \frac{5n}{2}[2a_1 + (5n-1)d] \quad ...(iii)$

R.H.S $= 5(S_3 - S_2)$

$= 5\left[\frac{3n}{2}(2a_1 + (3n-1)d) - \frac{2n}{2}(2a_1 + (2n-1)d)\right]$

$= \frac{5n}{2}[3(2a_1 + (3n-1)d) - 2(2a_1 + (2n-1)d)]$

$= \frac{5n}{2}[6a_1 + 3(3n-1)d - 4a_1 - 2(2n-1)d]$

$= \frac{5n}{2}[2a_1 + [3(3n-1) - 2(2n-1)]d]$

$= \frac{5n}{2}[2a_1 + (9n - 3 - 4n + 2)d]$

$= \frac{5n}{2}[2a_1 + (5n-1)d]$

$= S_5 = L.H.S$ from (iii)

Hence,

$S_5 = 5(S_3 - S_2)$ proved.

Question#9

Obtain the sum of all integers in the first 1000 integers which are neither divisible by 5 nor by 2.

Solution:

$1 + 3 + 7 + 9 + 11 + 13 + 17 + 19 + 21 + 23 + 27 + 29 +, ... + 991 + 993 + 997 + 999$ (400 terms)

$(1 + 3 + 7 + 9) + (11 + 13 + 17 + 19) + (21 + 23 + 27 + 29) +, ... + (991 + 993 + 997 + 999)$ (100 terms)

$20 + 60 + 100 +, ... + 3980$ (100 terms)

Here,

$a_1 = 20$

$d = a_2 - a_1 = 60 - 20 = 40$

$n = 100$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$S_{100} = \frac{100}{2}[2(20) + (100-1)(40)]$

$= 50[40 + 3960]$

$= 50[4000]$

$= 2000$

Question#10

S_8 and S_9 are the sums of the first eight and nine terms of an A.P., find S_9 if $50S_9 = 63S_8$ and $= a\ 2$ $(Hint = S_8 + S_9 = a_9)$

Solution:

$\Rightarrow 50\left[\frac{9}{2}(2a_1 + (9-1)d)\right] = 63\left[\frac{8}{2}(2a_1 + (8-1)d)\right]$

$\Rightarrow 50\left[\frac{9}{2}(2a_1 + 8d)\right] = 63[4(2a_1 + 7d)]$

$\Rightarrow 225(2a_1 + 8d) = 252(2a_1 + 7d)$

$\Rightarrow 225(2(2) + 8d) = 252(2(2) + 7d)$

$\because a_1 = 2$

$\Rightarrow 225(4 + 8d) = 252(4 + 7d)$

$\Rightarrow 900 + 1800d = 1008 + 1764d$

$\Rightarrow 1800d - 1764d = 1008 - 900$

$\Rightarrow 36d = 108$

$\Rightarrow d = \frac{108}{36} = 3$

Now,

$S_9 = \frac{9}{2}[2a_1 + (9-1)d]$

$\Rightarrow S_9 = \frac{9}{2}[2(2) + (8)(3)]$

$= \frac{9}{2}[4 + 24]$

$= \frac{9}{2}[28]$

$= 126$

Question#11

The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.

Solution:

Since,

$S_9 = 171$
$\Rightarrow \frac{9}{2}[2a_1 + (9-1)d] = 171$
$\Rightarrow \frac{9}{2}[2a_1 + 8d] = 171$
$\Rightarrow \frac{9}{2}.2[a_1 + 4d] = 171$
$\Rightarrow 9a_1 + 36d = 171 \quad ...(i)$
Now,
$a_8 = 31$
$\Rightarrow a_1 + 7d = 31 \quad ...(ii)$
Multiplying eq. (ii) *by* 9 *and Subtracting from* (i)
$9a_1 + 36d = 171$
$9a_1 + 63d = 279$
$- \quad - \quad -$

$-27d = -108$
$\Rightarrow d = \frac{-108}{-27}$
$\Rightarrow d = 4$
Putting values of $d = 4$ *in* (ii)
$a_1 + 7(4) = 31$
$\Rightarrow a_1 + 28 = 31$
$\Rightarrow a_1 = 31 - 28$
$\Rightarrow a_1 = 3$
So,
$a_2 = a_1 + d = 3 + 4 = 7$
$a_3 = a_2 + d = 3 + 2(4) = 3 + 8 = 11$
$a_4 = a_3 + d = 3 + 3(3) = 15$
Hence the requires series $3 + 7 + 11 + 15 +, ...$

Question#12,

The sum of S_9 and S_7 is 203 and $S_9 - S_7 = 49$, S_7 and S_9 being the sums of the first 7 and 9 terms of an A.P. respectively. Determine the series.

Solution:

$S_9 + S_7 = 203 \quad ...(i)$
Also,
$S_9 - S_7 = 49 \quad ...(ii)$
Adding (i) *and* (ii)
$S_9 + S_7 = 203$
$S_9 - S_7 = 49$

$2S_9 = 252$
$\Rightarrow S_9 = 126$
If a_1 *be the first term and d be the common difference then,*
$\Rightarrow \frac{9}{2}[2a_1 + (9-1)d] = 126$
$\Rightarrow 9[2a_1 + 8d] = 252$
$\Rightarrow 18a_1 + 72d = 252$
$\Rightarrow 18[a_1 + 4d] = 252$
$\Rightarrow a_1 + 4d = 14 \quad ...(i)$
Now,
Subtracting (i) *and* (ii)
$S_9 + S_7 = 203$
$S_9 - S_7 = 49$
$- \quad + \quad -$

$2S_7 = 154$

$\Rightarrow S_7 = 154$
$\Rightarrow \frac{7}{2}[2a_1 + (7-1)d] = 77$
$\Rightarrow 7[2a_1 + 6d] = 154$
$\Rightarrow 14a_1 + 42d = 154$
$\Rightarrow 14[a_1 + 3d] = 154$
$\Rightarrow a_1 + 3d = 11 \quad ...(ii)$
Subtracting (iii) *from* (iv)
$a_1 + 4d = 14$
$a_1 + 3d = 11$
$- \quad - \quad -$

$d = 3$
$\Rightarrow d = 3$
Putting values of $d = 3$ *in* (ii)
$a_1 + 4(3) = 14$
$\Rightarrow a_1 + 12 = 14$
$\Rightarrow a_1 = 14 - 12$
$\Rightarrow a_1 = 2$
So,
$a_2 = a_1 + d = 2 + 3 = 5$
$a_3 = a_2 + d = 2 + 2(3) = 2 + 6 = 8$
$a_4 = a_3 + d = 2 + 3(3) = 2 + 9 = 11$
Hence the requires series $2 + 5 + 8 + 11 +, ...$

Question#13

S_7 and S_9 are the sums of the first 7 and 9 terms of an A.P .respectively. If $\frac{S_9}{S_7} = \frac{18}{11}$ and $a_7 = 20$ find the series.

Solution:

$\frac{S_9}{S_7} = \frac{18}{11}$
$\Rightarrow 11S_9 = 18S_7$
$\Rightarrow 11.\frac{9}{2}[2a_1 + (9-1)d] = 18.\frac{7}{2}[2a_1 + (7-1)d]$
$\Rightarrow \frac{99}{2}[2a_1 + 8d] = 18.\frac{7}{2}[2a_1 + 6d]$
$\Rightarrow 99a_1 + 396d = 126a_1 + 378d$
$\Rightarrow 99a_1 - 126a_1 = 378d - 396d$
$\Rightarrow -27a_1 = -18d$
$\Rightarrow a_1 = \frac{-18d}{-27}$
$\Rightarrow a_1 = \frac{2}{3}d \quad ...(i)$
Also,
$\Rightarrow a_7 = 20$
$\Rightarrow a_1 + 6d = 20 \quad ...(ii)$
Putting values of a_1 *in above*
$\frac{2}{3}d + 6d = 20$
$\Rightarrow \frac{20}{3}d = 20$
$\Rightarrow d = 20.\frac{3}{20}$
$\Rightarrow d = 3$
Putting in (i)

$a_1 = \frac{2}{3}(3)$
$\Rightarrow a_1 = 2$
Now,
$a_2 = a_1 + d = 2 + 3 = 5$
$a_3 = a_2 + d = 2 + 2(3) = 2 + 6 = 8$
$a_4 = a_3 + d = 2 + 3(3) = 2 + 9 = 11$
Hence the requires series $2 + 5 + 8 + 11 + , \ldots.$

Question#14

The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.

Solution:

Let the number in A.P. are $a - d, a, a + d$
By the given condition
$a - d + a + a + d = 24$
$\Rightarrow 3a = 24$
$\Rightarrow a = 8$
Also, by the given condition
$(a - d).a.(a + d) = 440$
$\Rightarrow a(a^2 - d^2) = 440$
Putting $a = 8$ *in above*
$\Rightarrow 8((8)^2 - d^2) = 440$
$\Rightarrow 8(64 - d^2) = 440$
$\Rightarrow 512 - 8d^2 = 440$
$\Rightarrow 512 - 440 = 8d^2$
$\Rightarrow 8d^2 = 72$
$\Rightarrow d^2 = \frac{72}{8}$
$\Rightarrow d^2 = 9$
$\Rightarrow d = \pm 3$
When $a = 8$ *and* $d = 3$
$a - d = 8 - 3 = 5$
$a + d = 8 + 3 = 11$
When $a = 5$ *and* $d = -3$
$a - d = 8 - (-3) = 8 + 3 = 11$
$a + d = 8 + (-3) = 8 - 3 = 5$
Hence $5, 8, 11$ *or* $11, 8, 5$ *are the required number*

Question#15

Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.

Solution:

Consider four numbers $a - 3d, a - d, a + d, a + 3d$ *are in A.P. Thus,*
By the given condition
$a - 3d + a - d + a + d + a + 3d = 32$
$\Rightarrow 4a = 32$
$\Rightarrow a = 8$
Also, by the given condition
$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 276$
$\Rightarrow a^2 - 6ad + 9d^2 + a^2 - 2ad + d^2 = 276$
$\Rightarrow 4a^2 + 20d^2 = 276$
Putting $a = 8$ *in above*
$\Rightarrow 4(8)^2 + 20d^2 = 276$
$\Rightarrow 256 + d^2 = 276$
$\Rightarrow 20d^2 = 276 - 256$
$\Rightarrow d^2 = 1$
$\Rightarrow d = \pm 1$
When $a = 8$ *and* $d = 1$
$a - 3d = 8 - 3(1) = 5$
$a - d = 8 - 1 = 8$
$a + d = 8 + 1 = 9$
$a + d = 8 + 3(1) = 11$
When $a = 8$ *and* $d = -1$
$a - 3d = 8 - 3(-1) = 8 + 3 = 11$
$a - d = 8 - (-1) = 8 + 1 = 9$
$a + d = 8 + (-1) = 8 - 1 = 7$
$a + d = 8 + 3(-1) = 8 - 3 = 5$
Hence $5, 8, 11$ *or* $11, 9, 7, 5$ *are the required number*

Question#16

Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

Solution:

Suppose

Five numbers in A.P are $a_1 - 2d, a_1 - d, a_1 + d, a_1 + d$
I condition
$a_1 - 2d + a_1 - d + a_1 + a_1 + d + a_1 + 2d = 25$
$\Rightarrow 5a_1 = 25 \Rightarrow a_1 = 5$
II condition
$(a_1 - 2d)^2 + (a_1 - d)^2 + a_1^2 + (a_1 + d)^2 + (a_1 + 2d)^2 = 135$
$a_1^2 - 4a_1 d + 4d^2 + a_1^2 - 2a_1 d + d^2 + a_1^2 + a_1^2 + 2a_1 d$
$d^2 + a_1^2 + 4d^2 + 4a_1 d = 135$
$\Rightarrow 5a_1^2 + 10d^2 = 135$
$\Rightarrow a_1^2 + 2d^2 = 27$
or $(5)^2 + 2d^2 = 27 \quad \because a_1 = 5$
$\Rightarrow 2d^2 + 27 - 25 \Rightarrow 2d^2 = 2$
or $d^2 = 1$ or $d = \pm 1$
When $d = 1$
$a_1 - 2d = 5 - 2(1) = 3$, $a_1 - d = 5 - 1 = 4$
$a_1 = 5,\ a_1 + d = 5 + 1 = 6$
$a_1 + 2d = 5 + 2(1) = 5 + 2 = 7$
Hence $3,4,5,7$ required *numbers. When* $d = -1$
$a_1 - 2d = 5 - 2(-1) = 5 + 2 = 7$
$a_1 - d = 5 - -1 = 5 + 1 = 6$
$a_1 = 5,\ a - 1 + 2d = 5 + 2(-1) = 5 - 2 = 3$
$a_1 + d = 5 + (-1) = 5 - 1 = 4$
Hence $7,6,5,4,3$ *req. numbers.*

Question#17

The sum of the 6th and 8th terms of an A.P. is 40 and the product of 4th and 7th term is 220. Find the A.P.

Solution:

Since,

$a_6 + a_8 = 40$

$\Rightarrow a_1 + 5d + a_1 + 7d = 40$

$\Rightarrow 2a_1 + 12d = 40$

$\Rightarrow 2(a_1 + 6d) = 40$

$\Rightarrow a_1 + 6d = 20 \quad \dots(i)$

Also,

$a_4 + a_7 = 220$

$\Rightarrow (a_1 + 3d)(a_1 + 6d) = 220$

$\Rightarrow (a_1 + 3d)(20) = 220 \quad from (i)$

$\Rightarrow a_1 + 3d = \frac{220}{20}$

$\Rightarrow a_1 + 3d = 11 \; \dots(ii)$

Subtracting (i) from (ii)

$a_1 + 6d = 20$

$a_1 + 3d = 11$

$- \quad - \quad -$

$3d = 9$

$\Rightarrow d = 3$

Putting values of $d = 3$ in (ii)

$a_1 + 3(3) = 11$

$\Rightarrow a_1 + 9 = 11$

$\Rightarrow a_1 = 11 - 9$

$\Rightarrow a_1 = 2$

So,

$a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_2 + d = 2 + 2(3) = 2 + 6 = 8$

$a_4 = a_3 + d = 2 + 3(3) = 2 + 9 = 11$

Thus, the requires series $2 + 5 + 8 + 11 + , \dots$

Question#18

If a^2 , b^2 and c^2 are in A.P., show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

Solution:

Since,

a^2 , b^2 and c^2 are in A.P.

Therefore,

$b^2 - a^2 = c^2 - b^2$

$\Rightarrow (b - a)(c - a) = (c - b)(c + b) \quad \dots(i)$

Now to show $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$d = \frac{1}{c+a} - \frac{1}{b+c}$

$= \frac{b+c-c+a}{(c+a)(b+c)}$

$\Rightarrow d = \frac{b-a}{(c+a)(b+c)} \quad \dots(ii)$

$d = \frac{1}{a+b} - \frac{1}{c+a}$

$= \frac{c+a-a-b}{(a+b)(c+a)}$

From eq.(i)

$\frac{(b-a)(b+a)}{(c+b)} = c - b$

Putting in above

$d = \frac{\frac{(b-a)(b+a)}{(c+b)}}{(a+b)(c+a)}$

$= \frac{(b-a)(b+a)}{(c+b)(a+b)(c+a)}$

$\Rightarrow d = \frac{b-a}{(c+b)(c+a)} \quad \dots(iii)$

From (ii) and (iii)

$d = d$

Hence, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Word problems on A.P

Exercise 6.5

Question#1

A man deposit in a bank Rs. 10 in the first month; Rs. 15 in the second month; Rs. 20 in the third month and so on. Find how much he will have deposited in the bank by the 9th month.

Solution:

The sequence of the deposits is

$10, 15, 20, \dots \dots . to\ 9\ terms$

Here,

$a_1 = 10$

$d = a_2 - a_1 = 15 - 10 = 5$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow S_9 = \frac{9}{2}[2(10) + (9-1)(5)]$

$= \frac{9}{2}[20 + 40]$

$= \frac{9}{2}[60] = 270$

Hence the total amount he deposits is Rs. 270.

Question#2

378 trees are planted in rows in the shape of an isosceles triangle, the numbers in successive rows decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle?

Solution:

The sequence of the trees from top to base row is $1, 2, 3, \dots \dots$

Let n be the total number of tress in base row then

Here,

$a_1 = 1$

$d = a_2 - a_1 = 2 - 1 = 1$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 378 = \frac{n}{2}[2(1) + (n-1)(1)]$

$\Rightarrow 756 = n[2 + n - 1]$

$\Rightarrow 756 = n(n + 1)$

$\Rightarrow 756 = n^2 + n$

$\Rightarrow n^2 + n - 756 = 0$

So,

$n = \frac{-1\pm\sqrt{(-1)^2-4(1)(-756)}}{2(1)}$

$n = \frac{-1\pm\sqrt{1+3024}}{2} = \frac{-1\pm\sqrt{3025}}{2} = \frac{-1\pm 55}{2}$

So,

$n = \frac{-1+55}{2} = \frac{54}{2} = 27$ ***or*** $n = \frac{-1-55}{2} = \frac{-56}{2} = -28$

Since n can never be negative therefore $n = 27$

Now,

$a_n = a_1 + (n-1)d$

$\Rightarrow a_n = 1 + (27-1)(1) = 1 + 26 = 27$

Thus the numbers of trees in the base row are 27.

Question#3

A man borrows Rs. 1100 and agree to repay with a total interest of Rs. 230 in 14 instalments, each instalment being less than the preceding by Rs. 10. What should be his first instalment?

Solution:

Let the first instalment be x then the sequence of instalment will be $x, x-10, x-20, \ldots.$

Here,

$a_1 = x$

$d = -10$

$n = 14$

$\boldsymbol{S_n = 1100 + 230 = 1330}$

Now,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 1330 = \frac{14}{2}[2(x) + (14-1)(-10)]$

$\Rightarrow 1330 = 7[2x - 130]$

$\Rightarrow 1330 = 14x - 910$

$\Rightarrow 1330 + 910 = 14x$

$\Rightarrow 2240 = 14x$

$\Rightarrow x = \frac{2440}{14} = 160$

Hence the first instalment is 160.

Question#4

A clock strikes once when its hour hand is at one, twice when it is at two and so on. How many times does the clock strike in twelve hours ?

Solution:

The sequence of the strikes is $\mathbf{1, 2, 3, \ldots\ldots\ldots\ldots, 12}$

Here,

$a_1 = 1$

$d = a_2 - a_1 = 2 - 1 = 1$

$n = 12$

$a_n = 12$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow S_n = \frac{12}{2}[2(1) + (12-1)(1)]$

$= \frac{12}{2}[2 + 11]$

$= \frac{12}{2}[13]$

$= 78$

Hence clock strikes 78 hours in twelve strikes.

Question#5

A student saves Rs.12 at the end of the first week and goes on increasing his saving Rs.4 weekly. After how many weeks will he be able to save Rs.2100?

Solution:

The sequence of the savings is $12, 16, 20, \ldots\ldots\ldots\ldots\ldots$

Total Savings $= 2100$

Here,

$a_1 = 12$

$d = a_2 - a_1 = 16 - 12 = 4$

$S_n = 2100$

$n = ?$

Since,

$\boldsymbol{S_n = \frac{n}{2}[2a_1 + (n-1)d]}$

$\Rightarrow 2100 = \frac{n}{2}[2(12) + (n-1)(4)]$

$\Rightarrow 4200 = n[24 + 4n - 4]$

$\Rightarrow 4200 = n(4n + 20)$

$\Rightarrow 4200 = 4n^2 + 20n$

$\Rightarrow 4n^2 + 20n - 4200 = 0$

$\Rightarrow n^2 + 5n - 1050 = 0$

So,

$n = \frac{-5\pm\sqrt{(5)^2-4(1)(-1050)}}{2(1)}$

$n = \frac{-5\pm\sqrt{25+4200}}{2} = \frac{-5\pm\sqrt{4225}}{2} = \frac{-5\pm 65}{2}$

So,

$n = \frac{-5+65}{2} = \frac{60}{2} = 30$ ***or*** $n = \frac{-5-65}{2} = \frac{-70}{2} = -35$

Since n can never be negative therefore $n = 30$

Thus, student will save Rs. 2100 in 30 weeks

Question#6

An object falling from rest, falls 9 meters during the first second, 27 meters during the next second, 45 meters during the third second and so on.

i) How far will it fall during the fifth second?

Solution:

The sequence of the falls is $9, 27, 45, \ldots\ldots\ldots\ldots\ldots\ldots$

$a_1 = 5$

$d = 27 - 9 = 18$

$a_5 = ?$

Since,

$a_5 = a_1 + 4d$

$= 9 + 4(18)$

$= 9 + 72$

$= 81$

Hence in fifth second the object will fall 81 meters.

ii) How far will it fall up to the fifth second?

Solution:

Here,

$a_1 = 9$

$d = a_2 - a_1 = 27 - 9 = 18$

$n = 5$

$S_5 = ?$

Since,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow S_n = \frac{5}{2}[2(9) + (5-1)(18)]$

$= \frac{5}{2}[18 + 72]$

$= \frac{5}{2}[90]$

$= 225$

Thus, up to 5th second the object will fall 225 meters.

Question#7

An investor earned Rs.6000 for year 1980 and Rs. 12000 for year 1990 on the same investment. If his earning has increased by the same amount each year, how much income he has received from the investment over the past eleven years?

Solution:

Here,

$a_1 = 6000$

$a_{11} = 12000$

$n = 11$

Since,

$S_n = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow S_n = \frac{11}{2}[6000 + 12000]$

$= \frac{11}{2}[18000]$

$= 99000$

Hence, he will receive Rs. 99000 in past eleven years

Question#8

The sum of interior angles of polygon having sides 3,4,5,etc. form an A.P. Find the sum of the interior angles for a 16-sided polygon.

Solution:

Since the sum of angels of 3-sided polygon (triangle) $= a_1 = \pi$

Sum of angels of 4-sided polygon (quadrilateral) $= a_2 = 2\pi$

Sum of the angels of 5 sided polygon (pentagon) $= a_3 = 3\pi$

So,

The sum of interior angels of 16 side polygon $=$

$a_{14} = ?$

Here,

$a_1 = \pi$

$d = a_2 - a_1 = 2\pi - \pi = \pi$

$n = 14$

Since,

$a_n = a_1 + (n-1)d$

$\Rightarrow a_{14} = \pi + (14-1)(\pi)$

$= \pi + 13\pi$

$= 14\pi$

Hence sum of interior angels of 16 side polygon is 14π

Question#9

The prize money Rs. 60,000 will be distributed among the eight teams according to their positions determined in the match-series. The award increases by the same amount for each higher position. If the last place team is given Rs.4000, how much will be awarded to the first-place team?

Solution:

Let a_1 ***denotes the prize money for the last position***

Here,

$a_1 = 4000$

$S_N = 60000$

$n = 8$

$a_n = ?$

Since,

$S_n = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 60000 = \frac{8}{2}[4000 + a_n]$

$\Rightarrow 60000 = 4[4000 + a_n]$

$\Rightarrow 60000 = 16000 + 4a_n$

$\Rightarrow 60000 - 16000 = 4a_n$

$\Rightarrow 44000 = 4a_n$

$\Rightarrow a_n = \frac{44000}{4} = 11000$

Hence the team at 1st place will get 11000 Rs

Question#10

An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row. After this base layer, second layer is formed by placing 7 balls in its first row, 6

balls in its second row and so on with one ball in its last row. Continuing this process, a pyramid of balls is formed with one ball on top. How many balls are there in the pyramid?

Solution:

Balls in the first layer $= 8+7+6+\cdots+2+1$

$=\frac{8}{2}[2(8)+(8-1)(-1)]=4(16-7)=36$

Balls in the second layer $= 7+6+5+\ldots\ldots\ldots+2+1$

$=\frac{7}{2}[2(7)+(7-1)(-1)]$

$=\frac{7}{2}[14-6]$

$=\frac{7}{2}[8]=28$

Balls in the third layer $= 6+5+4+3+2+1=21$

Balls in the fourth layer $=5+4+3+2+1=15$

Balls in the fifth layer $= 4+3+2+1=10$

Balls in the sixth layer $=3+2+1+6$

Balls in the seventh layer $=2+1=3$

Balls in the eighth layer $= 1$

Hence the number of balls in pyramid

$= 36+28+21+15+10+6+3+1=\mathbf{120}$

Geometric progression(G.P)

A sequence $\{a_n\}$ *is a geometric sequence if* $\frac{a_n}{a_{n-1}}$ *is same for all terms when* $n\in N$ *and* $n>1$.

- *The quotient* $\frac{a_n}{a_{n-1}}$ *is denoted by r. and is called common ratio of the G.P*
- *The common ratio r=*$\frac{a_n}{a_{n-1}}$ *is defined only if* $a_{n-1}\neq 0$ *i.e No term of G.P is zero.*
- *Note: when common ratio of any two consecutive terms of a sequence is called a geometry sequence or geometric progression.*

Prove that $\boldsymbol{a_n=a_1r^{n-1}}$

Proof:

We know that

$$r=\frac{a_n}{a_{n-1}} \quad \to (i) \quad n>1$$

Put $n=2,3,,4,\ldots,$ *in* (i)

$$\Rightarrow r=\frac{a_2}{a_1}\Rightarrow a_2=a_1r$$

$$r=\frac{a_3}{a_2}\Rightarrow a_3=a_2r$$

$$or \quad a_3=(a_1r)r\Rightarrow a_3=a_1r^2(\because a_2=a_1r)$$

$$r=\frac{a_4}{a_3}\Rightarrow a_4=a_3r=(a_1r^2)r$$

$$\Rightarrow a_4=a_1r^3(\because a_3=a_1r^2)$$

In similar way , we have

$$a_n=a_1r^{n-1}$$

Hence proved.

Exercise 6.6

Question#1

Find the 5th term of the G.P.: $\mathbf{3,6,12,\ldots}$

Solution:

Here,

$a_1 = 3$

$r=\frac{a_2}{a_1}=\frac{6}{3}=2$

$n=5$

Since,

$a_n = a_1r^{n-1}$

$\Rightarrow a_5=(3)(2)^{5-1}$

$=(3)(2)^4$

$=(3)(16)$

$=48$

Question#2

Find the 11th term of the sequence, $\mathbf{1+i\,,2\,,\frac{4}{1+i}\,,\ldots}$

Solution:

Here,

$a_1 = 1+i$

$r=\frac{a_2}{a_1}=\frac{2}{1+i}$

$n=11$

Since,

$a_n=a_1r^{n-1}$

$\Rightarrow a_{11}=(1+i)\left(\frac{2}{1+i}\right)^{11-1}$

$=(1+i)\left(\frac{2}{1+i}\right)^{10}$

$=(1+i)\left(\frac{2}{1+i}\cdot\frac{1-i}{1-i}\right)^{10}$

$=(1+i)\left(\frac{2(1-i)}{(1)^2-(i)^2}\right)^{10}$

$=(1+i)\left(\frac{2(1-i)}{1+1}\right)^{10}$

$=(1+i)\left(\frac{2(1-i)}{2}\right)^{10}$

$=(1+i)(1-i)^{10}$

$=(1+i)[(1-i)^2]^5$

$=(1+i)[(1)^2-2(1)(i)+(i)^2]^5$

$=(1+i)[1-2i-1]^5$

$=(1+i)(-2i)^5$

$=(1+i)(-2)^5(i)^5$

$=(1+i)(-32)i^4.i$

$=(1+i)(-32)(i^2)^2.i$

$=(1+i)(-32)(-1)^2.i$

$=(1+i)(-32)(1).i$

$=-32i(1+i)$

$=-32i-32\,i^2$

$=-32i-32\,(-1)$

$=-32i+32$

$= 32(-i+1)$
$= 32(1-i)$

Question#3

Find the 12th term of $1+i, 2i, -2+2i, \dots$

Solution:

Here,

$a_1 = 1+i$

$r = \frac{a_2}{a_1} = \frac{2i}{1+i}$

$n = 12$

Since,

$a_n = a_1 r^{n-1}$

$\Rightarrow a_{11} = (1+i)\left(\frac{2i}{1+i}\right)^{12-1}$

$= (1+i)\left(\frac{2i}{1+i}\right)^{11}$

$= (1+i)\left(\frac{2i}{1+i}\cdot\frac{1-i}{1-i}\right)^{10}$

$= (1+i)\left(\frac{2i(1-i)}{(1)^2-(i)^2}\right)^{11}$

$= (1+i)\left(\frac{2i-2i^2}{1+1}\right)^{11}$

$= (1+i)\left(\frac{2i+2}{2}\right)^{11}$

$= (1+i)\left(\frac{2(i+1)}{2}\right)^{11}$

$= (1+i)(1+i)^{11}$

$= (1+i)^{12}$

$[(1+i)^2]^6$

$= [(1)^2 + 2(1)(i) + (i)^2]^6$

$= [1+2i-1]^5$

$= (2i)^6$

$= (2)^6(i)^6$

$= (64)i^6$

$= (64)(i^2)^3$

$= 64(-1)^3$

$= -64$

Question#4

Find the 11th term of the sequence, $1+i, 2, 2(1-i), \dots$

Solution:

Here

$a_1 = 1+i,\quad r = \frac{2}{1+i}, a_{11} = ?$

$\because a_{11} = a_1 r^{10} \qquad \because a_n = a_1 r^{n-1}$

$= (1+i)\left(\frac{2}{1+i}\right)^{10}$

$= \frac{(1+i)2^{10}}{(1+i)^{10}}$

$= \frac{(1+i)2^{10}}{[(1+i)^2]^5}$

$= \frac{(1+i)1024}{(2i)^5}$

$= \frac{(1+i)2^{10}}{[(1+i)^2]^5}$

$= \frac{(1+i)1024}{(2i)^5}$

$= \frac{(1+i)32}{i^4.i}$

$\because (1+i)^2$
$= (1)^2 + (i)^2 + 2(1)(i)$
$= 1 + i^2 + 2i$
$= 1 - 1 + 2i$
$= 2i \quad \because \quad i^2 = -1$

$= \frac{(1+i)32}{(i^2)^2.i}$

$= \frac{(1+i)32}{(-1)^2.i} = \frac{(1+i)32}{i}$

$a_{11} = \frac{(i-1)32}{-1} = -32(i-1)$

$a_{11} = 32(1-i) \quad (\because i^2 = -1)$

Question#5

If an automobile depreciates in value 5% every year, at the end of 4 years what is the value of the automobile purchased for Rs.12,000?

Solution:

Here,

$a_1 = 12000$

depreciation $= 5\%$

$r = 1 - \frac{5}{100} = 1 - 0.05 = 0.95$

$n = 5$

Since,

$a_n = a_1 r^{n-1}$

$\Rightarrow a_5 = (12000)(0.95)^{5-1}$

$= (12000)(0.95)^4$

$= (12000)(0.8145)$

$= 9774.08$

The value of the automobile at the end of 4 years is 9774.08

Question#6

Which term of the sequence $x^2 - y^2, x + y, \frac{x+y}{x-y}, \dots$ ***is*** $\frac{x+y}{(x-y)^9}$?

Solution:

Here,

$a_1 = x^2 - y^2$

$r = \frac{a_2}{a_1} = \frac{x+y}{x^2-y^2} = \frac{x+y}{(x-y)(x+y)} = \frac{1}{x-y}$

$n = ?$

$a_n = \frac{x+y}{(x-y)^9}$

Since,

$a_n = a_1 r^{n-1}$

$\Rightarrow \frac{x+y}{(x-y)^9} = (x^2-y^2)\left(\frac{1}{x-y}\right)^{n-1}$

$\Rightarrow \frac{x+y}{(x-y)^9} = (x+y)(x-y)\frac{1}{(x-y)^{n-1}}$

$\Rightarrow \frac{1}{(x-y)^9} = \frac{1}{(x-y)^{n-1-1}}$

$\Rightarrow \left(\frac{1}{x-y}\right)^9 = \left(\frac{1}{x-y}\right)^{n-2}$

$\Rightarrow 9 = n - 2$

$\Rightarrow 9+2=n$

$\Rightarrow n=11$

Question#7

If a, b, c, d are in G.P, prove that

Solution:

Since , a, b, c, d are in G.P

Therefore,

$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$

So,

$\frac{b}{a}=\frac{c}{b}\Rightarrow b^2=ac \quad ....(i)$

$\frac{c}{b}=\frac{d}{c}\Rightarrow c^2=bd \quad ....(ii)$

$\frac{b}{a}=\frac{d}{c} \quad \Rightarrow bc=ad \quad ....(iii)$

$(\boldsymbol{i})$. $\boldsymbol{a-b, b-c, c-d}$ **are in G.P.**

Solution:

To show, $a-b, b-c, c-d$ are in G.P

So,

$r=\frac{b-c}{a-b} \quad ...(i)$

Also,

$r'=\frac{c-d}{b-c} \quad ...(ii)$

$=\frac{c-d}{b-c}\cdot\frac{a-b}{a-b}$

$=\frac{ac-ad-bc+bd}{(b-c)(a-b)}\cdot\frac{a-b}{a-b}$

$=\frac{b^2-bc-bc+c^2}{(b-c)(a-b)}$

$\because ac=b^2$

$ad=bc$

$bd=c^2$

$=\frac{(b-c)^2}{(b-c)(a-b)}$

$=\frac{(b-c)}{(a-b)}=\frac{b-c}{a-b} \quad ....(ii)$

From (i) & (ii)

$r=r'$

Therefore,

$a-b, b-c, c-d$ are in G.P

$(\boldsymbol{ii})$. $\boldsymbol{a^2-b^2, b^2-c^2, c^2-d^2}$ **are in G.P.**

Solution:

To show, $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P

So,

$r=\frac{b^2-c^2}{a^2-b^2} \quad ...(i)$

Also,

$r'=\frac{c^2-d^2}{b^2-c^2} \quad ...(ii)$

$=\frac{c^2-d^2}{b^2-c^2}\cdot\frac{a^2-b^2}{a^2-b^2}$

$=\frac{a^2c^2-a^2d^2-b^2c^2+b^2d^2}{(b^2-c^2)(a^2-b^2)}$

$=\frac{(ac)^2-(ad)^2-(bc)^2+(ac)^2}{(b^2-c^2)(a^2-b^2)}$

From (i), (ii), (iii)

$=\frac{(b^2)^2-(bc)^2-(bc)^2+(c^2)^2}{(b^2-c^2)(a^2-b^2)}$

$=\frac{b^4-2b^2c^2+c^4}{(b^2-c^2)(a^2-b^2)}$

$=\frac{(b^2-c^2)^2}{(b^2-c^2)(a^2-b^2)}$

$=\frac{b^2-c^2}{a^2-b^2} \quad ...(ii)$

From (i) & (ii)

$r=r'$

Therefore, $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P

$(\boldsymbol{iii})$. $\boldsymbol{a^2+b^2, b^2+c^2, c^2+d^2}$ **are in G.P.**

Solution:

So,

$r=\frac{b^2+c^2}{a^2+b^2} \quad ...(i)$

Also,

$r'=\frac{c^2+d^2}{b^2+c^2}$

$=\frac{c^2+d^2}{b^2+c^2}\cdot\frac{a^2+b^2}{a^2+b^2}$

$\Rightarrow \quad \boldsymbol{(b^2+c^2)^2=(c^2+d^2)(a^2+b^2)}$

$\boldsymbol{R.H.S=(a^2+b^2)(c^2+d^2)}$

$\boldsymbol{=(ac)^2+(ad)^2+b^2c^2+b^2d^2}$

$\boldsymbol{(b^2)^2+b^2c^2+b^2c^2+(c^2)^2}$

$\boldsymbol{=(b^2)^2+2b^2c^2+(c^2)^2}$

$\boldsymbol{=(b^2+c^2)^2=L.H.S}$

Hence $a^2+b^2, b^2+c^2+d^2$ are in G.P

Question#8

Show that the reciprocals of the terms of the geometric sequence $a_1, a_1r^2, a_1r^4...$ form another geometric sequence

Solution:

The sequence of reciprocal of the term is,

$\frac{1}{a_1}, \frac{1}{a_1r^2}, \frac{1}{a_1r^4}$

To show this is in G.P.

$r_1=\frac{a_2}{a_1}$

$=\frac{\frac{1}{a_1r^2}}{\frac{1}{a_1}}$

$=\frac{1}{a_1r^2}\cdot a_1$

$r_1=\frac{1}{r^2} \quad ...(i)$

Also,

$r_1'=\frac{a_3}{a_2}$

$=\frac{\frac{1}{a_1r^4}}{\frac{1}{a_1r^2}}$

$=\frac{1}{a_1r^4}\cdot a_1r^2$

$r'=\frac{1}{r^2} \quad ...(ii)$

From (i) & (ii)

$r=r'$

The sequence of reciprocal of the term is, also in $G.P$

Question#9

Find the nth term of the geometric sequence if $\frac{a_5}{a_3}=\frac{4}{9}$ ***and*** $a_2=\frac{4}{9}$

Solution:

Let a_1 *be the first term and* r *be the common ratio*

Since,

$\frac{a_5}{a_3}=\frac{4}{9}$

$\frac{a_1r^{5-1}}{a_1r^{3-1}}=\frac{4}{9}$

$\frac{r^4}{r^2}=\frac{4}{9}$

$\Rightarrow r^2=\frac{4}{9}$

$\Rightarrow r=\pm\frac{2}{3}$

Also,

$a_2=\frac{4}{9}$

$\Rightarrow a_1r^{2-1}=\frac{4}{9}$

$\Rightarrow a_1r=\frac{4}{9}$

When $r=\frac{2}{3}$

$\Rightarrow a_1\left(\frac{2}{3}\right)=\frac{4}{9}$

$\Rightarrow a_1=\frac{4}{9}.\frac{3}{2}$

$\Rightarrow a_1=\frac{2}{3}$

When $r=-\frac{2}{3}$

$\Rightarrow a_1\left(-\frac{2}{3}\right)=\frac{4}{9}$

$\Rightarrow a_1=\frac{4}{9}.\left(-\frac{3}{2}\right)$

$\Rightarrow a_1=-\frac{2}{3}$

Since,

$a_n=a_1r^{n-1}$

When $a_1=\frac{2}{3}$ *and* $r=\frac{2}{3}$

$\Rightarrow a_n=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1}$

$=\left(\frac{2}{3}\right)^n$

When $a_1=-\frac{2}{3}$ *and* $r=-\frac{2}{3}$

$\Rightarrow a_n=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1}$

$=\left(-\frac{2}{3}\right)^n$ *or* $(-1)^n\left(\frac{2}{3}\right)^n$

Question#10

Find three, consecutive numbers in G.P whose sum is 26 and their product is 216.

Solution:

Consider $\frac{a_1}{r}, a_1, a_1r$ *are three consecutive numbers in G.P.*

By the given condition

$\frac{a_1}{r}+a_1+a_1r=26$

$\Rightarrow a_1\left(\frac{1}{r}+1+r\right)=26$

Multiplying by r

$\Rightarrow a_1(1+r+r^2)=26r \; ...(i)$

Also given,

$\left(\frac{a_1}{r}\right)(a_1)(a_1r)=216$

$\Rightarrow {a_1}^3=216$

$\Rightarrow {a_1}^3=(6)^3$

$\Rightarrow a_1=6$

Putting in eq. (i)

$\Rightarrow 6(1+r+r^2)=26r$

$\Rightarrow 6+6r+6r^2-26r=0$

$\Rightarrow 6r^2-20r+6=0$

$\Rightarrow 2(r^2-10r+3)=0$

$\Rightarrow r^2-10r+3=0$

$r=\frac{10\pm\sqrt{10^2-4(3)(3)}}{2(3)}$

$r=\frac{10\pm\sqrt{100-36}}{2(3)}$

$=\frac{10\pm\sqrt{64}}{6}$

$=\frac{10\pm8}{6}$

$r=\frac{10+8}{6}$ *or* $r=\frac{10-8}{6}$

$r=\frac{18}{6}=3$ $\quad r=\frac{2}{6}=\frac{1}{3}$

When $a_1=6$ *and* $r=3$

$\frac{a_1}{r}=\frac{6}{3}=2$

$a_1=6$

$a_1r=(6)(3)=18$

When $a_1=6$ *and* $r=\frac{1}{3}$

$\frac{a_1}{r}=\frac{6}{\frac{1}{3}}=6\times3=18$

$a_1=6$

$a_1r=(6)\left(\frac{1}{3}\right)=2$

Hence $2,6,18$ *or* $18,6,2$ *are required numbers in G.P.*

Question#11

If the sum of the four consecutive terms of a G.P is 80 and A.M of the second and the fourth of them is 30. Find the terms.

Solution:

Let the four terms in $a_1, a_1r, a_1r^2, a_1r^3$

By the given condition

$a_1,a_1r, a_1r^2, a_1r^3=80$

$\Rightarrow a_1(1,r,r^2,r^3)=80$

$\Rightarrow a_1[1(1+r)+r^2(1+r)]=80$

$\Rightarrow a_1(1+r)(1+r^2)=80 \quad ...(i)$

Also, we have given

$\frac{a_1+a_1r^3}{2}=30$

$\Rightarrow \frac{a_1r(1+r^2)}{2}=30$

$\Rightarrow a_1 r(1 + r^2) = 60 \quad \dots(iii)$

From eq. (i)

$1 + r^2 = \frac{80}{a_1(1+r)}$

Putting in (iii)

$a_1 r \cdot \frac{80}{a_1(1+r)} = 60$

$\Rightarrow \frac{80r}{1+r} = 60$

$\Rightarrow 80r = 60(1 + r)$

$\Rightarrow 80r - 60r = 60$

$\Rightarrow 20r = 60$

$\Rightarrow r = \frac{60}{20}$

$\Rightarrow r = 3$

Putting values of r in eq. (i)

$a_1(1 + 3)(1 + (3)^2) = 80$

$\Rightarrow a_1(4)(10) = 80$

$\Rightarrow 40a_1 = 80$

$\Rightarrow a_1 = 2$

So,

$a_1 r^2 = (2)(3)^2 = 2 \times 9 = 18$

$a_1 r^3 = (2)(3)^3 = 2 \times 27 = 54$

Hence 2,6,18,54 *are the required numbers.*

Question#12

If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P. show that the common ratio is $\pm\sqrt{\frac{a}{c}}$

Solution:

Since, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in G.P.*

Therefore,

$r = \frac{1/c}{1/b} = \frac{b}{c} \quad \dots(ii)$

Multiplying (i) *and* (ii)

$r.r = \frac{a}{b} \cdot \frac{b}{c}$

$\Rightarrow r^2 = \frac{a}{c}$

$\Rightarrow r = \pm\sqrt{\frac{a}{c}}$

Proved.

Question#13

If the numbers 1, 4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the numbers if their sum is 21.

Solution:

Let $a_1 - d, a_1, a_1 + d$ *are three numbers in A.P.*

$\Rightarrow 3a_1 = 21$

$\Rightarrow a_1 = 7$

Now,

$a_1 - d - 1, \ a_1 - 4, \ a_1 + d - 3$ *are in G.P.*

therefore

$r = \frac{a_1 - 4}{a_1 - d - 1} = \frac{a_1 + d - 3}{a_1 - d - 1}$

$\Rightarrow (a_1 - 4)^2 = (a_1 + d - 3)(a_1 - d - 1)$

Put $a_1 = 7$

$\Rightarrow (7 - 4)^2 = (7 + d - 3)(7 - d - 1)$

$\Rightarrow (3)^2 = (7 + d - 3)(7 - d - 1)$

$\Rightarrow 9 = (d + 4)(8 - d)$

$\Rightarrow 9 = 6d + 24 - d^2 + 4d$

$\Rightarrow d^2 - 2d - 15 = 0$

$\Rightarrow d(d - 5) + 3(d - 5) = 0$

$\Rightarrow (d - 5)(d + 3) = 0$

$\Rightarrow d - 5 = 0 \qquad d + 3 = 0$

$d = 5 \qquad d = -3$

When $a_1 = 7$ *and* $d = 5$

$a_1 - d = 7 - 5 = 2$

$a_1 = 7$

$a_1 + d = 7 + 5 = 182$

When $a_1 = 7$ *and* $d = 3$

$a_1 - d = 7 - 3 = 4$

$a_1 = 7$

$a_1 + d = 7 + 3 = 10$

Hence 2,7,12, *or* 10,7,4 *are the required numbers.*

Question#14

If three consecutive numbers in A.P. are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

Solution:

Consider numbers $a_1 - d, a_1, a_1 + d$ *then,*

$a_1 - d + 1, \ a_1 + 4, \ a_1 + d + 1$ *are in G.P*

Condition $II \Rightarrow a_1 - d + a_1 + a_1 + d = 6$

$2a_1 = 6 \Rightarrow a_1 = 2$

Condition I

$a_1 - d + 1, a_1 + d + 15$ *are in* $G.P$

or $2 - d + 1, \quad 2 + 4, \quad 2 + d + 15$ *are in* $G.P$

$\Rightarrow 3 - d, 6, 17 + d$ in $G.P \quad (\because a_1 = 2)$

$\Rightarrow \frac{17+d}{6} = \frac{6}{3-d} \quad (common\ ratio)$

$\Rightarrow (17 + d)(3 - d) = 36$

$51 - 17d + 3d - d^2 = 36$

$-d^2 - 14d - 15 = 0$

or $-d^2 - 14d + 15 = 0$

$\Rightarrow d^2 + 14d - 15 = 0$

$d^2 + 15d - d - 15 = 0$

$d(d + 15) - 1(d + 15) = 0$

$(d + 15) = 0, d - 1 = 0$

$d = -15, \ d = 1$

When $d = -15, a_1 = 2$

$a_1 + d = 2 + (-15) = -13, \ a_1 - d = 2 - (15) = 17$

When $d = 1, \ a_1 = 2$

$a_1 + d = 2 + 1 = 3 \quad , a_1 - d = 2 - 1 = 1$

required numbers are $-13, 2, 17$ *or* $1, 2, 3$

Geometric means(G.M)

A number G is said to be geometric mean a and b if a, G, b are in $G.P$"

In this case

$$r = \frac{G}{a} \quad and\ r = \frac{b}{G}$$

$$\Rightarrow \frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab$$

$$or \quad G = \pm\sqrt{ab}$$

n Geometric means between two given numbers:

the numbers $G_1, G_2, G_3, \dots, G_n$ are called n geometric means between a and b if

$a, G_1, G_2, G_3, \dots, G_n, b$ are in G.P

Now

$a, G_1, G_2, G_3, \dots, G_n, b$ are in $G.P$

Here $a_1 = a, \quad n = n+2, \quad a_{n+2} = b$

⇨ $a_{n+2} = a_1 r^{n+2-1}$ $\quad (\because a_n = a_1 r^{n-1})$

⇨ $b = ar^{n+1}$

$$or \quad \frac{b}{a} = r^{n+1} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Thus

$$r = \frac{G_1}{a} \Rightarrow G_1 = a_r = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$r = \frac{G_1}{G_2} \Rightarrow G_2 = G_1 r = ar.r$$

$$\Rightarrow G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

$\vdots$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Note :

$$G_1, G_2, G_3, \dots, G_n = a^n\left(\frac{b}{a}\right)^{\frac{1+2+3+\cdots+n}{n+1}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$$

And

$$\sqrt[n]{G_1, G_2, G_3, \dots, G_n} = a\left(\frac{b}{a}\right)^{\frac{1}{2}} = \sqrt{ab}$$

$$= G = G.M \ \ between\ a\ and\ b$$

Remembers that

- If the number of required $G.M's$ is even, there is only one set of G.M's we talk value of r only positive.
- If the number of required G.M 's is odd, there is two sets of G.M's i.e; we take value of r both positive and negative.

Exercise 6.7

Question#1

Find G.M. between

(i). -2 and 8

Solution:

$a = -2$ *and* $b = 8$

Since,

$G.M = \pm\sqrt{ab}$

$= \pm\sqrt{(-2)(8)}$

$= \pm\sqrt{-16}$

$= \pm 4i$

(ii). $-2i$ and $8i$

Solution:

$a = -2i$ *and* $b = 8i$

Since,

$G.M = \pm\sqrt{ab}$

$= \pm\sqrt{(-2i)(8i)}$

$= \pm\sqrt{-16i^2}$

$= \pm\sqrt{16} \qquad \because i^2 = -1$

$= \pm 4$

Question#2

Insert two G.Ms. between

(i). 1 and 8

Solution:

Let $G_1, G_2,$ *are the two G.Ms. between* 1 *and* 8.

Here,

$a_1 = 1$

$a_4 = 8$

$\Rightarrow a_1 r^3 = 8$

$\Rightarrow (1)r^3 = 8$

$\Rightarrow r^3 = (2)^3$

$\Rightarrow r = 2$

Now,

$G_1 = a_2 = a_1 r = (1)(2) = 2$

$G_2 = a_3 = a_1 r^2 = (1)(2)^2 = (1)(4) = 4$

Hence, 2,4 *are the two G.Ms. between* 1 *and* 8

(ii). 2 and 16

Solution:

Let G_1 and G_2 be two G.Ms between 2 and 16 then

$2, G_1, G_2, 16$ are in $G.P$

Here $a_1 = 2 \ , n = 4 \ , a_4 = 16$

So $a_4 = a_1 r^3 \qquad \because a_n = a_1 a^{n-1}$

⇨ $16 = (2)^3$

⇨ $8 = r^3$ or $(r)^3 = (2)^3$

⇨ $r = 2$

Thus $G_1 = a_1 r = (2)(2) = 4$

$G_2 = a_1 r^2 = (2)(2)^2 = 8$

Question#3

Insert three G.Ms. between

(i). **1 and 16**

Solution:

Let G_1, G_2, G_3 *are the two G.Ms. between* 1 *and* 16.

Here,

$a_1 = 1$

$a_5 = 16$

$\Rightarrow a_1r^4 = 16$

$\Rightarrow (1)r^4 = 16$

$\Rightarrow r^4 = (2)^4$

$\Rightarrow r = 2$

Now,

$G_1 = a_2 = a_1r = (1)(2) = 2$

$G_2 = a_3 = a_1r^2 = (1)(2)^2 = (1)(4) = 4$

$G_3 = a_4 = a_1r^3 = (1)(2)^3 = (1)(8) = 8$

Hence, $2,4,8$ *are the three G.Ms. between* 1 *and* 16

(ii). **2 and 32**

Solution:

let $g_1, G_2, G_3,$ *be three G.Ms between* $2,32$ *then*

$2\ G_1, G_2, G_3, 32$ *are in* $G.P$

Here $a_1 = 2, n = 5,\ a_5 = 32$

$\Rightarrow\ a_5 = a_1r^4 \quad \because\ a_n = a_1r^{n-1}$

or $32 = (2)r^4$

$\Rightarrow\ r^4 = 16$ *or* $(r)^4 = (2)^4$

so $r = 2$

So $G_1 = a_1r = (2)(2) = 4$

$G_2 = a_1r^2 = (2)(2)^2 = 8$

$G_3 = a_1r^3 = (2)(2)^3 = 16$

Question#4

Insert four real geometric means between 3 and 96.

Solution:

Let G_1, G_2, G_3 *and*, G_4 *are the two G.Ms. between* 1 *and* 16.

Here,

$a_1 = 3$

$a_6 = 96$

$\Rightarrow a_1r^5 = 96$

$\Rightarrow (3)r^5 = 96$

$\Rightarrow r^5 = \frac{96}{3} = 32$

$\Rightarrow r^5 = 32$

$\Rightarrow r^5 = (2)^5$

$\Rightarrow r = 2$

Now,

$G_1 = a_2 = a_1r = (3)(2) = 6$

$G_2 = a_3 = a_1r^2 = (3)(2)^2 = (3)(4) = 12$

$G_3 = a_4 = a_1r^3 = (3)(2)^3 = (3)(8) = 24$

$G_4 = a_5 = a_1r^4 = (3)(2)^4 = (3)(16) = 48$

Hence, $6,12,24,48$ *are the four G.Ms. between* 1 *and* 8

Question#5

If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their arithmetic mean.

Solution:

Suppose,

$A > G$

Then

$\Rightarrow \frac{x+y}{2} > \pm\sqrt{xy}$

$\Rightarrow x + y > \pm 2\sqrt{xy}$

$\Rightarrow x + y \mp 2\sqrt{xy} > 0$

$\Rightarrow (\sqrt{x})^2 + (\sqrt{y})^2 \mp 2\sqrt{xy} > 0$

$\Rightarrow (\sqrt{x} \mp \sqrt{y})^2 > 0$

Which is true as square is always $+ve$

Hence, $A > G$

Question#6

For what value of n, $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the positive geometric mean between a and b?

Solution:

Since G.M. between a and b. $= \sqrt{ab}\ \ldots.(i)$

But we have given $A.M. = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}\ \ldots(ii)$

Comparing (i) *and* (ii)

$\frac{a^n+b^n}{a^{n-1}+b^{n-1}} = \sqrt{ab}$

By cross multiplying

$\Rightarrow (a^n + b^n) = a^{\frac{1}{2}}b^{\frac{1}{2}}(a^{n-1} + b^{n-1})$

$\Rightarrow a^n + b^n = a^{n-\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{n-\frac{1}{2}}$

$\Rightarrow a^n - a^{n-\frac{1}{2}}b^{\frac{1}{2}} = a^{\frac{1}{2}}b^{n-\frac{1}{2}} - b^n$

$\Rightarrow a^{n-\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n-\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)$

$\Rightarrow a^{n-\frac{1}{2}} = b^{n-\frac{1}{2}}$

$\Rightarrow \frac{a^{n-\frac{1}{2}}}{b^{n-\frac{1}{2}}} = 1$

$\Rightarrow \left(\frac{a}{b}\right)^{n-\frac{1}{2}} = \left(\frac{a}{b}\right)^0 \quad \because \left(\frac{a}{b}\right)^0 = 1$

$\Rightarrow n - \frac{1}{2} = 0$

$\Rightarrow n = \frac{1}{2}$

Question#7

The A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers

Solution:

Let a and b be two positive integers then by given condition

$A.M. = G.M. + 2$

$\Rightarrow \frac{a+b}{2} = \sqrt{ab} + 2$

$\Rightarrow a + b = 2(\sqrt{ab} + 2)\ \ldots(i)$

Also, we have given

$a+b=20 \quad ....(ii)$

Comparing (i) *and* (ii)

$2(\sqrt{ab}+2)=20$

$\Rightarrow 2\sqrt{ab}+4=20$

$\Rightarrow 2\sqrt{ab}=20-4$

$\Rightarrow 2\sqrt{ab}=16$

$\Rightarrow \sqrt{ab}=\frac{16}{2}$

$\Rightarrow \sqrt{ab}=8$

By squaring

$\Rightarrow ab=64$

$\Rightarrow b=\frac{64}{a} \quad ...(iii)$

Putting in (ii)

$a+\frac{64}{a}=20$

$\Rightarrow a^2+64=20\,a$

$\Rightarrow a^2-20a+64=0$

$\Rightarrow a^2-16a-4a+64=0$

$\Rightarrow a(a-16)-4(a-16)=0$

$\Rightarrow (a-16)(a-4)=0$

$a-16=0 \quad , a-4=0$

$a=16 \quad , a=4$

Putting in (iii)

$\Rightarrow b=\frac{64}{16} \quad , \quad b=\frac{64}{4}$

$\Rightarrow b=4 \quad , \quad b=16$

Hence , $16{,}4$ *or* $4{,}16$ *are the required numbers*

Question#8

The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

Solution:

Let a and b be two positive integers then

$A.M.=5$

$\Rightarrow \frac{a+b}{2}=5$

$\Rightarrow a+b=10 \quad ...(i)$

Also, we have given

$G.M.=4 \quad ....(ii)$

$\Rightarrow \sqrt{ab}=4$

By squaring

$\Rightarrow ab=16$

$\Rightarrow b=\frac{16}{a} \quad ...(ii)$

Putting in (i)

$a+\frac{16}{a}=10$

$\Rightarrow a^2+16=10a$

$\Rightarrow a^2-10a+16=0$

$\Rightarrow a^2-8a-2a+16=0$

$\Rightarrow a(a-8)-2(a-8)=0$

$\Rightarrow (a-8)(a-2)=0$

$a-8=0 \quad , a-2=0$

$a=8 \quad , a=2$

Putting in (ii)

$\Rightarrow b=\frac{16}{8} \quad , \quad b=\frac{16}{2}$

$\Rightarrow b=2 \quad , \quad b=8$

Hence , $8{,}2$ *or* $2{,}8$ *are the required numbers*

Sum of n terms of a Geometric series

$$\boldsymbol{S_n=\frac{a_1(r^n-1)}{r-1}, |r|>1}$$

$$\textbf{and} \quad \boldsymbol{S_n=\frac{a_1(1-r^n)}{1-r}, \quad |r|<1}$$

Proof:

We know that if the sequence $\{a_n\}$ *is a geometric sequence then*

$$S_n=a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1}$$

$'\times'$ *noth sides by* ***(1-r)***

$$(1-r)S_n=(1-r)(a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1})$$

$$=a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1}-a_1r^n$$

$$(1-r)S_n=a_1-a_1r^n$$

$$(1-r)S_n=a_1-a_1r^n$$

$$\Rightarrow \quad S_n=\frac{a_1(1-r^n)}{1-r} \quad if \quad |r|<1$$

$$and \quad S_n=\frac{a_1(r^n-1)}{r-1} \quad if\ |r|>1$$

The infinite Geometric series

$$\boldsymbol{S_\infty=\frac{a_1}{1-r} \quad if\ |r|<1}$$

Proof:

We know that

$$\boldsymbol{a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1}+\cdots\infty}$$

Is called an infinite geometric series. It is denoted by $\boldsymbol{S_\infty}$

$$\boldsymbol{S_n=\frac{a_1(1-r^n)}{1-r} \quad if\ |r|<1}$$

But we do not know how to odd infinite many terms of the series. So for this purpose applying limit as

$$\boldsymbol{n\to\infty \quad i.e}$$

$$\boldsymbol{\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{a_1(1-r^n)}{1-r}}$$

$$\boldsymbol{\because \quad |r|<1, \quad it\ follow\ that}$$

$\boldsymbol{r^n}$ ***gets smaller and smaller as n gets larger and larger so***

When $\boldsymbol{n\to\infty}$ ***then*** $\boldsymbol{r^n\to 0}$

Now

$$S_\infty = \frac{a_1(1-o)}{1-r}$$

$$\Rightarrow S_\infty = \frac{a_1}{1-r}$$

Note S_∞ is only possible if $|r| < 1$

If $|r| > 1$

When r^n does not tend to zero when $n \to \infty$

⇨ S_n *does not tend to a limit and the series does not converge in this series does not converge in this case so the series is divergent.*

$$if\ r = 1$$

Then the series becomes as

$$a_1 + a_1 + a_1 + \cdots + and$$

$S_n = na_1$ *in this case* S_n *does not tend to a limit When* $n \to \infty$ *and the series does not converge.*

$$if\ r = -1$$

Then the series becomes as $a_1 - a_1 + a_1 + a_1 - a_1 + \cdots$

and $S_n = \frac{a_1-(-1)^n a_1}{2}$

i.e $S_n = a_1$ *if* n *is* $+ve$ *odd number.*

$$S_n = \frac{a_1 - (-1)^n a_1}{2}$$

i.e $S_n = a_1$ *if* n *is* $+ve$ *odd number.* $S_n = 0$ *if n is +ve even number.*

Thus S_n *does not tend to a definite number when* $n \to \infty$

In such a case we say that the series is oscillatory.

Exercise 6.8

Question No.1

Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}$

Solution:

$$here\ a_1 = 1, r = \frac{\frac{1}{3}}{1} = \frac{1}{3} < 1$$

$$\because S_n = \frac{a_1(1-r^n)}{1-r} \quad if\ r < 1$$

$$\Rightarrow S_{15} = \frac{(1)\left(1-\left(\frac{1}{3}\right)^{15}\right)}{1-\frac{1}{3}}$$

$$= \frac{1-\frac{1}{3^{15}}}{\frac{3-1}{3}} = \frac{\frac{3^{15}-1}{3^{15}}}{\frac{2}{3}}$$

$$S_{15} = \frac{3}{2}\left[\frac{14348907-1}{3^{15}}\right]$$

$$S_{15} = \frac{14348906}{2.3^{14}}$$

$$S_{15} = \frac{14348906}{2(478269)} = \frac{7174453}{4782969}$$

Question No.2 Sum to n terms, the series

$(i)\ .2 + .22 + .222 + \cdots$

$(ii) 3 + 33 + 333 + \cdots$

Solution:

(i)

$$0.2 + 0.22 + 0.222 + \cdots n\ terms$$
$$= 2[0.1 + 0.11 + 0.111 + \cdots + nterms]$$
$$= \frac{2}{9}(0.9 + 0.99 + 0.999 + \cdots + nterms)$$
$$= \frac{2}{9}\left(\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \cdots + nterms\right)$$
$$= \frac{2}{9}\left[\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{100}\right) + \left(1-\frac{1}{1000}\right) + \cdots + n\ terms\right]$$
$$= \frac{2}{9}\left[(1+1+1+\cdots n\ terms) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots n\ terms\right)\right]$$
$$= \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^n\right)}{\frac{9}{10}}\right] \quad \because a_1 = \frac{1}{10} \quad r = \frac{1}{10} < 1$$
$$= \frac{2}{9}\left(n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right)$$

$(ii)\ 3 + 33 + 333 + \cdots$

Solution:

$$= 3(1 + 11 + 111 + \cdots nterms)$$
$$= \frac{3}{9}(9 + 99 + 999 + \cdots n\ terms)$$
$$= \frac{1}{3}[(10-1) + (100-1) + (1000-1) + \cdots n\ terms]$$
$$= \frac{1}{3}[(10 + 100 + 1000 + \cdots n\ terms] - (1 + 1 + 1 + \cdots nterms)$$
$$a_1 = 10, \ r = 10 > 1$$
$$= \frac{1}{3}\left[\frac{(10)(10)^n - 1}{10-1} - n\right]$$
$$= \frac{1}{3}\left(\frac{10(10^n - 1}{9} - n\right)$$
$$= \frac{1}{3}\left(\frac{10}{9}(10^n - 1) - n\right)$$

Question No.3

Sum to n terms the series

$(i)\ 1 + (a+b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^2 + \cdots$

Solution:

$$1 + (a+b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^3 + \cdots$$

$$'\times'\ and \div\ by\ (a-b)$$

$$= \frac{1}{a-b}[(a-b) + (a^2 - b^2) + (a^3 - b^3) + \cdots n\ terms]$$

$$=\frac{1}{a-b}[(a+a^2+a^3+\cdots n\ terms)-(b+b^2+b^3+\cdots n\ terms]$$

$$\because a_1=a \quad r=a$$

$$=\frac{1}{a-b}\left[\frac{a(1-a^n)}{1-a}-\frac{b(1-b^n)}{1-b}\right]$$

$$=\frac{1}{a-b}\left[\frac{a(1-b)(1-a^n)-b(1-a)(1-b^n)}{(1-a)(1-b)}\right]$$

$$=\frac{a(1-b)(1-a^n)-b(1-a)(1-b^n)}{(a-b)(1-a)(1-b)}$$

ii) $\boldsymbol{r+(1+k)(r^2)+(1+k+k^2)r^3+\cdots}$

Solution:

$$r+(1+k)(r^2)+(1+k+k^2)r^3+\cdots$$

$$'\times'\ and\ \div\ by\ (a-b)$$

$$=\frac{1}{1-k}[(1-k)r+(1-k^2)r^2+(1-k^3)r^3+\cdots n\ terms]$$

$$=\frac{1}{1-k}[r-rk+r^2-r^2k^2+r^3-r^3k^3+\cdots nterms]$$

$$=\frac{1}{-k}[r+r^2+r^3+\cdots nterms]-(kr+k^2r^2+k^3r^3+\cdots+n\ terms)$$

$$=\frac{1}{1-k}\left[\frac{r(1-r^n)}{1-r}-\frac{kr(1-(kr)^n}{1-kr}\right]$$

$$=\frac{r}{1-k}\left[\frac{1-r^n}{1-r}-\frac{k(1-k^nr^n}{1-kr}\right]$$

Question No. 4

Sum the series $\boldsymbol{2+(1-i)+\frac{1}{i}+\cdots}$ ***to 8 terms***

$$2+(1-i)+\frac{1}{i}+\cdots to\ 8\ terms$$

Here $a_1=2,\quad r=\frac{1-i}{2},\quad n=8<1$

$$\because S_n=\frac{a_1(1-r^n)}{1-r}$$

$$S_8=\frac{2\left[1-\left(\frac{1-i}{2}\right)^8\right]}{1-\frac{1-i}{2}}$$

$$S_8=\frac{2[2^8-(1-i)^8]}{2^8\left(\frac{2-1+i}{2}\right)}$$

$$S_8=\frac{2^8-(1-i)^8}{2^6(1+i)}$$

$$S_8=\frac{256-[(1-i)^2]^4}{64(1+i)}$$

$$S_8=\frac{256[1+i^2-2i]^4}{64(1+i)} \quad \because i^2=-1$$

$$S_8=\frac{256-(-2i)^4}{64(1+i)}$$

$$S_8=\frac{256-16i^4}{64(1+i)}$$

$$S_8=\frac{256-16}{64(1+i)} \quad \because i^4=(i^2)^2=(-1)^2=1$$

$$S_8=\frac{240}{64(1+i)}=\frac{15}{4(1+i)}$$

$$S_8=\frac{15}{4(1+i)}\times\frac{1-i}{1-i}$$

$$S_8=\frac{15(1-i)}{4(1-i^2)}=\frac{15(1-i)}{4(1-(-1))}$$

$$S_8=\frac{15(1-i)}{4(2)}=\frac{15(1-i)}{8}$$

$$Thus\ \ S_8=\frac{5(1-i)}{8}$$

Question No. 5

Find the sums of the following infinite geometric series

$$(\boldsymbol{i})\ \frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$$

Solution:

Here $a_1=\frac{1}{5},\quad r=\frac{\frac{1}{25}}{\frac{1}{5}}=\frac{1}{25}\times\frac{5}{1}$

$$\because S\infty=\frac{a_1}{1-r}=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{\frac{1}{5}}{\frac{4}{5}}$$

$$S_\infty=\frac{1}{5}\times\frac{5}{4}=\frac{1}{4}$$

$(ii)\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

Solution:

$$here\ a_1=\frac{1}{2},r=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{4}\times\frac{2}{1}=\frac{1}{2}$$

$$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$$

$(iii)\ \frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\cdots$

Solution:

$$a_1=\frac{9}{4},\quad r=\frac{\frac{3}{2}}{\frac{9}{4}}=\frac{\frac{3}{2}}{\frac{9}{4}}=\frac{3}{2}\times\frac{4}{9}=\frac{2}{3}$$

$$\because S_\infty=\frac{a_1}{1-r}=\frac{\frac{9}{4}}{1-\frac{2}{3}}=\frac{\frac{9}{4}}{\frac{3-2}{3}}$$

$$S_\infty=\frac{\frac{9}{4}}{1/3}=\frac{9}{4}\times\frac{3}{1}=\frac{27}{4}$$

$(iv)\ 2+1+0.5+\cdots$

Solution:

Here $a_1=2,\quad r=\frac{1}{2}$

$$\because\quad S_\infty=\frac{a_1}{1-r}=\frac{2}{1-\frac{1}{2}}=\frac{2}{\frac{1}{2}}=2\times\frac{2}{1}$$

$$S_\infty=4$$

$(v)\quad 4+2\sqrt{2}+2\sqrt{2}+1+\cdots$

Solution:

Here $a_1=4,\quad r=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$$S_\infty = \frac{1}{\sqrt{2}}$$

$$\because \quad S_\infty = \frac{a_1}{1-r} = \frac{4}{1-\frac{1}{\sqrt{2}}}$$

$$S_\infty = \frac{4}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{4\sqrt{2}}{\sqrt{2}-1}$$

$$= \frac{4\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{4\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2})^2-(1)^2} = \frac{4(2+\sqrt{2})}{2-1}$$

$$S_\infty = \frac{4(2+\sqrt{2})}{1} = 4(2+\sqrt{2}) = 4(2+\sqrt{2})$$

(vi) $0.1 + 0.05 + 0.025 + \cdots$

Solution:

$$a_1 = 0.1, \qquad r = \frac{0.05}{0.1} = o.5$$

$$\because S_\infty = \frac{a_1}{1-r} = \frac{0.1}{1-0.5} = \frac{0.1}{0.5}$$

$$\Rightarrow S_\infty = 0.2$$

Question No.6

Find vulgar fractions equivalent to the following recurring decimal's

(i) $1.\dot{3}\dot{4}$

Solution:

$$1.\dot{3}\dot{4} = 1.34\ 34\ 34\ \ldots$$

$$= 1 + 0.34\ 34\ 34\ \ldots$$

$$= 1 + (0.34 + 0.0034 + \cdots)$$

$$a_1 = 0.34, \quad r = \frac{0.0034}{0.34} = 0.01$$

$$= 1 + \frac{a_!}{1-r}$$

$$= 1 + \frac{0.34}{1-0.01} = 1 + \frac{0.34}{0.99}$$

$$= 1 + \frac{34}{99} = \frac{99+34}{99} = \frac{133}{99}$$

(ii) $0.\dot{7}$

Solution:

$$0.\dot{7} = 0.7777\ \ldots$$

$$= 0.7 + 0.07 + 0.007 + \cdots$$

$$a_1 = 0.7 \quad , r = \frac{0.07}{0.7} = 0.1$$

$$\because \quad S_\infty = \frac{a_1}{1-r} = \frac{0.7}{1-0.1} = \frac{0.7}{0.9}$$

$$S_\infty = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$$

(iii) $0.\dot{2}5\dot{9}$

Solution:

$$0.\dot{2}5\dot{9} = 0.259\ 259\ 259\ \ldots$$

$$0.259 + 0.000259 + \cdots$$

$$a_1 = 0.259, r = \frac{0.000259}{0.259} = 0.001$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.259}{1-0.001}$$

$$= \frac{0.259}{0.999} = \frac{\frac{259}{1000}}{\frac{999}{1000}} = \frac{259}{999}$$

(iv) $1.\dot{5}\dot{3}$

Solution:

$$1.\dot{5}\dot{3} = 1.53\ 53\ 53\ \ldots$$

$$= 1 + 0.535353\ \ldots$$

$$= 1 + (0.53 + 0.0053 + \cdots)$$

$$a_1 = 0.53, \quad r = \frac{0.0053}{0.53} = 0.01$$

$$1 + \frac{a_1}{1-r} = 1 + \frac{0.53}{1-0.01}$$

$$= 1 + \frac{0.53}{0.99} = 1 + \frac{\frac{53}{100}}{\frac{99}{100}}$$

$$= 1 + \frac{53}{99} = \frac{99+53}{99} = \frac{152}{99}$$

(v) $0.\dot{1}5\dot{9}$

Solution:

$$0.\dot{1}5\dot{9} = 0.159\ 159\ 159\ \ldots$$

$$0.159 + 0.000159 + \cdots$$

$$a_1 = 0.159, r = \frac{0.000159}{0.159} = 0.001$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.159}{1-0.001}$$

$$= \frac{0.159}{0.999} = \frac{\frac{159}{1000}}{\frac{999}{1000}} = \frac{159}{999}$$

(vi) $1.\ 1\dot{4}\dot{7}$

Solution:

$$1.\ 1\dot{4}\dot{7} = 1.1474747\ \ldots$$

$$1.1 + 0.0474747\ \ldots$$

$$= 1.1 + 0.47 + 0.00047 + \cdots$$

$$a_1 = 0.047, \quad r = \frac{0.00047}{0.047} = 0.01$$

$$= 1.1 + \frac{a_1}{1-r}$$

$$= 1.1 + \frac{0.047}{1-0.01} = 1.1 + \frac{0.047}{0.99}$$

$$= 1.1 + \frac{\frac{47}{1000}}{\frac{99}{100}} = 1.1 + \frac{47}{1000} \times \frac{100}{99}$$

$$= 1.1 + \frac{47}{990} = \frac{11}{10} + \frac{47}{990}$$

$$= \frac{1089+47}{990} = \frac{1136}{990}$$

Vuigar fraction:

A fraction whose numerator and denominator both are integers is called vulgar fraction or common fraction.

Proper fraction:

If the numerator of the fraction is less than the denominator, the fraction is called proper fraction.

Question No.7

Find the sum to infinity of the series

$\boldsymbol{r+(1+k)r^2+(1+k+k^2)r^3+\cdots}$

$\boldsymbol{r \text{ and } k \text{ being proper fractions.}}$

Solution:

$r+(1+k)r^2+(1+k+k^2)r^3+\cdots$

$'\times'$ and $\div$ by $1-k$

$$=\frac{1}{k-1}[(1-k)r+(1-k^2)r^2+(1-k^3)r^3+\cdots n\ terms]$$

$$=\frac{1}{k-1}[r-rk+r^2-k^2r^2+r^3-k^3r^3+\cdots n\ terms]$$

$$=\frac{1}{1-k}[(r+r^2+r^3+\cdots nterms)-(kr+k^2r^2+k^3r^3+\cdots n\ terms]$$

$$=\frac{1}{1-k}\left[\frac{r}{1-r}-\frac{kr}{1-kr}\right] \quad \because \quad S_\infty=\frac{a_1}{a-r}$$

$$=\frac{1}{1-k}\left[\frac{r(1-kr)-kr(1-r)}{(1-r)(1-kr)}\right]$$

$$=\frac{1}{1-k}\left[\frac{r-kr^2-kr+kr^2}{(1-r)(1-kr)}\right]$$

$$=\frac{1}{1-k}\left(\frac{r-kr}{(1-r)(1-kr)}\right)$$

$$=\frac{1}{1-k}\left[\frac{r(1-k)}{(1-r)(1-kr)}\right]=\frac{r}{(1-r)(1-kr)}$$

Question No.9

If $y=\frac{x}{2}+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$ *and if* $0<x<2$, *then*

Prove that

$$x=\frac{2y}{1+y}$$

Solution:

$$y=\frac{x}{2}+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$$

$$a_1=\frac{x}{2}\quad,\quad r=\frac{\frac{1}{4}x^2}{\frac{1}{2}x}=\frac{1}{4}x^2\times\frac{2}{x}$$

$$\Rightarrow r=\frac{x}{2}$$

$$\because \quad S\infty=\frac{a_1}{1-r}$$

$$\Rightarrow y=\frac{\frac{x}{2}}{1-\frac{x}{2}}=\frac{\frac{x}{2}}{\frac{2-x}{2}}$$

$$\Rightarrow y=\frac{x}{2-x}$$

$$\Rightarrow y(2-x)x$$

$$2y-xy=x\Rightarrow 2y=x+xy$$

$$or\quad 2y=x(1+y)$$

$$or\quad x=\frac{2y}{1}+y$$

Hence proved.

Question No.9

$\boldsymbol{if\ \ y=\frac{2}{3}x+\frac{4}{9}x^2+\frac{8}{27}x^3+\cdots\ \ and\ if\ 0<x<\frac{3}{2},}$

Then show that $x=\frac{3y}{2(1+y)}$

Solution:

$$y=\frac{2}{3}x+\frac{4}{9}x^2+\frac{8}{27}x^3+\cdots$$

$$a_1=\frac{2}{3}x,\quad r=\frac{\frac{4}{9}x^2}{\frac{2}{3}x}=\frac{4}{9}x^2\times\frac{3}{2x}$$

$$\Rightarrow r=\frac{2x}{3}$$

$$\because \quad S_\infty=\frac{a_1}{1-r}$$

$$y=\frac{\frac{2}{3}x}{1-\frac{2x}{3}}=\frac{\frac{2x}{3}}{\frac{3-2x}{3}}$$

$$or\quad y=\frac{2x}{3-2x}$$

$$y=(3-2x)=2x$$

$$\Rightarrow 3y-2xy=2x$$

$$or\quad 3y=2x+2xy$$

$$3y=2x(1+y)$$

$$\Rightarrow \frac{3y}{1+y}=2x$$

$$or\quad x=\frac{3y}{2(1+y)}$$

Hence proved.

Question No.10

A ball is dropped from a height of 27 meters and distance it falls. If it continues to fall in the same way what distance will be ball travel before coming to rest.?

Solution:

According to given condition

$$27,2(27)\times\frac{2}{3},\qquad 2(27)\times\left(\frac{2}{3}\right)^2,\ldots$$

So

$$S_\infty=27+2\times27\times\frac{2}{3}+2\times27\left(\frac{2}{3}\right)^2+\cdots$$

$$=27+2(18+12+\cdots)\left(a_1=18,\ r=\frac{12}{18}=\frac{2}{3}\right)$$

$$=27+2\left(\frac{a_1}{1-r}\right)$$

$$=27+2\left(\frac{18}{1-\frac{2}{3}}\right)$$

$$=27+2\left(\frac{8}{\frac{3-2}{3}}\right)$$

$$= 27 + 2\left(\frac{8}{\frac{1}{3}}\right) = 27 + 2 \times 18 \times \frac{3}{1}$$

$$S_\infty = 27 + 108 = 135m$$

Question No.11

What distance will a ball travel before coming to rest if it is dropped from a height of 75meters and after each fall it rebound $\frac{2}{5}$ of the distance it fell?

Solution:

According to the given condition

$$75, 2(75) \times \frac{2}{5}, \quad 2(75) \times \left(\frac{2}{5}\right)^2, \ldots$$

So

$$S_\infty = 75 + 2 \times 75 \times \frac{2}{5} + 2 \times 75\left(\frac{2}{5}\right)^2 + \cdots$$

$$= 75 + 2(30 + 12 + \cdots)\left(a_1 = 30,\ r = \frac{12}{30} = \frac{2}{5}\right)$$

$$= 75 + 2\left(\frac{a_1}{1-r}\right)$$

$$= 75 + 2\left(\frac{30}{1-\frac{2}{5}}\right)$$

$$= 75 + 2\left(\frac{30}{\frac{5-2}{5}}\right)$$

$$= 75 + 2\left(\frac{30}{\frac{3}{5}}\right) = 75 + 2 \times 30 \times \frac{5}{3}$$

$$S_\infty = 75 + 100 = 175m$$

Question No.12

$$\text{if } 1 + 2x + 4x^2 + 8x^3 + \cdots$$

$$(i)\ \text{show that } x = \frac{y-1}{2y}$$

(ii) find the interval in which the series is convergent

Solution:

$$y = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

$$a_1 = 1, \quad r = \frac{2x}{1} = 2x$$

$$\because\ S\infty = \frac{a_1}{1-r}$$

$$\Rightarrow y = \frac{1}{1-2x}$$

$$or \quad y(1-2x)$$

$$y - 2xy = 1$$

$$\Rightarrow y - 1 = 2xy$$

$$\Rightarrow \frac{y-1}{2y} = x$$

$$or \quad x = \frac{y-1}{2y} \quad \text{hence proved}$$

For interval series will be convergent if

$$|r| < 1 \Rightarrow |2x| < 1$$

$$or\ |x| < \frac{1}{2}$$

$$\Rightarrow\ -\frac{1}{2} < x < \frac{1}{2}$$

Question No.13

$$if\ y = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots$$

(i) *show that* $x = \frac{y-1}{2y}$

(ii) find the interval in which the series is convergent.

Solution:

$$y = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots$$

$$a_1 = 1 \quad r = \frac{\frac{x}{2}}{1} = \frac{x}{2}$$

$$\because S_\infty = \frac{a_1}{1-r}$$

$$\Rightarrow y = \frac{1}{1-\frac{x}{2}} = \frac{1}{\frac{2-x}{2}}$$

$$or \quad 2y - xy = 2$$

$$\Rightarrow 2y - 2 = xy$$

$$2(y-1) = xy$$

$$\Rightarrow \frac{2(y-1)}{y} = x$$

$$or\ x = \frac{2(y-1)}{y}$$

Hence proved.

For interval series will be convergent if

$$|r| < 1 \Rightarrow 2 \Rightarrow -2 < x < 2$$

Question No.14

The sum of infinite geometric series is 9 and the sum of the sequences of its terms is $\frac{81}{5}$ .find the series?

Suppose infinite series

$$I\ condition \Rightarrow S_\infty = \frac{a_1}{1-r} = 9$$

$$\Rightarrow a_1 = 9(1-r) \rightarrow (i)$$

$$II\ condition \Rightarrow$$

$$a_1^2 + a_1^2r^2a_1^2r^4 + \cdots = \frac{81}{5}$$

$$\Rightarrow \frac{a_1^2}{1-r^2} = \frac{81}{5}$$

$$\Rightarrow 5a_1^2 = 81(1-r^2)$$

$$5[9(1-r)]^2 = 81(1-r)(1-r) from\ (i)$$

$$\Rightarrow 5(81)(1-r)^2 = 81(1-r)(1+r)$$

$$or \quad 5(1-r) = 1 + r$$

$$5 - 5r = 1 + r$$

$$5 - 1 = r + 5r$$

$$\Rightarrow 4 = 6r$$

$$\Rightarrow r = \frac{4}{6} \Rightarrow r = \frac{2}{3}\ put\ in\ (i)$$

$$a_1 = 9\left(1 - \frac{2}{3}\right) = 9\left(\frac{3-2}{3}\right) = 3(1) = 3$$

$\Rightarrow a_1 = 3$ *so infinite series*

Is $3 + 3\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \cdots$

or $3 + 2 + \frac{4}{3} + \cdots$

Word problems on G.P

Exercise 6.9

Question#1

A man deposit in a bank Rs. 8 in the first year, Rs. 24 in the second year Rs.72 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.

Solution:

The sequence of deposit is $8, 24, 72, \dots \dots$

Here,

$a_1 = 8$

$r = \frac{24}{8} = \frac{72}{24} = 3$, $n = 5$

Since,

$S_n = \frac{a_1(r^n -)}{r-1}$

$\Rightarrow S_5 = \frac{8(3^5-1)}{3-1}$

$= \frac{8(243-1)}{2}$

$= 4(242)$

$= 968$

Thus, he has to deposited Rs. 968 up to the fifth year.

Question#2

A man borrows Rs. 32760 without interest and agrees to repay the loan in instalments, each instalment being twice the preceding one. Find the amount of the last instalment, if the amount of the first instalment is Rs.8.

Solution:

Here,

$a_1 = 8$

$r = 3$,

$S_n = 32760$

$n = ?$

$a_n = ?$

Since,

$S_n = \frac{a_1(r^n -)}{r-1}$

$\Rightarrow 32760 = \frac{8(2^n-1)}{2-1}$

$\Rightarrow 32760 = \frac{8(2^n-1)}{1}$

$\Rightarrow 4095 = (2^n - 1)$

$\Rightarrow 4095 + 1 = 2^n$

$\Rightarrow 4096 = 2^n$

$\Rightarrow 2^{12} = 2^n$

$\Rightarrow n = 12$

Now,

$a_{12} = a_1 r^{11} \Rightarrow (8)(2)^{11} = (8)(2048) = 16384$

Hence the last instalment is Rs. 16384

Question#3

The population of a certain village is 62500. What will be its population after 3 years if it increases geometrically at the rate of 4% annually?

Solution:

Here,

$a_1 = 62500$

$r = 1 + \frac{4}{100} = 1 + 0.04 = 1.04$,

$n = 4$

Since,

$a_n = a_1 r^{n-1} \Rightarrow a_4 = (62500)(1.04)^{4-1}$

$= (62500)(1.04)^3$

$= (62500)(1.1249)$

$= 70304$

Thus, the population after 3 years is 70304.

Question#4

The enrolment of a famous school doubled after every eight years from 1970 to 1994. If the enrolment was 6000 in 1994, what was its enrolment in 1970?

Solution:

Let the enrolment in 1970 is a_1

Also,

$a_n = 6000$

$r = 2$

$n = 4$

Since,

$a_n = a_1 r^{n-1}$

$\Rightarrow 6000 = a_1(2)^{4-1}$

$\Rightarrow 6000 = a_1(2)^3$

$\Rightarrow 6000 = a_1(8)$

$\Rightarrow \frac{6000}{8} = a_1$

$\Rightarrow a_1 = 750$

Thus, the enrolment was 750.

Question#5

A singular cholera bacterium produces two complete bacteria in $\frac{1}{2}$hour. If we start with a colony of a bacteria, how many bacteria will we have in n hours?

Solution:

The colony of bacteria in the start $= a_1 = A$

Then,

$r = 2$

$n = 2n + 1$

Since,

$a_n = a_1 r^{n-1}$
$\Rightarrow a_{2n+1} = (A)(2)^{2n+1-1}$
$= (A)(2)^{2n}$
Thus, bacteria after n hours will be $(A)(2)^{2n}$

Question#6

Joining the mid points of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested perimeter $\frac{3}{2}$ .What will be the total perimeter of all the triangles formed in this way?

Solution:

Here,
$a_1 = \frac{3}{2}$
$r = \frac{1}{2}$
So, the series is $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} +, \dots$
Which is infinite geometric series
Now
$S = \frac{a_1}{1-r}$
$= \frac{\frac{3}{2}}{1-\frac{1}{2}}$
$= \frac{\frac{3}{2}}{\frac{1}{2}}$
$= 3$

Harmonic progression (H.P)

A sequence of numbers is called a harmonic sequence or Harmonic progression if the reciprocals of its terms are in arithmetic of its terms are in arithmetic progression (A.P)

**the reciprocal of zero is not defined so zero cannot be the term of a harmonic sequence.*

**the general form of H.P is* $\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+3d}, \dots$ *where*

$$a_1, a_1 + d, a_1 + 2d, \dots is\ A.P$$

Harmonic Mean:

***A number H is said to be harmonic mean** (H.M) between two numbers a and b if*

$$a, H, b \text{ are in } H.P$$

$$\Rightarrow \frac{1}{a}, \frac{1}{H}, \frac{1}{b} \quad \text{are in } A.P$$

In this case

$$d = \frac{1}{H} - \frac{1}{a} \quad \text{and } d = \frac{1}{b} - \frac{1}{H}$$

$$\Rightarrow \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\Rightarrow \frac{1}{H} + \frac{1}{H} = \frac{1}{b} + \frac{1}{a}$$

$$\Rightarrow \frac{1+1}{H} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{2}{H} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{H}{2} = \frac{ab}{a+b}$$

$$or \quad H = \frac{2ab}{a+b}$$

N harmonic Means between two numbers:

$H_1, H_2, H_3, \dots, H_n$ *are n harmonic means between a and b if* $a, H_1, H_2, H_3, \dots, H_n, b$ *are in* $H.P$

If we want to find n H.M between a and b, we first find n A.Ms $A_1, A_2, A_3, \dots A_n$ *between* $\frac{1}{a}$ and $\frac{1}{b}$ then take reciprocal *to get n* $H.M$ between a *and b that is* $\frac{1}{A_1}, \frac{1}{A_2}, \dots, \frac{1}{A_n}$ *will be required n H.Ms between a and b.*

Relationship between A.M , G.M and H.M

Prove that A, G, H are in $G.P$

$$or\ G^2 = A \times H$$

$$or\ \frac{G}{A} = \frac{H}{A}$$

Proof:

We know that

$$A = \frac{a+b}{2}, G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}$$

Now

$$G^2 = (\sqrt{ab})^2 = ab \rightarrow (i)$$

$$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$A \times H = ab \rightarrow (ii)$$

By (i) and (ii)

$$G^2 = A \times H \text{ or } G \times G = A \times H$$

$$\Rightarrow \frac{G}{A} = \frac{H}{A} \text{ it is clears that}$$

$$\Rightarrow A, G, H \text{ are in } G.P$$

Prove that $A > G > H$

If a, b are numbers any two distinct +ve numbers.

Proof:

We know that for two +ve distinct numbers a and b

$$A=\frac{a+b}{2}, G=\sqrt{ab} \ \ , H=\frac{2ab}{a+b}$$

Let $A>G \quad \Rightarrow \frac{a+b}{2}>\sqrt{ab}$

Squaring both sides

$$\left(\frac{a+b}{4}\right)^2>\left(\sqrt{ab}\right)^2$$

$$\frac{(a+b)^2}{4}>ab$$

$$(a+b)^2>4ab$$

$$a^2+b^2+2ab-4ab>0$$

$$a^2+b^2-2ab>0$$

$$\Rightarrow (a-b)^2>0 \ \ \text{which is always} \ \ \text{true.}$$

$$\text{so } A>G\rightarrow(i)$$

Now let $G>H$

$$\text{squring bothsides}$$

$$\Rightarrow ab>\frac{4a^2b^2}{(a+b)^2} \ \ \text{squaring both sides}$$

$$\Rightarrow \ (a+b)^2>\frac{4a^2b^2}{ab}$$

$$\Rightarrow a^2+b^2+2ab>4ab$$

$$a^2+b^2+2ab-4ab>0$$

$$a^2+b^2-2ab>0$$

$$\Rightarrow (a-b)^2>0 \ \ \text{which is always true.}$$

So

$$G>H\rightarrow(ii)$$

From (i) *and* (ii)

$$A>G>H \ \ \text{hence proved}$$

Prove that

A < G < H if a, b are any two distinct

– ve numbers.

***Proof*:**

$$\text{let } a=-m \ \ \text{and } b=-n$$

$$\text{where } m,n\in R^+$$

$$\because \ \ A=\frac{a+b}{2}\Rightarrow A=\frac{-m+(-n)}{2}$$

$$G=-\sqrt{ab}\Rightarrow G=-\sqrt{(-m)(-n)}$$

$$G=-\sqrt{mn}$$

$$H=\frac{2ab}{a+b}=\frac{2(-m)(-n)}{-m\pm(-n)}=\frac{2mn}{-m-n}$$

Let $A<G$

$$\Rightarrow \frac{-m-n}{2}<-\sqrt{mn}$$

$$-\left(\frac{m+n}{2}\right)<-\sqrt{mn} \quad (\because \ -2<-1\Rightarrow 2>1)$$

$$\Rightarrow \frac{m+n}{2}>\sqrt{mn}$$

$$\left(\frac{m+n}{2}\right)^2>mn \quad (squaring)$$

$$m^2+n^2+2mn>4mn$$

$$m^2+n^2+2mn-4mn>0$$

$$m^2+n^2-2mn>0$$

$$\Rightarrow (m-n)^2>0 \ \ \text{always true}$$

$$\text{so } A<G\rightarrow(i)$$

$$\text{now let } \ G<H$$

$$\Rightarrow -\sqrt{mn}<\frac{2mn}{-m-n}$$

$$\Rightarrow -\sqrt{mn}<-\left(\frac{2mmn}{m+n}\right)$$

$$\Rightarrow \sqrt{mn}>\frac{2mn}{m+n} \qquad (\because \ \ -2<-1 \ \ , \qquad 2>1)$$

$$mn>\frac{4m^2n^2}{(m+n)^2}(squsring)$$

$$(m+n)^2>\frac{4m^2n^2}{mn}$$

$$m^2+n^2+2mn>4mn$$

$$\Rightarrow m^2+n^2+2mn-4mn>0$$

$$m^2+n^2-2mn-4mn>0$$

$$(m-n)^2>0 \quad (\text{always true})$$

So $G<H\rightarrow(ii)$

From (i) and (ii)

$$A<G<H \ \ \text{hence proved}$$

Prove that A > H (page#223)

For any two distinct a and b

Solution:

$$A>H \ \ if$$

$$\frac{a+b}{2}>\frac{2ab}{a+b}$$

$$\Rightarrow (a+b)^2>4ab$$

$$a^2+b^2+2ab-4ab>0$$

$$a^2+b^2-2ab>0$$

$$(a-b)^2>0 \ \ \text{always true}$$

Hence $A>H$

Exercise 6.10

Question#1

Find the 9th term of the harmonic sequence

(i). $\frac{1}{3},\frac{1}{5},\frac{1}{7},\dots$

Solution:

$3,5,7,\dots$ *is in A.P.*

$a_1=3$

$d=5-3=2$

$n=9$

Since,

$a_n=a_1+(n-1)d$

$\Rightarrow a_9=3+(9-1)(2)$

$=3+(8)(2)$

$=3+16$

$=19$

So, 9th term of A.P. is 19

Hence 9th term of H.P. is $\frac{1}{19}$

(ii). $\frac{-1}{5},\frac{-1}{3},-1,\dots\ ,a_9=?$

Solution:

$$\because -\frac{1}{5},-\frac{1}{3},-1,\dots \text{ in } H.P$$
$$\Rightarrow -5,-3,-1,\dots \text{ in } A.P$$
$$a_1=-5,\ d=-3-(-5)=-3+5=2$$
$$\because\ a_9=a_1+8d$$
$$=-5+8(2)=-5+16$$
$$a_9=11 \text{ in } A.P$$
$$a_9=\frac{1}{11} \text{ in } H.P$$

Question#2

Find the 12th term of the following harmonic sequences

(i). $\frac{1}{2},\frac{1}{5},\frac{1}{8},\dots$

Solution:

$2,5,8,\dots$ *is in A.P.*

$a_1=2$

$d=5-2=3$

$n=12$

Since,

$a_n=a_1+(n-1)d$

$\Rightarrow a_{12}=2+(12-1)(3)$

$=2+(11)(3)$

$=2+33$

$=35$

So, 12th term of A.P. is 35

Hence 12th term of H.P. is $\frac{1}{35}$

(ii). $\frac{1}{3},\frac{2}{9},\frac{1}{6},\dots\quad a_{12}=?$

Solution:

$\because\ \frac{1}{3},\frac{2}{9},\frac{1}{6},\dots$ in $A.P$

$\Rightarrow 3,\frac{9}{2},6,\dots$ in $A.P$

$a_1=3, d=\frac{9}{2}-3=\frac{9-6}{2}=\frac{3}{2}$

$\because\ a_{12}=a_1+11d\quad \because\ a_n=a_1+(n-1)d$

$=3+11\left(\frac{3}{2}\right)=3+\frac{33}{2}$

$a_{12}=\frac{6+33}{2}=\frac{39}{2}$

$a_{12}=\frac{39}{2}$ in $A.P$

$so,\ a_{12}=\frac{2}{39}$ in $H.P$

Question#3

Insert five harmonic means between the following given numbers,

(i). $\frac{-2}{5}$ *and* $\frac{2}{13}$

Solution:

Let H_1, H_2, H_3, H_4 *and* H_5 *are the five H.Ms. between* 2 *and* 5.

Then $\frac{-2}{5}, H_1, H_2, H_3, H_4, H_5, \frac{2}{13}$ *are in H.P.*

Then $\frac{-5}{2},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{H_3},\frac{1}{H_4},\frac{1}{H_5},\frac{13}{2}$ *are in A.P.*

Here

$a_1=\frac{-5}{2}$

$a_7=\frac{13}{2}$

$\Rightarrow a_1+6d=\frac{13}{2}$

$\Rightarrow \frac{-5}{2}+6d=\frac{13}{2}$

$\Rightarrow 6d=\frac{13}{2}+\frac{5}{2}=9$

$\Rightarrow d=\frac{9}{6}=\frac{3}{2}$

So,

$\frac{1}{H_1}=a_2=a_1+d=\frac{-5}{2}+\frac{3}{2}=-1$

$\Rightarrow H_1=-1$

$\frac{1}{H_2}=a_3=a_1+2d=\frac{-5}{2}+2\left(\frac{3}{2}\right)=\frac{-5}{2}+3=\frac{1}{2}$

$\Rightarrow H_2=2$

$\frac{1}{H_3}=a_4=a_1+3d=\frac{-5}{2}+3\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{9}{2}=2$

$\Rightarrow H_3=\frac{1}{2}$

$\frac{1}{H_4}=a_5=a_1+4d=\frac{-5}{2}+4\left(\frac{3}{2}\right)=\frac{-5}{2}+6=\frac{7}{2}$

$\Rightarrow H_4=\frac{7}{2}$

$\frac{1}{H_5}=a_6=a_1+5d=\frac{-5}{2}+5\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{15}{2}=5$

$\Rightarrow H_5=\frac{1}{5}$

Hence , $-1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$ *are the five H.Ms. between* $\frac{-2}{5}$ *and* $\frac{2}{13}$.

(ii). $\frac{1}{4}$ *and* $\frac{1}{24}$

Solution:

Let H_1, H_2, H_3, H_4 *and* H_5 *are the five H.Ms. between* $\frac{1}{4}$ *and* $\frac{1}{5}$

Then $\frac{1}{4}, H_1, H_2, H_3, H_4, H_5, \frac{1}{24}$ *are in H.P.*

Then $4, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 24$ are in A.P.

Here

$a_1 = 4$

$a_7 = 24$

$\Rightarrow a_7 = a_1 + 6d = 24$

$\Rightarrow 4 + 6d = 24$

$\Rightarrow 6d = 24 - 4 = 20$

$$\Rightarrow d = \frac{20}{6} = \frac{10}{3}$$

So,

$\frac{1}{H_1} = a_2 = a_1 + d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3}$

$\Rightarrow H_1 = \frac{3}{22}$

$\frac{1}{H_2} = a_3 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = \frac{22+10}{3} = \frac{32}{3}$

$\Rightarrow H_2 = \frac{3}{32}$

$\frac{1}{H_3} = a_4 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = \frac{32+10}{3} = \frac{42}{3}$

$\Rightarrow H_3 = \frac{3}{42}$

$\frac{1}{H_4} = a_5 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = \frac{42+!0}{3} = \frac{52}{3} =$

$\Rightarrow H_4 = \frac{3}{52}$

$\frac{1}{H_5} = a_6 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = \frac{52+10}{3} = \frac{62}{3}$

$\Rightarrow H_5 = \frac{3}{62}$

Hence , $\frac{3}{22}, \frac{3}{32}, \frac{3}{42}, \frac{3}{52}, \frac{3}{62}$ are the five H.Ms. between $\frac{1}{4}$, and $\frac{1}{24}$

Question#4

Insert four harmonic means between the following given numbers.

(i). $\frac{1}{3}$ **and** $\frac{1}{23}$

Solution:

Let H_1, H_2, H_3, H_4 are the four H.Ms. between $\frac{1}{3}$ and $\frac{1}{23}$

Then $\frac{1}{3}, H_1, H_2, H_3, H_4, H_5, \frac{1}{23}$ are in H.P.

Then $3, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 23$ are in A.P.

Here

$a_1 = 3 \;, n = 6 \;, a_6 = 23$

$a_6 = 23$

$\Rightarrow a_1 + 5d = 23$

$\Rightarrow 3 + 5d = 23$

$\Rightarrow 5d = 23 - 3 = 20 \Rightarrow d = 4$

$\Rightarrow d = \frac{9}{6} = \frac{3}{2}$

So,

$\frac{1}{H_1} = a_1 + d = 3 + 4 = 7 \Rightarrow H_1 = \frac{1}{7}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = 7 + 4 = 11 \Rightarrow H_2 = \frac{1}{11}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = 11 + 4 = 15 \Rightarrow H_3 = \frac{1}{15}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = 15 + 4 = 19 \Rightarrow H_4 = \frac{1}{19}$

Thus $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$ are required 4 H.Ms

(ii). $\frac{7}{3}$ **and** $\frac{7}{11}$

Solution:

Let H_1, H_2, H_3, H_4 are the four H.Ms. between $\frac{7}{3}$ and $\frac{7}{11}$ then

Then $\frac{7}{3}, H_1, H_2, H_3, H_4, \frac{7}{11}$ are in H.P.

Then $3, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 23$ are in A.P.

Here

$a_1 = \frac{3}{7} \;, n = 6 \;, a_6 = \frac{11}{7}$

$a_6 = a_1 + 5d = \frac{11}{7}$

$\Rightarrow \frac{3}{7} + 5d = \frac{11}{7}$

$\Rightarrow 5d = \frac{11}{7} - \frac{3}{7} = \frac{8}{7} \Rightarrow d = \frac{8}{35}$

$\Rightarrow d = \frac{8}{35}$

So,

$\frac{1}{H_1} = a_1 + d = \frac{3}{7} + \frac{8}{35} = \frac{3(5)+8}{35} \Rightarrow H_1 = \frac{3(5)+8}{35}$

$\frac{1}{H_1} = \frac{15+8}{35} = \frac{23}{35} \Rightarrow H_1 = \frac{35}{23}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{23}{35} + \frac{8}{35} = \frac{31}{35} \Rightarrow H_2 = \frac{35}{31}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = \frac{31}{35} + \frac{8}{35} = \frac{39}{35} \Rightarrow H_3 = \frac{35}{39}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = \frac{39}{35} + \frac{8}{35} = \frac{47}{35} \Rightarrow H_4 = \frac{35}{47}$

Thus $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}$ and $\frac{35}{47}$ are required 4 H.Ms

(iii) 4 and 20

Solution:

Let H_1, H_2, H_3, H_4 are the four H.Ms. between 4 and 20

Then 4, H_1, H_2, H_3, H_4, H_5, 20 are in H.P.

Then $\frac{1}{4}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{20}$ are in A.P.

Here

$a_1 = \frac{1}{4}, n = 6 \;, a_6 = \frac{1}{20}$

$a_6 = \frac{1}{20}$

$\Rightarrow a_1 + 5d = \frac{1}{20}$

$\Rightarrow \frac{1}{4} + 5d = \frac{1}{20}$

$\Rightarrow 5d = \frac{1}{20} - \frac{1}{4} = \frac{1-5}{20} = -\frac{4}{20}$

$\Rightarrow 5d = -\frac{1}{5} \Rightarrow d = -\frac{1}{25}$

Now

$$\frac{1}{H_1} = a_1 + d = \frac{1}{4} + \left(-\frac{1}{25}\right) = \frac{1}{4} - \frac{1}{25}$$

$$\frac{1}{H_1} = \frac{25-4}{100} = \frac{21}{100} \Rightarrow H_1 = \frac{100}{21}$$

$$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{21}{100} + \left(-\frac{1}{25}\right)$$

$$= \frac{21}{100} - \frac{1}{25} = \frac{21-4}{100}$$

$\frac{1}{H_2} = \frac{17}{100} \Rightarrow H_2 = \frac{100}{17}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = \frac{17}{100} + \left(-\frac{1}{25}\right)$

$\frac{1}{H_3} = \frac{17}{100} - \frac{1}{25} = \frac{17-4}{100}$

$\frac{1}{H_3} = \frac{13}{100} \Rightarrow H_3 = \frac{100}{13}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = \frac{13}{100} + \left(-\frac{1}{25}\right)$

$\frac{13}{100} - \frac{1}{25} = \frac{13-4}{100}$

$\frac{1}{H_4} = \frac{9}{100} \Rightarrow H \Rightarrow_4 = \frac{100}{9}$

Hence four H.Ms between 4 and 20 are

$\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$

Question#5

If the 7th and 10th terms of an H.P. are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th term.

Solution:

$a_7 = \frac{1}{3} \quad in\ H.P.$

$a_7 = 3 \quad in\ A.P.$

$\Rightarrow a_1 + 6d = 3 \ \dots(i)$

Also,

$a_{10} = \frac{5}{21} \quad in\ H.P.$

$a_{10} = \frac{21}{5} \quad in\ A.P.$

$\Rightarrow a_1 + 9d = \frac{21}{5} \ \dots(ii)$

Subtracting (i) *from* (ii)

$a_1 + 6d = 3$

$a_1 + 9d = \frac{21}{5}$

$- \quad - \quad -$

$-3d = -\frac{6}{5}$

$\Rightarrow d = \left(\frac{6}{5}\right)\left(\frac{1}{3}\right)$

$\Rightarrow d = \frac{2}{5}$

Putting values of d in (i)

$a_1 + 6\left(\frac{2}{5}\right) = 3$

$\Rightarrow a_1 + \frac{12}{5} = 3$

$\Rightarrow a_1 = 3 - \frac{12}{5}$

$\Rightarrow a_1 = \frac{3}{5}$

Now,

$a_{14} = a_1 + 13d$

$= \frac{3}{5} + 13\left(\frac{2}{5}\right)$

$= \frac{3}{5} + \frac{26}{5} = \frac{29}{5}$

$\Rightarrow a_{14} = \frac{5}{29}$

Thus, 14^{th} terms of an H.P. is $\frac{5}{29}$

Question#6

The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Solution:

$a_1 = -\frac{1}{3} \quad in\ H.P.$

$a_1 = -3 \quad in\ A.P.$

$a_5 = \frac{1}{5} \quad in\ H.P.$

$a_5 = 5 \quad in\ A.P.$

$\Rightarrow a_1 + 4d = 5$

Putting values of $a_1 = -3$ *in above*

$-3 + 4d = 5$

$\Rightarrow 4d = 5 + 3$

$\Rightarrow 4d = 8$

$\Rightarrow d = 2$

Now,

$a_9 = a_1 + 8d$

$= -3 + 18(2)$

$= -3 + 16 = 13$

So,

$\Rightarrow a_9 = \frac{1}{13}$

Thus, 9^{th} terms of an H.P. is $\frac{1}{13}$

Question#7

If 5 is the harmonic mean between 2 and b, find b.

Solution:

H.M. $= 5$

$a = 2$

$b = b$

Since,

H.M. $= \frac{2ab}{a+b}$

$\Rightarrow 5 = \frac{2(2)(b)}{2+b}$

$\Rightarrow 5(2+b) = 2(2)(b)$

$\Rightarrow 10 + 5b = 4b$

$\Rightarrow 5b - 4b = -10$

$\Rightarrow b = -10$

Question#8

If the numbers are $\frac{1}{k}, \frac{1}{2k+1}$ and $\frac{1}{4k-1}$ in harmonic sequence, find k.

Solution:

Since, $\frac{1}{k}, \frac{1}{2k+1}$ *and* $\frac{1}{4k-1}$ *are in H.P.*

So,

$k, 2k+1, 4k-1$ *are in A.P.*

So,

$d = 2k + 1 - k = 4k - 1 - 2k - 1$

$\Rightarrow k + 1 = 2k - 2$

$\Rightarrow k - 2k = -2 - 1$

$\Rightarrow -k = -3$

$\Rightarrow k = 3$

Question#9

Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M. between a and b.

Solution:

Since H.M. between a and b. $=\frac{2ab}{a+b}$ ….(i)

But we have given $H.M.=\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ ….(ii)

Comparing (i) and (ii)

$\frac{2ab}{a+b}=\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

By cross multiplying

$\Rightarrow 2ab(a^n+b^n)=(a+b)(a^{n+1}+b^{n+1})$

$\Rightarrow 2a^{n+1}b+2ab^{n+1}=a^{n+2}+a^{n+1}b+ab^{n+1}+b^{n+2}$

$\Rightarrow 2a^{n+1}b+2b^{n+1}b-a^{n+1}b-ab^{n+1}=a^{n+2}+b^{n+2}$

$\Rightarrow a^{n+1}b+ab^{n+1}=a^{n+2}+b^{n+2}$

$\Rightarrow a^{n+1}b-a^{n+2}=b^{n+2}-ab^{n+1}$

$\Rightarrow a^{n+1}(b-a)=b^{n+1}(b-a)$

$\Rightarrow a^{n+1}=b^{n+1}$

$\Rightarrow \frac{a^{n+1}}{b^{n+1}}=1$

$\Rightarrow \left(\frac{a}{b}\right)^{n+1}=\left(\frac{a}{b}\right)^0 \quad \because \left(\frac{a}{b}\right)^0=1$

$\Rightarrow n+1=0$

$\Rightarrow n=-1$

Question#10

If a^2, b^2, c^2 are in A.P. show that $a+b,\ c+a$ and $b+c$ are in H.P.

Solution:

Since, a^2, b^2, c^2 *are in A.P.*

Therefore,

$d=b^2-a^2=c^2-b^2$

$\Rightarrow (b-a)(b+a)=(c-b)(c+b)$ …(i)

Now, to prove

$a+b,\ c+a$ *and* $b+c$ *are in H.P.*

We will prove

$\frac{1}{a+b},\frac{1}{c+a},\frac{1}{b+c}$ *are in A.P.*

Now,

$d=\frac{1}{c+a}-\frac{1}{a+b}$

$=\frac{a+b-c-a}{(c+a)(a+b)}$

$=\frac{b-c}{(c+a)(a+b)}$ …(i)

Also,

$d=\frac{1}{b+c}-\frac{1}{c+a}$

$=\frac{c+a-b-c}{(b+c)(c+a)}$

$=\frac{a-b}{(b+c)(c+a)}$ …(ii)

From eq. (i)

$c+b=\frac{(b-a)(b+a)}{(c-b)}$

Putting in above

$d=\frac{a-b}{\frac{(b-a)(b+a)}{(c-b)}.(c+a)}$

$=\frac{(a-b)(c-b)}{-(a-b)(b+a)(c+a)}$

$=\frac{-(c-b)}{-(c+a)(a+b)}$ …(iii)

From (ii) and (iii)

$d=d$

i.e.

$\frac{1}{a+b},\frac{1}{c+a},\frac{1}{b+c}$ *are in A.P.*

$\Rightarrow a+b,\ c+a$ *and* $b+c$ *are in H.P.*

Question#11

The sum of the first and fifth term of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$. find the sequence

Solution:

Suppose the harmonic sequence

$\frac{1}{a_1},\frac{1}{a_1+d},\frac{1}{a_1+2d},\dots\dots\dots$

By the given condition

$\frac{1}{a_1}+\frac{1}{a_1+4d}=\frac{4}{7}$ …(i)

Also, we have given

$\frac{1}{a_1}=\frac{1}{2}$

$\Rightarrow a_1=2$

Putting in (i)

$\frac{1}{2}+\frac{1}{2+4d}=\frac{4}{7}$

$\Rightarrow \frac{1}{2+4d}=\frac{4}{7}-\frac{1}{2}$

$\Rightarrow \frac{1}{2+4d}=\frac{1}{14}$

$\Rightarrow 2+4d=14$

$\Rightarrow 4d=14-2$

$\Rightarrow 4d=12$

$\Rightarrow d=\frac{12}{4}$

$\Rightarrow d=3$

Now,

$\frac{1}{a_1+d}=\frac{1}{2+3}=\frac{1}{5}$

$\frac{1}{a_1+2d}=\frac{1}{2+2(3)}=\frac{1}{2+6}=\frac{1}{8}$

$\frac{1}{a_1+3d}=\frac{1}{2+3(3)}=\frac{1}{2+9}=\frac{1}{11}$

Thus, the required sequence is $\frac{1}{2},\frac{1}{5},\frac{1}{8},\frac{1}{11},\dots\dots$

Question#12

If A, G and H are the arithmetic, geometric and harmonic means between a and b respectively, show that $G^2=AH$.

Solution:

Since,

$A=\frac{a+b}{2}$

$G=\pm\sqrt{ab}$

$H=\frac{2ab}{a+b}$

Now,

$G^2 = (\pm\sqrt{ab})^2 = ab \; \ldots .(i)$

$AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab \; \ldots(ii)$

From (i) *and* (ii)

$G^2 = AH$ *proved.*

Question#13

Find A, G, H and show that $G^2 = AH$. ***if***

(i). $a = -2, b = 6$

Solution:

$\because A = \frac{a+b}{2} = \frac{-2-8}{2} = -\frac{8}{2} = -4$

$G = \pm\sqrt{ab} = \pm\sqrt{(-2)(-6)} = \pm\sqrt{12}$

$H = \frac{2ab}{a+b}$

$= \frac{2(-2)(-6)}{-2-6} = \frac{24}{-8} = -3$

Now

$G^2 = (\pm\sqrt{12})^2 = 12 \to (i)$

$AH = (-4)(-3) = 12 \to (ii)$

By (i) and (ii) Hence

$G^2 = AH$

(ii). $a = 2i, b = 4i$

Solution:

$A = \frac{a+b}{2}$

$= \frac{2i+4i}{2}$

$= \frac{6i}{2}$

$= 3i$

$G = \pm\sqrt{ab}$

$G = \pm\sqrt{(2i)(4i)}$

$= \pm\sqrt{8i^2}$

$= \pm 2\sqrt{2i}$

$H = \frac{2ab}{a+b} = \frac{2(2i)(4i)}{2i+4i} = \frac{16i^2}{6i} = \frac{8i}{3}$

Now,

$G^2 = (\pm 2\sqrt{2i})^2 = 4(2)(-1) = -8$

$AH = (3i)\left(\frac{8i}{3}\right) = 8i^2 = -8$

From (i) *and* (ii)

$G^2 = AH$

(iii). $a = 9, b = 4$

Solution:

$A = \frac{a+b}{2} = \frac{9+4}{2} = \frac{13}{2}$

$G = \pm\sqrt{ab} = \pm\sqrt{(9)(4)} = \pm\sqrt{36} = \pm 6$

$H = \frac{2ab}{a+b} = \frac{2(9)(4)}{9+4} = \frac{72}{13}$

Now

$G^2 = (\pm 6)^2 = 36 \to (i)$

$A.H = \left(\frac{13}{2}\right)\left(\frac{72}{13}\right) = \frac{72}{2} = 36 \to (ii)$

From (i) and (ii)

$G^2 = AH$

Question#14

Find A, G, H ***and verify that*** $A > G > H \; (G > 0)$, ***if***

(i). $a = 2, b = 8$

Solution:

$A = \frac{a+b}{2}$

$= \frac{2+8}{2}$

$= \frac{10}{2}$

$= 5$

$G = \pm\sqrt{ab}$

$G = \sqrt{(2)(8)}$

$= \sqrt{16}$

$= 4$

$H = \frac{2ab}{a+b}$

$= \frac{2(2)(8)}{2+8}$

$= \frac{32}{10}$

$= \frac{16}{5}$

$= 3\frac{1}{5}$

Since,

$5 > 4 > 3\frac{1}{5}$

$\Rightarrow A > G > H$ *proved.*

(ii). $a = \frac{2}{5}, b = \frac{8}{5}$

Solution:

$\because A = \frac{a+b}{2} = \frac{\frac{2}{5}+\frac{8}{5}}{2} = \frac{1}{2}\left(\frac{2+8}{5}\right)$

$A = \frac{1}{2}\left(\frac{10}{5}\right) = \frac{10}{10} = 1$

$G = \sqrt{ab} = \sqrt{\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)} = \sqrt{\frac{16}{25}}$

$G = \frac{4}{5} (G > 0)$

$H = \frac{2ab}{a+b} = \frac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}}$

$= \frac{\frac{32}{25}}{\frac{10}{5}} = \frac{32}{25} \times \frac{5}{10}$

$H = \frac{16}{25}$

Clearly $1 > \frac{4}{5} > \frac{16}{25}$

$A > G > H$

Question#15

Find A, G, H ***and verify that*** $A < G < H \; (G < 0)$, ***if***

(i). $a = 2, b = 8$

Solution:

$\because A = \frac{a+b}{2} = \frac{-2-8}{2} = -\frac{10}{2} = -5$

$G = -\sqrt{ab} = -\sqrt{(-2)(-8)} = -\sqrt{16} = -4 (G < 0)$

$H = \frac{2ab}{a+b} = \frac{2(-2)(-8)}{-2-8} = \frac{-32}{-10} = \frac{32}{10}$

Clearly $-5 < -4 < \frac{32}{10}$

$\Rightarrow A < G < H$

(ii). $a = \frac{-2}{5}, b = \frac{-8}{5}$

Solution:

$A = \frac{a+b}{2}$

$= \frac{\frac{-2}{5} - \frac{8}{5}}{2}$

$= \frac{\frac{-10}{5}}{2}$

$= \frac{-10}{10}$

$= -1$

$G = -\sqrt{ab} \quad \therefore G > 0$

$G = -\sqrt{\left(\frac{-2}{5}\right)\left(\frac{-8}{5}\right)}$

$= -\sqrt{\frac{16}{25}}$

$= -\frac{4}{5}$

$H = \frac{2ab}{a+b}$

$= \frac{2\left(\frac{-2}{5}\right)\left(\frac{-8}{5}\right)}{\frac{-2}{5} - \frac{-8}{5}}$

$= \frac{\frac{32}{25}}{\frac{-10}{5}} = \frac{\frac{32}{25}}{2}$

$= \frac{-32}{50} = \frac{-16}{25}$

Since,

$-1 < \frac{-4}{5} < \frac{-16}{25}$

$\Rightarrow A < G < H$ *proved.*

Question#16

If the H.M and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers

Solution:

Let a & b be two numbers

Since,

H.M. $= 4$

$\Rightarrow \frac{2ab}{a+b} = 4$

$\Rightarrow 2ab = 4(a+b)$

$\Rightarrow ab = 2(a+b) \quad ....(i)$

Also,

A.M. $= 4$

$\Rightarrow \frac{a+b}{2} = \frac{9}{2}$

$\Rightarrow a+b = 9 \quad ....(ii)$

Putting the value of $a+b$ *in* (i)

$\Rightarrow ab = 2(9)$

$\Rightarrow ab = 18$

$\Rightarrow a = \frac{18}{b} \quad ...(iii)$

Putting in (ii)

$\frac{18}{b} + b = 9$

$\Rightarrow \frac{18+b^2}{b} = 9$

$\Rightarrow 18 + b^2 = 9b$

$\Rightarrow b^2 - 9b + 18 = 0$

$\Rightarrow b^2 - 6b - 3b + 18 = 0$

$\Rightarrow b(b-6) - 3(b-6) = 0$

$\Rightarrow (b-6)(b-3) = 0$

$\Rightarrow b - 6 = 0$ *or* $b - 3 = 0$

$\Rightarrow b = 6$ *or* $b = 3$

Putting in (iii)

$\Rightarrow a = \frac{18}{6} = 3$ *or* $a = \frac{18}{3} = 6$

Thus, 3,6 or 6,3 are the required numbers.

Question#17

If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$ find the numbers.

Solution:

Let a & b be two numbers

Since,

G.M. $= 4$

$\Rightarrow \sqrt{ab} = 4$

On squaring,

$\Rightarrow ab = 16 \quad ...(i)$

Also,

H.M. $= \frac{16}{5}$

$\Rightarrow \frac{2ab}{a+b} = \frac{16}{5} \rightarrow (ii)$

$(ii) \Rightarrow \frac{2(16)}{a+b} = \frac{16}{5} (\because ab = 16 \text{ from } (ii)$

$\Rightarrow \frac{2}{a+b} = \frac{1}{5}$

or $a + b = 10$

$\Rightarrow a = 10 - b$ put in (ii)

$(10-b)b = 16$

$10b - b^2 = 16$

$\Rightarrow b^2 = 10b + 16 = 0$

$b^2 - 8b - 2b + 16 = 0$

$b(b-8) - 2(b-8) = 0$

$\Rightarrow (b-8)(b-2) = 0$

$b - 8 = 0, \quad b - 2 = 0$

$b = 8 \quad , \quad b = 2$

When $b = 8$, so (ii) $\Rightarrow a(8) = 16 \Rightarrow a = 2$

When $b = 2$, so (ii) $\Rightarrow a(2) = 16$

$a = 8$

Hence numbers are 2,8 *or* 8,2

Question#18

If the numbers $\frac{1}{4}, \frac{4}{21}$ And $\frac{1}{36}$ are subtracted from the three consecutive terms of a G.P., the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.

Solution:

Let three consecutive terms of G.P are

a_1, a_1r, a_1r^2

I condition ⇒

$(a_1)(a_1r)(a_1r^2) = \frac{1}{27} \to a_1^3r^3 = \frac{1}{27}$

$\Rightarrow (a_1r)^3 = \frac{1}{3^3}$

So $a_1r = \frac{1}{3} \Rightarrow r = \frac{1}{3a_1}$

Now we have H.P is

$a_1 - \frac{1}{2}, a_1\left(\frac{1}{3a_1}\right) - \frac{4}{21}, a_1\left(\frac{1}{3a_1}\right)^2 - \frac{1}{36}$ in H.P

$a_1 - \frac{1}{2}, \frac{1}{3} - \frac{4}{21}, \frac{1}{9a_1} - \frac{1}{36}$ in H.P

$\frac{2a_1-1}{2}, \frac{7-4}{21}, \frac{4-a_1}{36a_1}$ in H.P

$\frac{2a_1-1}{2}, \frac{7}{1}, \frac{4-a_1}{36a_1}$ in A.P

$\Rightarrow \frac{2}{2a_1-1}, \frac{7}{1}, \frac{36a_1}{4-a_1}$ in A.P

So $7 - \frac{2}{2a_1-1} = \frac{36a_1}{4-a_1} - 7$

$\Rightarrow 7 + 7 = \frac{36a_1}{4-a_1} + \frac{2}{2a_1-1}$

$\Rightarrow 14 = \frac{36a_1(2a_1-1) + 2(4-a_1)}{(4-a_1)(2a_1-1)}$

$\Rightarrow 14 = \frac{72a_1^2 - 36a_1 + 8 - 2a_1}{8a_1 - 4 - 2a_1^2 + a_1}$

$14 = \frac{71a_1^2 - 38a_1 + 8}{-2a_1^2 + 9a_1 - 4}$

$\Rightarrow 14(-2a_1^2 + 9a_1 - 4) = 72a_1^2 - 38a_1 + 8$

$-28a_1^2 + 126a_1 - 56 = 72a_1^2 - 38a_1 + 8$

$\Rightarrow 72a_1^2 - 38a_1 + 8 + 28a_1^2 - 126a_1 + 56 = 0$

$100a_1^2 - 164a_1 + 64 = 0$

$25a_1^2 - 41a_1 + 16 = 0$ (÷ by 4)

$\Rightarrow 25a_1^2 - 25a_1 - 16a_1 + 16 = 0$

$25a_1(a_1 - a) - 16(a_1 - 1) = 0$

$(a_1 - a)(25a_1 - 16) = 0$

$(a_1 - 1)(25a_1 - 16) = 0$

$a_1 - 1 = 0, \ 25a_1 - 16 = 0$

$\Rightarrow a_1 = 1, \ a_1 = \frac{16}{25}$

If $a_1 = 1$ then $r = \frac{1}{3(1)} = \frac{1}{3}$

If $a_1 = \frac{16}{25}$ then $r = \frac{1}{3\left(\frac{16}{25}\right)} = \frac{25}{48}$

If $a_1 = 1, \ r = \frac{1}{3}$ then numbers

a_1, a_1r, a_1r^2

$1, \frac{1}{3}, \frac{1}{9}$

If

$a_1 = \frac{16}{25}, r = \frac{25}{48}$ then numbers are a_1, a_1r, a_1r^2

$\frac{16}{25}, \frac{16}{25}\left(\frac{25}{48}\right), \left(\frac{16}{25}\right)\left(\frac{25}{48}\right)^2$

or $\frac{16}{25}, \frac{1}{3}, \frac{25}{144}$

Sigma Notation(or Summation Notation)

The Greek letter $\sum$(sigma) is used to sum a sequence of numbers. We write the sum in sigma notation as,

$$a_1, a_2, a_3, + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Here $\sum$ indicate the sum and k is called index of summation. The summation begins from $k = 1$ and ends with k=n

$k = 1$ is called lower limit while $k = n$ is called upper limit.

Remember that

(i) $\sum_{k=1}^{n} 1 = n(1) = n$

(ii) $\sum_{k=1}^{n} C = nC$ (where c is constant)

(iii) $\sum_{k=1}^{n} (a_{k+b_k}) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(iv) $\sum_{k=1}^{n} (a_{k-b_k}) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(v) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

we may write

$$\sum_{k=1}^{n} a_k = c\sum_{1}^{k} a_k$$

And

$$\sum_{k=1}^{n} [k^m - (k-1)^m] = n^m$$

To find the formulae for the sums

(i) $\sum_{k=1}^{n} k$

We know that

$$(k-1)^2 = k^2 - 2k + 1$$

$$\Rightarrow 2k - 1 = k^2 - (k-1)^2$$

$$or\ k^2 = (k-1)^2 = 2k - 1$$

Taking summation on the both sides

$$\sum_{k=1}^{n}[k^2-(k-1)^2] = \sum_{k=1}^{n}(2k-1)$$

$$\sum_{k=1}^{n}[k^2-(k-1)^2] = \sum_{k=1}^{n}2k - \sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n}[k^2-(k-1)^2] = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n}[k^2-(k-1)^2] + \sum_{k=1}^{n}1 = 2\sum_{k=1}^{n}k$$

$$\because \sum_{k=1}^{n}[k^m-(k-1)^m] = n^m$$

$$\sum_{k=1}^{n}1 = n$$

$$\Rightarrow 2\sum_{k=1}^{n}k = n^2 + n$$

$$\Rightarrow \sum_{k=1}^{n}k = \frac{n(n-1)}{2}$$

$$(ii)\quad \sum_{k=1}^{n}k^2$$

We know that

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1$$

$$\Rightarrow 3k^2 - 3k + ! = k^3 - (k-1)^3$$

$$or\ \ k^3(k-1)^3 = 3k^2 - 3k + 1$$

Taking summation on both sides

$$\sum_{k=1}^{n}[k^3-(k-1)^3] = \sum_{k=1}^{n}(3k^2-3k+1)$$

$$\sum_{k=1}^{n}[k^3-(k-1)^3] = 3\sum_{k=1}^{n}k^2 - 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$3\sum_{k=1}^{n}k^2 = \sum_{k=1}^{n}[k^3-(k-1)^3] + 3\sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n}[k^m-(k-1)^m] = n^m$$

$$\Rightarrow 3\sum_{k=1}^{n}k^2 = n^3 + \frac{3n(n+10}{2} - n$$

$$= \frac{2n^3 + 3n(n+1) - 2n}{2}$$

$$= \frac{n[2n^2 + 2n + 3 - 2n]}{2}$$

$$3\sum_{k=1}^{n}k^2 = \frac{n[2n^2+3n+1]}{2}$$

$$\Rightarrow \sum_{k=1}^{n}k^2 = \frac{n[2n^2+3n+1]}{2}$$

$$\frac{n(2n(n+1)+1(n+1)}{6}$$

$$= \frac{n(2n+1)(n+1)}{6}$$

$$\Rightarrow \sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$$

$(iii)\ \sum_{k=1}^{n}k^3$

$$(k-1)^4 = k^4 - 4k^3 + 6k^2 - 4k + 1$$

$$\because\quad (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^2 + b^4$$

$$\Rightarrow 4k^3 - 6k + 4k - 1 = k^4 - (k-1)^4$$

Taking summation on both sides

$$\sum_{k=1}^{n}k^4 - (k-1)^4 = \sum_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1)$$

$$= 4\sum_{k=1}^{n}k^3 - 6\sum_{k=1}^{n}k^2 + 4\sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

$$\Rightarrow 4\sum_{k=1}^{n}k^3 = \sum_{k=1}^{n}[k^4-(k-1)^4] + 6\sum_{k=1}^{n}k^2 + 4\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}1$$

$$= n^4 + \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$4\sum_{k=1}^{n}k^3 = n^4 + \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + n(n+1)(2n+1) - 2n(n+1) + n}{4}$$

$$\frac{n}{4}[n^3 + 2n^2 + n + 2n + 1 - 2n - 2 + 1]$$

$$= \frac{n}{4}[n^3 + 2n^2 + n]$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Exercise 6.11

Sum the following series up to n terms.

Question#1

$\mathbf{1 \times 1 + 2 \times 4 + 3 \times 7 + \ldots}$

Solution:

$1 \times 1 + 2 \times 4 + 3 \times 7 + \ldots$

$1 + (k-1)3$

$= 1 + 3k - 3$

$= 3k - 2$

If T_k *denotes the kth term of the series, then*

$T_k = k(3k-2) = 3k^2 - 2k$

Let S_n *denotes the sum of first n terms of the series then*

$$S_n = \sum_{k=1}^{n}(3k^2 - 2k)$$

$$= \sum_{k=1}^{n} 3k^2 - \sum_{k=1}^{n} 2k$$

$$= 3\left(\frac{n(n+1)(2n+1)}{6}\right) - 2\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)(2n+1)}{2} - n(n+1)$$

$$= \frac{n(n+1)}{2}(2n+1-2)$$

$$= \frac{n(n+1)(2n-1)}{2}$$

Question#2

$\mathbf{1 \times 3 + 3 \times 6 + 5 \times 9 + \ldots}$

Solution:

$1 \times 3 + 3 \times 6 + 5 \times 9 + \ldots$

$1 + (k-1)2$

$= 1 + 2k - 2$

$= 2k - 1$

$3 + (k-1)3$

$= 3 + 3k - 3$

$= 3k$

If T_k *denotes the kth term of the series, then*

$T_k = (2k-1)(3k) = 6k^2 - 3k$

Now

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(6K^2 - 3k\right)$$

$$= 6\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k$$

$$= 6\frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2}$$

$$= n(n+1)(2n+1) - \frac{3}{2}n(n+1)$$

$$= n(n+1)\left[2n + 1 - \frac{3}{2}\right]$$

$$= n(n+1)\left[\frac{4n+2-3}{2}\right]$$

$$= n(n+1)\left[\frac{4n+2-3}{2}\right]$$

$$S_n = \frac{n(n+1)(4n-1)}{2}$$

Question#3

$\mathbf{1 \times 4 + 2 \times 7 + 3 \times 10 + \ldots}$

Solution:

$\boldsymbol{T_k}$

$= (\textbf{\textit{kth term of }} 1,2,3,\ldots)(\textbf{\textit{kth term of }} 4,7,10)$

$= (\mathbf{1} + (\boldsymbol{k} - \mathbf{1})(\mathbf{1})(\mathbf{4} + (\boldsymbol{k} - \mathbf{1})\mathbf{3})$

$= (\mathbf{1} + \boldsymbol{k} - \mathbf{1})(\mathbf{4} + \mathbf{3}\boldsymbol{k} - \mathbf{3})$

$= \boldsymbol{k}(\mathbf{3}\boldsymbol{k} + \mathbf{1})$

$\boldsymbol{T_k} = \mathbf{3}\boldsymbol{k}^2 + \boldsymbol{K}$

$$\Rightarrow \boldsymbol{S_n} = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(3k^2 + k)$$

$$= 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{\mathbf{3n(n+1)(2n+1)}}{\mathbf{6}} + \frac{\mathbf{n(n+1)}}{\mathbf{2}}$$

$$= \frac{\mathbf{n(n+1)(2n+1)}}{\mathbf{6}} + \frac{\mathbf{n(n+1)}}{\mathbf{2}}$$

$$= \frac{\mathbf{n(n+1)(2n+1)}}{\mathbf{2}} + \frac{\mathbf{n(n+1)}}{\mathbf{2}}$$

$$= \frac{\mathbf{n(n+1)}}{\mathbf{2}}[\mathbf{2n+1+1}]$$

$$= \frac{\mathbf{n(n+1)}}{\mathbf{2}}(\mathbf{2n+2})$$

$$= \frac{\mathbf{2n(n+1)(n+1)}}{\mathbf{2}}$$

$\boldsymbol{S_n = n(n+1)^2}$

Question#4

$\mathbf{3 \times 5 + 5 \times 9 + 7 \times 13 + \ldots}$

Solution:

$\boldsymbol{T_k} =$

$(\textbf{\textit{kth term of }} 3,5,7,\ldots)(\textbf{\textit{kth term of }} 5,9,13\ldots)$

$= \big(\mathbf{3} + (\boldsymbol{k}-\mathbf{1})(\mathbf{2})\big)(\mathbf{5} + (\boldsymbol{k}-\mathbf{1})\mathbf{4})$

$= (\mathbf{3} + \mathbf{2}\boldsymbol{k} - \mathbf{1})(\mathbf{5} + \mathbf{4}\boldsymbol{k} - \mathbf{4})$

$= (\mathbf{2}\boldsymbol{k} + \mathbf{1})(\mathbf{4}\boldsymbol{k} + \mathbf{1})$

$= \left(\mathbf{8}\boldsymbol{k}^2 + \mathbf{2}\boldsymbol{k} + \mathbf{4}\boldsymbol{k} + \mathbf{1}\right)$

$\boldsymbol{T_k} = \mathbf{8}\boldsymbol{k}^2 + \mathbf{6}\boldsymbol{k} + \mathbf{1}$

$$\Rightarrow \boldsymbol{S_n} = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(8k^2 + 6k + 1)$$

$= 8\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= 8\frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$

$= \frac{4n(n-1)(2n+1)}{3} + 3n(n+1) + n$

$= \frac{n}{3}[4(2n^2 + n + 2n + 1)] + 9n + 9 + 3]$

$= \frac{n}{3}(8n^2 + 12n + 4 + 9n + 9 + 3)$

$= \frac{n}{3}(8n^2 + 21n + 16)$

$= \frac{n}{3}(8n^2 + 21n + 16)$

Question#5

$1^2 + 3^2 + 5^2 + \ldots$

Solution:

$1^2 + 3^2 + 5^2 + \ldots$

$1 + (k-1)2$

$= 1 + 2k - 2$

$= 2k - 1$

If T_k denotes the kth term of the series, then

$T_k = (2k-1)^2 = 4k^2 - 4k + 1$

Let S_n denotes the sum of first n terms of the series then

$S_n = \sum_{k=1}^{n}(4k^2 - 4k + 1)$

$= \sum_{k=1}^{n} 4k^2 - \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$

$= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - 4\left(\frac{n(n+1)}{2}\right) + n$

$= \frac{2n(n+1)(2n+1)}{3} - 2(n(n+1)) + n$

$= n\left(\frac{2(n+1)(2n+1)}{3} - 2((n+1)) + 1\right)$

$= n\left(\frac{2(2n^2 + 2n + n + 1)(2n+1)}{3} - 2n - 2 + 1\right)$

$= n\left(\frac{2(2n^2 + 3n + 1)}{3} - 2n - 1\right)$

$= n\left(\frac{4n^2 + 6n + 2 - 6n - 3}{3}\right)$

$= n\left(\frac{4n^2 - 1}{3}\right)$

$= \frac{n}{3}(4n^2 - 1)$

Question#6

$2^2 + 5^2 + 8^2 + \ldots$

Solution:

$2^2 + 5^2 + 8^2 + \ldots$

$2 + (k-1)3$

$= 2 + 3k - 3$

$= 3k - 1$

If T_k denotes the kth term of the series, then

$T_k = (3k-1)^2 = 9k^2 - 6k + 1$

Let S_n denotes the sum of first n terms of the series then

$S_n = \sum_{k=1}^{n}(9k^2 - 6k + 1)$

$= 9\sum_{k=1}^{n} k^2 - 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= 9\frac{n(n+1)(2n+1)}{6} - \frac{6n(n+1)}{2} + n$

$= \frac{3n(n+1)(2n+!)}{2} - 3n(n+1) + n$

$= \frac{n}{2}[3(2n^2 + n + 2n + 1] - 6(n+1) + 2]$

$= \frac{n}{2}(6n^2 + 9n + 3 - 6n - 6 + 2)$

$$S_n = \frac{n}{2}(6n^2 + 3n - 1)$$

Question#7

$2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + \ldots$

Solution:

$2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + \ldots$

$2 + (k-1)2$

$= 2 + 2k - 2$

$= 2k$

$1 + (k-1)1$

$= 1 + k - 1$

$= k$

If T_k denotes the kth term of the series, then

$T_k = (2k)(k)^3 = 2k^3$

Let S_n denotes the sum of first n terms of the series then

$S_n = \sum_{k=1}^{n}(3k^2 - 2k)$

$= \sum_{k=1}^{n}(2k^3) = 2\sum_{k=1}^{n} k^3$

$= 2\left(\frac{n(n+1)}{2}\right)^2$

$= 2.\frac{n^2(n+1)^2}{4}$

$= \frac{n^2(n+1)^2}{2}$

Question#8

$3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + \ldots$

Solution:

$3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + \ldots$

$3 + (k-1)2$

$= 3 + 2k - 2$

$= 2k + 1$

$2 + (k-1)1$

$= 2 + k - 1$

$= k + 1$

If T_k denotes the kth term of the series, then

$T_k = (2k+1)(k+1)^3 = (k+1)(k^2+2k+1)$

$= 2k^3 + 4k^2 + 2k + 1 + k^2 + 2k + 1$

$= 2k^3 + 5k^2 + 4k + 1$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n} (2k^3 + 5k^2 + 4k + 1)$$

$$= 2\sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 2\left(\frac{n(n+1)}{2}\right)^2 + 5\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right) + n$$

$$= 2.\frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2(n(n+1)) + n$$

$$= n\left(\frac{n(n^2+2n+1)}{2} + \frac{5n(2n^2+2n+n+1)}{6} + 2(n+1) + 1\right)$$

$$= n\left(\frac{n^3+2n^2+n}{2} + \frac{5(2n^2+2n+n+1)}{6} + 2n + 2 + 1\right)$$

$$= n\left(\frac{n^3+2n^2+n}{2} + \frac{10n^2+15n+5}{6} + 2n + 3\right)$$

$$= n\left(\frac{3n^3+6n^2+3n+10n^2+15n+5+12n+18}{6}\right)$$

$$= n\left(\frac{3n^3+16n^2+30n+23}{6}\right)$$

$$= \frac{n}{6}(3n^3 + 16n^2 + 30n + 23)$$

Question#9

$2\times4\times7 + 3\times6\times10 + 4\times8\times13 + \ldots$

Solution:

$2\times4\times7 + 3\times6\times10 + 4\times8\times13 + \ldots$

$\Rightarrow 2 + (k-1)1$

$= 2 + k - 1$

$= k + 1$

$\Rightarrow 7 + (k-1)3$

$= 7 + 3k - 3$

$= 3k + 4$

$\Rightarrow 4 + (k-1)2$

$= 4 + k - 2$

$= 2k + 2 = 2(k+1)$

If T_k denotes the kth term of the series, then

$T_k = 2(k+1)(k+1)(3k+4)$

$= 2(k+1)^2(3k+4)$

$= 2(k^2+2k+1)(3k+4)$

$= (3k^3 + 6k^2 + 3k + 4k^2 + 8k + 4)$

$T_k = (3k^3 + 10k^2 + 11k + 4)$

$$\Rightarrow S_n = \sum_{k=1}^{n} T_k = 2.\sum_{k=1}^{n}(3k^3 + 10k^2 + 11k + 4)$$

$$= 2\left[3\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 + 11\sum_{k=1}^{n} k + \sum_{k=1}^{n} 4\right]$$

$$= 2\left[3.\frac{n^2(n+1)^2}{4} + \frac{10n(n+1)(2n+1)}{6} + \frac{11n(n+1}{2} + 4n\right]$$

$$= 2\left[\frac{3n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{3} + \frac{11n(n+1)}{2} + 4n\right]$$

$$= 2n\left[\frac{9n(n^2+1+2n) + 20(2n^2+n+2n+1) + 66n + 66 + 48}{12}\right]$$

$$= \frac{n}{6}[9n^3 + 9n + 18n^2 + 40n^2 + 60n + 20 + 66n + 66 + 48]$$

$$= \frac{n}{6}(9n^3 + 58n^2 + 135n + 134)$$

Question#10

$1\times\ 4\times6 + 4\times7\times10 + 7\times10\times14 + \ldots$

Solution:

$\Rightarrow 2 + (k-1)1$

$= 2 + k - 1$

$= k + 1$

$\Rightarrow 7 + (k-1)3$

$= 7 + 3k - 3$

$= 3k + 4$

$\Rightarrow 4 + (k-1)2$

$= 4 + k - 2$

$= 2k + 2 = 2(k+1)$

If T_k denotes the kth term of the series, then

$T_k = 2(k+1)(k+1)(3k+4)$

$= 2(k+1)^2(3k+4)$

$= 2(k^2+2k+1)(3k+4)$

$= 3k^3 + 6k^2 + 3k + 4k^2 + 8k + 4$

$= 3k^3 + 10k^2 + 11k + 4$

Statement is wrong.

Question#11

$1 + (1+2) + (1+2+3) + \ldots$

Solution:

$1 + (1+2) + (1+2+3) + \ldots$

If T_k denotes the kth term of the series, then

$1 + 2 + 3 \ldots\ldots + k = \frac{k(k+1)}{2} = \frac{1}{2}(k^2 + k)$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{2}(k^2 + k)\right)$$

$= \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k$

$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$

$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3} + 1\right)$

$= \frac{n(n+1)}{4}\left(\frac{2n+1+3}{3}\right)$

$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)$

$= \frac{n(n+1)(2n+4)}{12}$

Question#12

$\mathbf{1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots}$

Solution:

$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

If T_k denotes the kth term of the series, then

$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

$= \frac{1}{6}k(2k^2 + 2k + k + 1)$

$= \frac{1}{6}k(2k^2 + 3k + 1)$

$= \frac{1}{6}(2k^3 + 3k^2 + k)$

$= \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right)$$

$= \frac{1}{3}\sum_{k=1}^{n} k^3 + \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$

$= \frac{1}{3}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{6}\left(\frac{n(n+1)}{2}\right)$

$= \frac{n^2(n+1)^2}{12} + \frac{n(n+1)(2n+1)}{12} + \frac{(n(n+1))}{12}$

$= \frac{n(n+1)}{12}(n(n+1) + 2(n+1) + 1)$

$= \frac{n(n+1)}{12}(n^2 + n + 2n + 1 + 2)$

$= \frac{n(n+1)(n^2+3n+3)}{12}$

Question#13

$\mathbf{2 + (2+5) + (2+5+8) + \dots}$

Solution:

$2 + (2+5) + (2+5+8) + \dots$

If T_k denotes the kth term of the series, then

$2 + 5 + 8 + \dots + upto\ k\ term.$

$= \frac{k}{2}[2(2) + (k-1)(3)]$

$= \frac{k}{2}[4 + 3k - 3]$

$= \frac{k}{2}[3k + 1]$

$= \frac{3}{2}k^2 + \frac{1}{2}k$

$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left[\frac{1}{2}(3k^2 + k)\right]$

$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{2} \cdot \frac{n(n+1)}{2}[2n + 1 + 1]$

$= \frac{n(n+1)}{4}[2n + 2]$

$= \frac{n(n+1)(n+1)}{2}$

$s_n = \frac{n(n+1)^2}{2}$

Question#14

Sum the series.

$(i).\ \mathbf{1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2}$

Solution:

$1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2$

If T_k denotes the kth term of the series, then

$T_k = 2(k-1)^2 - (2k)^2$

$= 2(k^2 - 2k + 1) - (4k^2)$

$= 4k^2 - 4k + 1 - 4k^2$

$= -4k + 1$

Now

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(-4k + 1)$$

$$= -4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= -4\frac{n(n-1)}{2} + n$$

$$= -n(n+1) + n = -2n^2 - 2n + n$$

$$S_n = -2n^2 - n = -n(2n+1)$$

$(ii).\quad \mathbf{1^2 - 3^2 + 5^2 - 7^2 + \dots + (4n-3)^2 - (4n-1)^2}$

Solution:

$1^2 - 3^2 + 5^2 - 7^2 + \dots + (4n-3)^2 - (4n-1)^2$

If T_k denotes the kth term of the series, then

$T_k = (4k-3)^2 - (4k-1)^2$

$= (16k^2 - 24k + 9) - (16k^2 - 8k + 1)$

$= 16k^2 - 24k + 9 - 16k^2 + 8k - 1$

$= -16k + 8$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(-16k + 8)$$

$$= -16\sum_{k=1}^{n} k + \sum_{k=1}^{n} 8$$

$$= -16\frac{n(n+1)}{2} + 8n$$
$$= -8n(n+1) + 8n$$
$$= -8n^2 - 8n + 8n$$
$$S_n = -8n^2$$

(iii). $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \dots$ *to n terms*

Solution:

If T_k denotes the kth term of the series, then

$$\frac{1^2+2^2+3^2+\dots+k^2}{k} = \frac{\frac{k(k+1)(2k+1)}{6}}{k} = \frac{k(k+1)(2k+1)}{6k}$$
$$= \frac{2k^2+2k+k+1}{6k}$$
$$= \frac{2k^2+3k+1}{6k}$$
$$= \frac{2}{6}k^2 + \frac{3}{6}k + \frac{1}{6}$$
$$= \frac{1}{3}k^2 + \frac{1}{2}k + \frac{1}{6}$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{3}k^2 + \frac{1}{2}k + \frac{1}{6}\right)$$
$$= \frac{1}{3}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k + \frac{1}{6}\sum_{k=1}^{n}(1)$$
$$= \frac{1}{3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\left(\frac{n(n+1)}{2}\right) + \frac{n}{6}$$
$$= \frac{n(n+1)(2n+1)}{18} + \frac{(n(n+1))}{4} + \frac{n}{6}$$
$$= \frac{n}{2}\left(\frac{(n+1)(2n+1)}{9} + \frac{(n+1)}{2} + \frac{1}{3}\right)$$
$$= \frac{n}{2}\left(\frac{(2n^2+2n+n+1)}{9} + \frac{n+1}{2} + \frac{1}{3}\right)$$
$$= \frac{n}{2}\left(\frac{2n^2+3n+1}{9} + \frac{n+1}{2} + \frac{1}{3}\right)$$
$$= \frac{n}{2}\left(\frac{4n^2+6n+2+9n+9+6}{18}\right)$$
$$= \frac{n}{2}\left(\frac{4n^2+15n+17}{18}\right)$$
$$= \frac{n}{36}(4n^2 + 15n + 17)$$

Question#15

Find the sum to n terms of the series whose nth terms are given.

(i). $3n^2 + n + 1$

Solution:

Since,

$T_n = 3n^2 + n + 1$

Therefore,

$T_k = 3k^2 + k + 1$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(3k^2 + k + 1\right)$$
$$= 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \sum_{k=1}^{n} 1$$
$$= 3\frac{n(n+1)(2n+1)}{6} + \frac{n9n+1)}{2} + n$$
$$= n\left[\frac{(n+1)}{2} + \frac{n+1}{2} + 1\right]$$
$$= \frac{n}{2}[2n^2 + n + 2n + 1 + n + 1 + 2]$$
$$\frac{n}{2}(2n^2 + 4n + 4) = \frac{2n}{2}(n^2 + n + 2)$$
$$S_n = n(n^2 + n + 2)$$

(ii). $n^2 + 4n + 1$

Solution:

$$T_n = n^2 + 4k + 1$$
$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2 + 4k + 1)$$
$$\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n$$
$$n\left[\frac{2n^2 + n + 2n + 1}{6} + 2n + 2 + 1\right]$$
$$= \frac{n}{6}[2n^2 + 3n + 1 + 12n + 12 + 6]$$
$$= \frac{n}{6}[2n^2 + 15n + 19]$$

Question#16

Given nth terms of the series, find the sum to 2n terms.

(i). $3n^2 + 2n + 1$

Solution:

$T_n = 3n^2 + 2n + 1$

$S_{2n} = ?$

$T_k = 3k^2 + 2k + 1$

$$\because S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(3k^2 + 2k + 1)$$
$$S_n = 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} + \sum_{k=1}^{n} 1$$
$$= 3\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n$$
$$= \frac{n(n+1)(2n+1)}{6} + n(n+1) + n$$
$$= n\left[\frac{2n^2 + n + 2n + 1}{2} + n + 1 + 1\right]$$
$$= \frac{n}{2}(2n^2 + 3n + 1 + 2n + 2 + 2)$$
$$= \frac{n}{2}(2n^2 + 5n + 5)$$
$$\Rightarrow S_n = \frac{n}{2}(2n^2 + 5n + 5)$$

So

$$S_n = \frac{n}{2}(2n^2 + 5n + 5)$$
$$\Rightarrow S_n = \frac{n}{2}(2n^2 + 5n + 5)$$

So

$$S_{2n} = \frac{2n}{2}[2(2n)^2 + 5n + 5]$$
$$\Rightarrow S_{2n} = n[8n^2 + 10n + 5]$$

(*ii*). $\boldsymbol{n^3 + 2n + 3}$

Solution:

Since,

$T_n = n^3 + 2n + 3$

Therefore,

$T_k = k^3 + 2k + 3$

Let S_n *denotes the sum of first n terms of the series then*

$$S_n = \sum_{k=1}^{n} (k^3 + 2k + 3\,)$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k + 3\sum_{k=1}^{n} (1)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\left(\frac{n(n+1)}{2}\right) + 3n$$

$$= n\left(\tfrac{n(n+1)^2}{4} + n + 1 + 3\right)$$

$$= n\left(\frac{n(n^2+2n+1)}{4} + n + 4\right)$$

$$= n\left(\frac{n^3+2n^2+n}{4} + n + 4\right)$$

$$= n\left(\frac{n^3+2n^2+n+4n+16}{4}\right)$$

$$= \tfrac{n}{4}(n^3 + 2n^2 + 5n + 16)$$

Now for sum of first 2n terms put n = 2n

$$S_n = \tfrac{2n}{4}((2n)^3 + 2(2n) + 5(2n) + 16)$$

$$= \tfrac{n}{2}(8n^3 + 8n^2 + 10n + 16)$$

$$= \tfrac{2n}{2}(4n^3 + 4n^2 + 5n + 8)$$

$$S_{2n} = n(4n^3 + 4n^2 + 5n + 8)$$