MATHEMATICS 11 INTERMEDIATE PART 1

Bilal Article

Chapter 3.

MATRICES AND DETERMINANTS

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3.1 Introduction

Matrix:

A rectangular array of numbers enclosed by a pair of bracket is called a matrix.

$e.g;\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$

Note:

i) Matrices are denoted by capital letters such as A,B,C…,X,Y,Z

ii) The elements or entries of a matrix as denoted by small letters such as $a, b, c, \dots, x, y, z.$

iii) The horizontal lines of elements are called rows of a matrix.

iv) The vertical lines of elements are called columns of a matrix.

Order of a matrix.

The number of rows and columns of a matrix is called order of a matrix.

i.e.; if a matrix has m rows and n column then its order is $m \times n$ (read as m-by-n).

$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 7 \end{bmatrix}$ $its\ order\ is\ 2 \times 3$

$B = [1 \quad 4 \quad 6]$ $its\ order\ is\ 1 \times 3$

*the matrix A is called real if all of its elements are real.

Types of Matrices

Row Matrix:

A matrix which has only one row i.e; a matrix of order $1 \times n$ is called row matrix.

e.g.;$[a_{11} \quad a_{12} \quad a_{13}]$,

$[1 \quad 2 \quad 3]$ etc.

Column Matrix:

A matrix which has only one column. i.e.; a matrix of order $m \times 1$ is called column matrix. e.g.;

$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $etc.$

Rectangular Matrix.

A matrix whose number of rows and columns are not equal is called rectangular matrix.eg.;

$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 7 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{@1} & a_{22} & a_{23} \end{bmatrix}$

Square Matrix:

A matrix whose number of rows and columns are equal is called square matrix. e.g.;

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Null Matrix:

A matrix whose all elements are zero is called null matrix.

e.g.; $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

*row matrix and column matrix are also called row vector and column.

Principal Diagonal:

The diagonal from upper left corner to the lower right corner of a square matrix is called principal or main diagonal or leading diagonal.

e.g.;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the entries a_{11}, a_{12}, a_{13} form the principal diagonal

Secondary diagonal:

The diagonal from lower left corner to the upper right corner of a square matrix is called secondary diagonal or leading diagonal.

e.g.;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The entries $a_{13}, a_{22}, and\ a_{31} form\ the\ secondary$ Diagonal.

Diagonal Matrix:

A square matrix in which all elements except the diagonal are zero is called a diagonal matrix.

e.g.; $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Identity Matrix:

A diagonal matrix whose all elements of the main diagonal are 1 is called identity matrix denoted by I_n

e.g.; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Scalar Matrix:

A diagonal matrix whose all elements of the main diagonal are same is called scalar matrix.

e.g.; $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}, \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}_{3\times 3}$

- **Needs to remember**

Square Matrix: A matrix having m rows and n columns with m=n is called square matrix.

Rectangular Matrix: A matrix having m rows and n columns with $m \neq n$ is called rectangular matrix.

Diagonal Matrix: let $A = [a_{ij}] be\ a\ square\ matrix$ of order n, if $a_{ij} = 0 \forall\ i \neq j\ and\ at\ least\ a_{ij} \neq 0\ for$ $i = j\ some\ elements\ of\ the\ principal\ diagonal\ of\ \ A\ may\ be\ zero\ but\ not\ all, then\ matrix\ A\ is\ called\ a\ diagonal\ matrix.$

Identity Matrix: let $A = [a_{ij}] be\ a\ square\ matrix\ of\ order\ n\ for\ all\ i \neq j\ and\ a_{ij} = 1\ for\ all\ i = j$
Then A is called unit matrix or identity matrix denoted by I_n.

Null Matrix: A matrix of order $m \times n$ with all elements zero is called null matrix.

Scalar Matrix: let $A = [a_{ij}] be\ a\ square\ matrix\ of$ Order n. if $a_{ij} = 0\ \forall\ i \neq j\ \ and\ a_{ij} = k(some\ non\ zero\ scalar)\ \forall\ i = j\ then\ the\ some\ matrix\ is\ called\ scalar\ matrix.$

Equal matrices:
Two matrices of the same order are said to be equal if corresponding entries are equal.

Addition of Matrices:
Two matrices can be added if both have same order.
*addition is done by adding corresponding entries of the matrices.
*Matrices of different orders cannot be added.

Transpose of matrix:
Transpose of a matrix is denoted by A^t can be obtained by interchanging rows into column or column into rows.

Scalar multiplication:
$let\ A = [a_{ij}] be\ an\ m \times n\ matrix\ and\ k\ be\ a\ sclar$
$then\ the\ product\ ok\ k\ and\ A\ can\ be\ obtained\ by\ multipling\ each\ entry\ of A\ by\ k.$
$i.e\ kA = [ka_{ij}]$
$order\ of\ kA\ is\ m \times n$
$*if\ n\ is\ a + ve\ integer\ ,$
$then\ A + A + \cdots, to\ nterms = nA$

Subtraction of matrices:
$if\ A = [a_{ij}] and\ B = [b_{ij}]\ are\ matrices\ of\ order\ m \times n, then\ we\ define\ subtraction\ of\ B\ from\ A$ As:

$$A - B = A + (-B) = [a_{ij}] + [-b_{ij}]$$
$$A - B = [a_{ij} - b_{ij}]$$
$$i = 1,2,3, \ldots, m, j = 1,2,3, \ldots, n$$

$A - b\ if\ formed\ by\ subtracting\ each\ entry\ of\ B\ from\ the\ corresponding\ entry\ of\ A.$

Multiplication of two Matrix:
Two matrix A and B are said to be conformable for product AB if the number of columns of A is equal to the number of rows of B.
*Matrix multiplication is not commutative $i.e;$.

$$AB \neq BA$$

- If the product AB is defined, then the order of the product can be illustrated as given below.

Order of A	$m \times n$
$order\ of\ B$	$n \times p$
Order of AB	$m \times n$

NOTE:
Powers of square matrices are defined as $A^2 = A \times A$
$A^3 = A \times A \times A \ldots \ldots to\ n\ factors.$

Determinant of 2×2 Matrix:
The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets.
e.g; $if\ A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\ then\ |A| = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

⇨ $|A| = ad - bc$
$for\ exampl$
$if\ A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\ then\ |A| = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$
$|A| = (2)(3) - (4)(-1) = 6 + 4 = 10$
Hence the determinant of a matrix is the difference of the product of the entries in the two diagonals.

Singular Matrix:
A square matrix A is said to be singular if $|A| = 0$
Example:
$if\ A = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}\ then\ |A| = \begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix}$
$|A| = (8)(1) - (4)(2) = 8 - 8 = 0$

Non-Singular Matrix:
A square matrix A is said to be Non- singular if $|A| \neq 0$
Example:
$if\ A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}\ then\ |A| = \begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix}$
$|A| = (1)(9) - (2)(4) = 9 - 8$
$= 1 \neq 0\ so\ A\ is\ non - singular$

Ad joint of 2×2 *Matrix*:
$the\ adjoint\ of\ a\ matrix\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ is\ denoted\ by\ adjA\ and\ defined\ as\ adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of 2×2 *Matrix*
Let A be a non-singular square matrix of order 2. If there exist of matrix B such that
$AB = BA = I_2\ where\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ then\ B\ is\ called\ multiplication\ inverse\ of\ A\ and\ is\ usually\ denoted\ by\ A^{-1}$
i.e.; $B = A^{-1}\ thus$
$$AA^{-1} = A^{-1}A = I_2$$

Solution Of Simultaneous Linear Equations By Using Matrices:
$let\ the\ system\ of\ linear\ eqs.\ be$
$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in R$

Given system in matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$let\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$if\,|A| \neq 0\ then\ A^{-1}\ exist\ so,$$

$$AX = B$$

$$pre\ multing\ by\ A^{-1}$$

⇨ $A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$I_2X = A^{-1}B \qquad \because A^{-1}A = I_2$

⇨ $X = A^{-1}B$

Or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$= \frac{1}{|A|}\begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_{22}b_1 - a_{12}b_2}{|A|} \\ \frac{-a_{21}b_1 + a_{11}b_2}{|A|} \end{bmatrix}$$

⇨ $x_1 = \frac{a_{22}b_1 - a_{12}b_2}{|A|}$ and

$x^2 = \frac{-a_{21}b_1 + a_{11}b_2}{|A|}$

thus

$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|}$ and

$x^2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix}}{|A|}$

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Exercise 3.1

Q1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$

then show that i) $4A - 3A = A$

ii) $3B - 3A = 3(B - A)$

Solution:

$4A - 3A = A$

$L.H.S \quad = 4A - 3A$

$$4\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6 & 12-9 \\ 4-3 & 20-15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= A = R.H.S\ proved$$

ii) $3B - 3A = 3(B - A)$

Solution:

$L.H.S \qquad 3B - 3A$

$$= 3\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 21 \\ 18 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3-6 & 21-9 \\ 18-3 & 12-15 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 12 \\ 15 & 15 \end{bmatrix} \rightarrow (1)$$

$R.H.S$

$$3(B - A) = 3\left[\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\right]$$

$$= 3\left[\begin{bmatrix} 1-2 & 7-3 \\ 6-1 & 4-5 \end{bmatrix}\right]$$

$$= 3\begin{bmatrix} -1 & 4 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 12 \\ 15 & 15 \end{bmatrix} \rightarrow (2)$$

hence $L.H.S = R.H.S$

Q2.

$$if\ A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}\ show\ that\ A^4 = I_2$$

Solution: $Note\ i = \sqrt{-1} = i^2 = -1$

$$\because A^2 = A \times A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$A^2 = \begin{bmatrix} i^2 + 0 & 0 - 0 \\ i - i & 0 + i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad Hence\ Proved$$

Q3.

find x and y if

$$\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

⇨ $x+3=2$, $3y-4=2$

⇨ $x=2-3$, $3y=2+4$

⇨ $x=-1$, $y=\frac{6}{3}=2$

⇨ So $x=-1$ and $y=2$

ii)

$$\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$$

Solution:

⇨ $x+3=y \qquad 3y-4=2x$

⇨ $2x-3y+4=0$

⇨ $2x-3(x+3)+4=0$

⇨ $2x-3x-9+4=0$

⇨ $-x-5=0$

⇨ $-x=5$

⇨ $x=-5$ put in(1)

⇨ $-5+3=y$

⇨ $y=-2$

So $x=-5$ and $y=-2$

Q.4

if $A=\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ ***and*** $B=\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

find the following matrices

(i)4A − 3B (ii) A + 3(B − A)

Solution:

$$4A-3B = 4\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 9 & 6 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-0 & 8-9 & 12-6 \\ 4-3 & 0+3 & 8-6 \end{bmatrix}$$

$$4A-3B = \begin{bmatrix} -4 & -1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

ii)

$A+3(B-A)$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\left(\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\left(\begin{bmatrix} 0+1 & 3-2 & 2-3 \\ 1-1 & -1-0 & 2-2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -3 \\ 0 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3 & 2+3 & 3-3 \\ 1+0 & 0-3 & 2-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 0 \\ 1 & -3 & 2 \end{bmatrix}$$

Q5. ***Find x and y if***

$$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2x & 2y \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2+2 & 0+2x & x+2y \\ 1+0 & y+4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2x & x+2y \\ 1 & 4+y & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

⇨ $2x=-2 \qquad \Rightarrow x=-1$

⇨ $4+y=6 \quad \Rightarrow y=2$

⇨ $x=-1$ *and* $y=2$

Q 6. ***if*** $A=[a_{ij}]_{2\times 2}$ ***show that***

i) $\lambda(\mu A)=(\lambda\mu)A$

Solution :

$$L.H.S = \lambda(\mu A) \qquad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \lambda\left(\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$= \lambda\left(\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix}$$

$$= \lambda\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (\lambda\mu)A \quad = R.H.S$$

hence proved.

ii) $(\lambda+\mu)A=\lambda A+\mu A$

$L.H.S$

$$= (\lambda+\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda+\mu)a_{11} & (\lambda+\mu)a_{12} & (\lambda+\mu)a_{13} \\ (\lambda+\mu)a_{21} & (\lambda+\mu)a_{22} & (\lambda+\mu)a_{23} \\ (\lambda+\mu)a_{31} & (\lambda+\mu)a_{32} & (\lambda+\mu)a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11}+\mu a_{11} & \lambda a_{12}+\mu a_{12} & \lambda a_{13}+\mu a_{13} \\ \lambda a_{21}+\mu a_{21} & \lambda a_{22}+\mu a_{22} & \lambda a_{23}+\mu a_{23} \\ \lambda a_{31}+\mu a_{31} & \lambda a_{32}+\mu a_{32} & \lambda a_{33}+\mu a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} + \begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

$$= \lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$= \lambda A+\mu A \quad = R.H.S$ *hence proved*

(iii) $\lambda A - A = (\lambda-1)A$

$$\lambda A - A = \lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11}-a_{11} & \lambda a_{12}-a_{12} & \lambda a_{13}-a_{13} \\ \lambda a_{21}-a_{21} & \lambda a_{22}-a_{22} & \lambda a_{23}-a_{23} \\ \lambda a_{31}-a_{31} & \lambda a_{32}-a_{32} & \lambda a_{33}-a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda-1)a_{11} & (\lambda-1)a_{12} & (\lambda-1)a_{13} \\ (\lambda-1)a_{21} & (\lambda-1)a_{22} & (\lambda-1)a_{23} \\ (\lambda-1)a_{31} & (\lambda-1)a_{32} & (\lambda-1)a_{33} \end{bmatrix}$$

$$= (\lambda-1)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (\lambda-1)A$$

Q.7

$$\textit{if } A = [a_{ij}]_{2\times3} \textit{ and } B = [b_{ij}]_{2\times3}$$

$$\textit{show that } \lambda(A+B) = \lambda A + \lambda B$$

$$A = [a_{ij}]_{2\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = [b_{ij}]_{2\times3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$L.H.S$

$$= \lambda\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right)$$

$$\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} & \lambda b_{13} \\ \lambda b_{21} & \lambda b_{22} & \lambda b_{23} \end{bmatrix}$$

$$= \lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \lambda\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \lambda A + \lambda B \quad R.H.S$$

Hence proved.

Q.8

$$\textit{if } A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \textit{ and } A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

find the valves of a and b

$$\because A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix}$$

$$\because A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \textit{ so}$$

$$\begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$1+2a = 0$$

⇨ $a = -\frac{1}{2}$

⇨ $= 2+2b = 0$

⇨ $b = -\frac{2}{2} = -1$

Q.9 $\textit{if } A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} \textit{ and } A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find the valves of a and b

$$\because A^2 = A \times A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}$$

$$\because A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{ so}$$

$$\begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1-a = 1 \quad \Rightarrow a = 0$$

$$-1-b = 0 \quad \Rightarrow b = -1$$

Q10.

$$\textit{if } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \textit{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

then show that

solution: $(A+B)^t = A^t + B^t$

$L.H.S = (A+B)^t$

$$= \left(\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}\right)^t$$

$$= \left(\begin{bmatrix} 1+2 & -1+3 & 2+0 \\ 0+1 & 3+2 & 1-1 \end{bmatrix}\right)^t$$

$$= \left(\begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}\right)^t$$

$$(A+B)^t = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix}$$

$$A^t + B^t = \left(\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\right)^t + \left(\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}\right)^t$$

$$A^t + B^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$A^t + B^t = \begin{bmatrix} 1+2 & 0+1 \\ -1+3 & 3+2 \\ 2+0 & 1-1 \end{bmatrix}$$

$$\boldsymbol{A^t + B^t} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{0} \end{bmatrix}$$

$\boldsymbol{L.H.S = R.H.S}$

Q.11 $\boldsymbol{find\ A^3\ if\ A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{2} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$

Ans.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

Q12. $\boldsymbol{find\ the\ matrix\ X\ if}$

Solution:

i) $\boldsymbol{X}\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{5} \\ \mathbf{12} & \mathbf{3} \end{bmatrix}$

Solution:

$$X\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

$$XA = B$$

⇨ $X = BA^{-1} \rightarrow (1)$

Now

$$|A| = \begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix} = 5+4 = 9 \neq 0$$

$$adjA = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$\because A^{-1} = \frac{adjA}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}}{9}$$

S0 (1) become

$$X = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix} \times \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} -1+10 & 2+25 \\ 12+6 & -24+15 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 9 & 27 \\ 18 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$
$$so\ X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

ii) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{10} \end{bmatrix}$

Solution:

$$\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{10} \end{bmatrix}$$
$$\boldsymbol{AX = B}$$
$$\boldsymbol{X = A^{-1}B} \rightarrow (\mathbf{1})$$
$$|A| = \begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix} = 5 + 4 = 9 \neq 0$$
$$adjA = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$
$$\because A^{-1} = \frac{adjA}{|A|}$$
$$A^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}}{9}$$

So (1)

$$X = \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{10} \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 2-10 & 1-20 \\ 4+25 & 2+50 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} -8 & -19 \\ 29 & 52 \end{bmatrix} = \begin{bmatrix} -\frac{8}{9} & -\frac{19}{9} \\ \frac{29}{9} & \frac{52}{9} \end{bmatrix}$$
$$so\ X = \begin{bmatrix} -\frac{8}{9} & -\frac{19}{9} \\ \frac{29}{9} & \frac{52}{9} \end{bmatrix}$$

Q13.

Find the matrix A if

$$i)\ \begin{bmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} \boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{7} & \mathbf{2} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$
$$suppose\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}\ then$$

$$\begin{bmatrix} 5a-c & 5b-d \\ 0+0 & 0+0 \\ 3a+c & 3b+d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 5a-c & 5b-d \\ 0 & 0 \\ 3a+c & 3b+d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$

⇨ $5a - c = 3 \rightarrow (i)$ $\quad 5b - d = -7 \rightarrow (iii)$

⇨ $3a + c = 7 \rightarrow (ii)$ $\quad 3b + d = 2 \rightarrow (iv)$

⇨ $(i) + (ii) 8a = 10$ $\quad (iii) + (iv)$

⇨ $a = \frac{10}{8}$ $\quad 8b = -5$

⇨ $a = \frac{5}{4}$ $\quad b = -\frac{5}{8}$

$$so\ (i)\quad 5\left(\frac{5}{4}\right) - c = 3$$
$$c = \frac{25}{4} - 3 = \frac{25-12}{4} = \frac{13}{4}$$
$$so\ (iv)\quad 3\left(-\frac{5}{8}\right) + d = 2$$

⇨ $-\frac{15}{8} + d = 2$

⇨ $d = 2 + \frac{15}{8} = \frac{16+15}{8} = \frac{31}{8}$

Hence $\boldsymbol{A} = \begin{bmatrix} \frac{5}{4} & -\frac{5}{8} \\ \frac{13}{4} & \frac{31}{8} \end{bmatrix}$

(ii)

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{-7} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$
$$BA = C$$

⇨ $A = B^{-1}C \rightarrow (1)$

$$|B| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$
$$adjB = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$\because B^{-1} = \frac{adjB}{|B|}$$
$$A^{-1} = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}{3}$$
$$A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$
$$A = \frac{1}{3}\begin{bmatrix} 0+3 & -6+3 & 16-7 \\ 0+6 & -3+6 & 8-14 \end{bmatrix}$$
$$A = \frac{1}{3}\begin{bmatrix} 3 & -3 & 9 \\ 6 & 3 & -6 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

Q.14

Show that

$$\begin{bmatrix} \boldsymbol{rcos\emptyset} & \mathbf{0} & \boldsymbol{-sin\emptyset} \\ \mathbf{0} & \boldsymbol{r} & \mathbf{0} \\ \boldsymbol{rsin\emptyset} & \mathbf{0} & \boldsymbol{cos\emptyset} \end{bmatrix}\begin{bmatrix} \boldsymbol{rcos\emptyset} & \mathbf{0} & \boldsymbol{-sin\emptyset} \\ \mathbf{0} & \boldsymbol{r} & \mathbf{0} \\ \boldsymbol{rsin\emptyset} & \mathbf{0} & \boldsymbol{cos\emptyset} \end{bmatrix} = \boldsymbol{rI_3}$$

Solution:

L.H.S= $\begin{bmatrix} r\cos\emptyset & 0 & -\sin\emptyset \\ 0 & r & 0 \\ r\sin\emptyset & 0 & \cos\emptyset \end{bmatrix}\begin{bmatrix} r\cos\emptyset & 0 & -\sin\emptyset \\ 0 & r & 0 \\ r\sin\emptyset & 0 & \cos\emptyset \end{bmatrix}$

$$= \begin{bmatrix} r\cos^2\emptyset + 0 + r\sin^2\emptyset & 0+0+0 & r\cos\emptyset\sin\emptyset - r\cos\emptyset\sin\emptyset \\ 0+0+0 & 0+r+0 & 0+0=0 \\ r\cos\emptyset\sin\emptyset - r\cos\emptyset\sin\emptyset & 0+0+0 & r\cos^2\emptyset + 0 + r\sin^2\emptyset \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos^2\emptyset + \sin^2\emptyset) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(\cos^2\emptyset + \sin^2\emptyset) \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad R.H.S$$

Hence proved

Properties of the matrix:

Addition, Scalar Multiplication and Matrix Multiplication:

If A, B and C are $m \times n$ matrices and c and d are scalars, then following properties are true:

1. Commutative property w.r.t addition:
   $A + B = B + A$
2. Associative property w.r.t addition
   $(A + B) + C = A + (B + C)$
3. Associative property w.r.t scalar multiplication:
   $(cd)A = c(dA)$
4. Existence of additive identity :
   $A + O = O + A = A \quad (O is\ null\ matrix$
5. Existence of Multiplicative identity:
   $IA + AI + A \quad (I\ is\ unit\ matrix)$
6. Distributive property w.r.t scalar Multiplicative:
   $(a) \quad c(A + B) = cA + cB$
   $(b) \quad (c + d)A = cA + dA$
7. Associative property w.r.t Multiplicative:
   $A(BC) + (AB)C$
8. Left Distributive property :
   $A(B + C) + AB + AC$
9. Right distributive property:
   $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$

Exercise 3.2

Q1. $if\ A = [a_{ij}]_{3\times4} then\ show\ that$

$i)\ \ I_3A = A \qquad ii) AI_4 = A$

Solution:

L.H.S= $I_3 = A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}0+0+0 & +0a_{32}+0+0 & 0+0+a_{33+0} & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

$hence\ I_3A = A$

ii) $AI_4 = A$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}0+0+0 & +0a_{32}+0+0 & 0+0+a_{33+0} & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A \quad hence \quad AI_4 = A$$

Q2.

$find\ the\ inverse\ of\ the\ following\ matricces.$

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

Solution:

Let A= $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$$A^{-1} = ? \quad A^{-1} = \frac{adjA}{|A|} \rightarrow (1)$$

$$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 3 + 2 = 5 \neq 0$$

$$adjA = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$\because A^{-1} = \frac{adjA}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}}{5}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$

Solution:

Let A= $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$

$$A^{-1} = ? \quad A^{-1} = \frac{adjA}{|A|} \rightarrow (1)$$

$$|A| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = -10 + 12 = 2 \neq 0$$

$$adjA = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$$
$$\because A^{-1} = \frac{adjA}{|A|}$$
$$A^{-1} = \frac{\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}}{2}$$
$$A^{-1} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{4}{2} & \frac{-2}{2} \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

iii)
$$\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$$

Solution:

Let A= $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$

$$A^{-1} = ? \quad A^{-1} = \frac{adjA}{|A|} \to (1)$$
$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix} = -2i^2 - i^2 = -3i^2 = 3 \neq 0$$
$$adjA = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$
$$\because A^{-1} = \frac{adjA}{|A|}$$
$$A^{-1} = \frac{\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}}{3}$$
$$A^{-1} = \frac{1}{3}\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix} = \begin{bmatrix} -\frac{i}{3} & -\frac{i}{3} \\ -\frac{i}{3} & \frac{2i}{3} \end{bmatrix}$$

iv) **Let A=** $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$$A^{-1} = ? \quad A^{-1} = \frac{adjA}{|A|} \to (1)$$
$$|A| = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 6 - 6 = 0$$

So $|A| = 0 \quad A^{-1}$ *does not exist.*

Q3.

Solve the following system of linear equations.

i) $2x_1 - 3x_2 = 5$
$5x_1 + x_2 = 4$

Solution:

In matrix form
$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$AX = B$$
$$X = A^{-1}B \to (1)$$
$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = 2 + 15 = 17 \neq 0$$

SOA^{-1}*exist.*

$$adjA = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$
$$\because A^{-1} = \frac{adjA}{|A|}$$
$$A^{-1} = \frac{\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}}{17}$$

So eq (1) *become*

$$X = \frac{\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}}{17}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$X = \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$X = \frac{1}{17}\begin{bmatrix} 5 + 12 \\ -25 + 8 \end{bmatrix} = \begin{bmatrix} \frac{17}{17} \\ -\frac{17}{17} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad so\ x_1 = 1, and\ x_2 = -1$$

ii) $4x_1 + 3x_2 = 5$
$3x_1 - x_2 = 7$

Solution:

In matrix form
$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$
$$AX = B$$
$$X = A^{-1}B \to (1)$$
$$A = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = -4 - 9 = -13 \neq 0$$

SOA^{-1}*exist.*

$$adjA = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$
$$\because A^{-1} = \frac{adjA}{|A|}$$
$$A^{-1} = \frac{\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}}{-13}$$

So eq (1) *become*

$$X = \frac{\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}}{-13}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$
$$X = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$
$$X = \frac{1}{-13}\begin{bmatrix} -5 - 21 \\ -15 + 18 \end{bmatrix} = \begin{bmatrix} \frac{-26}{-13} \\ -\frac{13}{13} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad so\ x_1 = 2, and\ x_2 = -1$$

iii) $3x - 5y = 1$
$-2x + y = -3$

Solution:

In matrix form
$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$AX = B$$

$$X = A^{-1}B \rightarrow (1)$$

$A = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = 3 - 10 = -7 \neq 0$

SO A^{-1} exist.

$$adjA = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

$$\because A^{-1} = \frac{adjA}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}{-7}$$

So eq (1) become

$$X = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}{-7} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$X = \frac{1}{-7} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$X = \frac{1}{-7} \begin{bmatrix} 1 - 15 \\ 2 - 9 \end{bmatrix} = \begin{bmatrix} \frac{-14}{-7} \\ \frac{-7}{-7} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so $x = 2$, and $y = 1$

Q4. if $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ then find (i) $(A - B)$ (ii) $B - A$ iii) $(A - B) - C$ (iv) $A - (B - C)$

Solution:

$$(i)(A - B) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1-1 & 2+1 \\ 3-1 & 2-3 & 5-4 \\ -1+1 & 0-2 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

(ii) $B - A$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+1 & -1-2 \\ 1-3 & 3-2 & 4-5 \\ -1+1 & 2-0 & 1-4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

iii) $(A - B) - C$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-1 & -2-3 & 3+2 \\ 2+1 & -1-2 & 1-0 \\ 0-3 & -2-4 & 3+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 & 5 \\ 3 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$$

(iv) $A - (B - C)$

$$A - \left(\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix} \right)$$

$$A - \left(\begin{bmatrix} 2-1 & 1-3 & -1+2 \\ 1+1 & 3-2 & 4-0 \\ -1-3 & 2-4 & 1+1 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1+2 & 2-1 \\ 3-2 & 2-1 & 5-4 \\ -1+4 & 0+2 & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

Q.5

if $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}, B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$

then show that (i) $(AB)C = A(BC)$

(ii) $(A + B)C = AC + BC$

(i) $(AB)C = A(BC)$

Solution:

$L.H.S = (AB)C$

$$= \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \right) \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 + 4i^2 & i + 2i^2 \\ -i - 2i^2 & 1 - i^2 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$\begin{bmatrix} 3i^2 & i + 2(-1) \\ -i - 2(-1) & 1 - (-1) \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$\begin{bmatrix} -3 & i - 2 \\ -i + 2 & 2 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -6i - i^2 + 2i & 3 + i^2 - 2i \\ 4i - 2i^2 - 2i & -2 + i + 2i \end{bmatrix}$$

$$= \begin{bmatrix} -4i - (-1) & 3 + (-1) - 2i \\ 2i - 2(-1) & -2 + 3i \end{bmatrix}$$

$$= \begin{bmatrix} -4i + 1 & 2 - 2i \\ 2i + 2 & -2 + 3i \end{bmatrix} \rightarrow (1)$$

$R.H.S = A(BC)$

$$= A\left(\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} \right)$$

$$= A\left(\begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix} \right)$$

$$= A\left(\begin{bmatrix} -2(-1) - (i) & 2i \\ 4(-1) - (-1) & -2i + (-1) \end{bmatrix} \right)$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2 - i & 2i \\ -3 & -2i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2i - i^2 - 6i & 2i^2 - 2i - 4i^2 \\ 2 - i + 3i & 2i + 2i^2 + i \end{bmatrix}$$

$$= \begin{bmatrix} -4i - (-1) & -2i^2 - 2i \\ 2 + 2i & 3i + 2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -4i + 1 & -2(-1) - 2i \\ 2 - 2i & 3i - 2 \end{bmatrix}$$

$$\begin{bmatrix}-4i+1 & 2-2i\\ 2+2i & 3i-2\end{bmatrix} \to (2)$$

From eq(1) and (2)

$L.H.S = R.H.S$

(ii) $(A+B)C = AC+BC$

$L.H.S = (A+B)C$

$$= \left(\begin{bmatrix} i & 2i\\ 1 & -i\end{bmatrix} + \begin{bmatrix}-i & 1\\ 2i & i\end{bmatrix}\right)\begin{bmatrix}2i & -1\\ -i & i\end{bmatrix}$$

$$= \begin{bmatrix} i-i & 2i+i\\ 1+2i & -i+i\end{bmatrix}C$$

$$= \begin{bmatrix} 0 & 3i\\ 1+2i & 0\end{bmatrix}\begin{bmatrix}2i & -1\\ -i & i\end{bmatrix}$$

$$= \begin{bmatrix} 0-i-2i^2 & o+i+2i^2\\ 2i+4i^2+0 & -1-2i+0\end{bmatrix}$$

$$\begin{bmatrix}-i+2 & i-2\\ 2i-4 & -1-2i\end{bmatrix} \to (1)$$

$R.H.S = AC+BC$

$$\begin{bmatrix} i & 2i\\ 1 & -i\end{bmatrix}\begin{bmatrix}2i & -1\\ -i & i\end{bmatrix} + \begin{bmatrix}-i & 1\\ 2i & i\end{bmatrix}\begin{bmatrix}2i & -1\\ -i & i\end{bmatrix}$$

$$\begin{bmatrix}2i^2-2i^2 & -i+2i^2\\ 2i+i^2 & -1-i^2\end{bmatrix} + \begin{bmatrix}-2i^2-i & i+i\\ 4i^2-i^2 & -2i+i^2\end{bmatrix}$$

$$= \begin{bmatrix}-2+2 & -i-2\\ 2i-1 & -1+1\end{bmatrix} + \begin{bmatrix}2-i & 2i\\ -4-1 & -2i-1\end{bmatrix}$$

$$= \begin{bmatrix}0 & -i-2\\ 2i-1 & 0\end{bmatrix} + \begin{bmatrix}2-i & 2i\\ -3 & -2i-1\end{bmatrix}$$

$$= \begin{bmatrix}0+2-i & -i-2+2i\\ 2i-1-3 & 0-2i-1\end{bmatrix}$$

$$= \begin{bmatrix}-i+2 & i-2\\ 2i-4 & -1-2i\end{bmatrix} \to ((2)$$

from eq(1)and eq (2)

$\boldsymbol{L.H.S = R.H.S}$

Q.6

if A nd B are square matrices of same order, then explain why in general;

$(\boldsymbol{i})(\boldsymbol{A}+\boldsymbol{B})^2 \neq \boldsymbol{A}^2 + 2\boldsymbol{Ab} + \boldsymbol{B}^2$

Solution:

$L.H.S = (A+B)^2$

$= (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

$\because AB \neq BA$ *in general so*

$AB + BA \neq 2AB$

Now

$L.H.S \neq A^2 + 2AB + B^2 = R.H.S$

hence $(A+B)^2 \neq A^2 + 2Ab + B^2$

ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

Solution:

$L.H.S = (A-B)^2$

$= (A-B)(A-B)$

$A^2 - AB - BA + B^2$

$\because AB \neq BA$ *in general so*

$-AB - BA \neq -2AB$

NOW $L.H.S \neq A^2 - 2AB + B^2 = R.H.S$

Hence $(A-B)^2 \neq A^2 - 2AB + B^2$

iii)

$A+B)(A-B) = A^2 - B^2$

$= A^2 - AB + BA - B^2$

$\because AB \neq BA$ *in general so*

$-AB + BA \neq 0$

Now

$L.H.S \neq A^2 - B^2 = R.H.S$

Hence

$A+B)(A-B) = A^2 - B^2$

$$Q.7\ if\ A = \begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$$

then find AA^t *and* A^tA

Solution:

$$\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}^t$$

$$AA^t = \begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & 1 & -3\\ -1 & 0 & 5\\ 3 & 4 & 2\\ 0 & -2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}4+1+9+0 & 2-0+12+0 & -6-5+6-0\\ 2-0+12+0 & 1+0+16+4 & -3+0+8+2\\ -6-5+6-0 & -3+0+8+2 & 9+25+4+1\end{bmatrix}$$

$$AA^t = \begin{bmatrix}14 & 14 & -5\\ 14 & 21 & 7\\ -5 & 7 & 39\end{bmatrix}$$

Also,

$$A^tA = \begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}^t\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}2 & 1 & -3\\ -1 & 0 & 5\\ 3 & 4 & 2\\ 0 & -2 & -1\end{bmatrix}\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}4+1+9 & -2+0-15 & 6+4-6 & 0-2+3\\ -2+0-15 & 1+0+25 & 1+0+25 & 0-0-5\\ 6+4-6 & -3+0+10 & -3+0+10 & 0-8-2\\ 0-2+3 & 0-0-5 & 0-8-2 & 0+4+1\end{bmatrix}$$

$$A^tA = \begin{bmatrix}14 & -17 & 4 & 1\\ -17 & 26 & 7 & -5\\ 4 & 7 & 29 & -10\\ 1 & -5 & -10 & 5\end{bmatrix}$$

Q8. solve the following matrix equations for X

$$i)\ 3X - 2A = B\ if\ A = \begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix}$$

$$and\ B = \begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$$

Solution:

$i)\ 3X - 2A = B$

$$3X - 2\begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix} = \begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$$

$$3X - \begin{bmatrix}4 & 6 & 2\\ -2 & 2 & 10\end{bmatrix} = \begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$$

$$3X = \begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix} + \begin{bmatrix}4 & 6 & 2\\ -2 & 2 & 10\end{bmatrix}$$

$$3X = \begin{bmatrix}2+4 & -3+6 & 1-4\\ 5-2 & 4+2 & -1+10\end{bmatrix}$$

$$3X = \begin{bmatrix}6 & 3 & -3\\ 3 & 6 & 9\end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix}6 & 3 & -3\\ 3 & 6 & 9\end{bmatrix}$$

$$X=\begin{bmatrix}\frac{6}{3} & \frac{3}{3} & -\frac{3}{3}\\ \frac{3}{3} & \frac{6}{3} & \frac{9}{3}\end{bmatrix}$$

$$X=\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 3\end{bmatrix}$$

ii) $2X-3A=B$

$$if\ A=\begin{bmatrix}1 & -1 & 2\\ -2 & 4 & 5\end{bmatrix}\ and\ B=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}$$

$$2X-3\begin{bmatrix}1 & -1 & 2\\ -2 & 4 & 5\end{bmatrix}=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}$$

$$2X-\begin{bmatrix}3 & -3 & 6\\ -6 & 12 & 15\end{bmatrix}=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}$$

$$2X=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}+\begin{bmatrix}3 & -3 & 6\\ -6 & 12 & 15\end{bmatrix}$$

$$2X=\begin{bmatrix}3+3 & -1-3 & 0+6\\ 4-6 & 2+12 & 1+15\end{bmatrix}$$

$$2X=\begin{bmatrix}6 & -4 & 6\\ -2 & 14 & 16\end{bmatrix}$$

$$X=\frac{1}{2}\begin{bmatrix}6 & -4 & 6\\ -2 & 14 & 16\end{bmatrix}$$

$$X=\begin{bmatrix}\frac{6}{2} & -\frac{4}{2} & \frac{6}{2}\\ -\frac{2}{2} & \frac{14}{2} & \frac{16}{2}\end{bmatrix}$$

$$X=\begin{bmatrix}3 & -2 & 3\\ -1 & 7 & 8\end{bmatrix}$$

Q.9 *solve the following matrix equations for A*

(i) $\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A-\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}-1 & -4\\ 3 & 6\end{bmatrix}$

Solution:

$$\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A-\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}-1 & -4\\ 3 & 6\end{bmatrix}$$

$$⇨\ \begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}-1 & -4\\ 3 & 6\end{bmatrix}+\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}$$

$$⇨\ \begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}-1+2 & -4+3\\ 3-1 & 6-2\end{bmatrix}$$

$$⇨\ \begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}$$

⇨ $BA=C$

⇨ $A=B^{-1}C\rightarrow(1)$

Here $B=\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}$ *and* $C=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}$

$$|B|=\begin{vmatrix}4 & 3\\ 2 & 2\end{vmatrix}=8-6=2\neq 0\ so\ B^{-1}\ exist$$

$$adjB=\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}$$

$$B^{-1}=\frac{adjB}{|B|}=\frac{1}{2}\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}$$

so eq (1) *become*

$$A=\frac{1}{2}\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}2-6 & -2-12\\ -2+8 & 2+16\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-4 & -14\\ 6 & 18\end{bmatrix}=\begin{bmatrix}-\frac{4}{2} & -\frac{14}{2}\\ \frac{6}{2} & \frac{18}{2}\end{bmatrix}=\begin{bmatrix}-2 & -7\\ 3 & 9\end{bmatrix}$$

$$⇨\ A=\begin{bmatrix}-2 & -7\\ 3 & 9\end{bmatrix}$$

ii)

$$A\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}-\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ -1 & 5\end{bmatrix}$$

$$A\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\ -1 & 5\end{bmatrix}+\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}$$

$$⇨\ A\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2-1 & 0+2\\ -1+3 & 5+1\end{bmatrix}$$

$$⇨\ A\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 6\end{bmatrix}$$

$AB=C$ *here* $B=\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}$ *and* $C=\begin{bmatrix}1 & 2\\ 2 & 6\end{bmatrix}$

⇨ $A=CB^{-1}\rightarrow(1)$

$$|B|=\begin{vmatrix}3 & 1\\ 4 & 2\end{vmatrix}=6-4=2\neq 0\ so\ B^{-1}\ exist$$

$$adjB=\begin{bmatrix}2 & -1\\ -4 & 3\end{bmatrix}$$

$$B^{-1}=\frac{adjB}{|B|}=\frac{1}{2}\begin{bmatrix}2 & -1\\ -4 & 3\end{bmatrix}$$

so eq (1) *become*

$$A=\frac{1}{2}\begin{bmatrix}2 & -1\\ -4 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 6\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}2-8 & -1+6\\ 4-24 & -2+18\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-6 & 5\\ -20 & 16\end{bmatrix}=\begin{bmatrix}-\frac{6}{2} & \frac{5}{2}\\ -\frac{20}{2} & \frac{16}{2}\end{bmatrix}=\begin{bmatrix}-3 & \frac{5}{2}\\ -10 & 8\end{bmatrix}=A$$

https://newsongoogle.com/

Determinants Minor Of An Element:

$Let\ A = [a_{ij}] be\ a\ matrix\ of\ order \times n. if\ we\ delete$ $the\ ith\ row\ and\ jth\ column\ of\ A, then\ we\ get\ a\ (n-1) \times (n-1) matrix. the\ determinant\ of\ the\ matrix (n-1)\times(n-1)\ matrix\ is\ called\ minor\ of the\ element\ a_{ij} denoted\ by\ M_{ij} for\ example$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{22} = M_{22} = \begin{vmatrix} a_{11} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

etc.

$$let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$Mionr\ of\ 2 = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

$$minor\ of\ 8 = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

Cofactors of an element:

$\boldsymbol{let\ A = [a_{ij}] be\ a\ square\ matrix\ .then\ cofector\ of\ a_{ij} is\ denoted\ by\ A_{ij} and\ denoted\ by\ A_{ij} and\ defined\ as}$

$\boldsymbol{Aij = (-1)^{i+j}\ M_{ij}}$

For example

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = cofector\ of\ a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{12} = cofector\ of\ a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Etc.

Determinant of a square matrix of order $n \geq 3$:

The determinant of a square matrix of order n is the sum of products of each element of a row or column by its cofactor.

For example , if A is matrix of order 3×3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad \textbf{then}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

(by expanding column first)

$$if\ A\ is\ a\ matrix\ of\ order\ n\ i.e$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & a_{2m} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & a_{in} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & a_{m} \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + .. + a_{ij}A_{1j} + \cdots + a_{1n}A_{n1}$$

Properties of Determinants which help their Evaluation:

1. For a square matrix $|A| = |A^t|$
2. if in a square matrix A, two rows or two column are interchanged, the determinant of resulting matrix is $-|A|$
3. if a square matrix A has two identical rows (or column)then $|A| = 0$
4. if all the entries of a row or column of a square matrix are zero, then $|A| = 0$
5. if the entries of row or a column in a square matrix A are multiply by a number $k \in R\ then\ the\ determinant\ of\ the\ resulting\ matrix$ is $k \in R$
6. if each entry of a row or a column of a square matrix consists of two terms then its determinant can be written as the sum of two determinants.
7. if two each entry of a row or column of a square matrix A is added a non- zero multiply of the corresponding entry of another row or column , then the determinant of the resulting matrix is $|A|$.
8. if a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Examples of above mentioned property of determinants.

1. Let A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad a_{11}a_{22} - a_{12}a_{21}$$

$$|A^t| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \quad a_{11}a_{22} - a_{12}a_{23}$$

⇨ $|A| = |A^t|$

2.

Let A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Now interchanged $R_1\ and\ R_2$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = a_{21}a_{12} - a_{11}a_{22}$$
$$= -(a_{11}a_{22} - a_{21}a_{12})$$
$$= -|A|$$

3. Let A=$\begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|A| = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$|A| = 0$$

4. $let\ A = \begin{bmatrix} a & b & c \\ a & b & c \\ x & y & z \end{bmatrix}$

$$|A| = a\begin{vmatrix} b & c \\ y & z \end{vmatrix} - b\begin{vmatrix} a & c \\ x & z \end{vmatrix} + c\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$
$$= a(bz - cy) - b(az - cx) + c(ay - by)$$
$$= abz - acy - baz + bcx + acy - bcx$$

⇨ $|A| = 0$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$Now'x'R_1 by\ K$

$$\begin{vmatrix} Ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = ka_{11}a_{22} - ka_{21}a_{12}$$

$$= k(a_{11}\,a_{22} - a_{31}\ \ a_{12}\,)$$

$$= k|A|$$

6.

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} + b_{11}a_{22} - a_{12}a_{21} - a_{12}b_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21} + b_{11}a_{22} - a_{12}b_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$$

$$= R.H.S$$

7. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix}$

$$L.H.S. \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$R.H.S \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix}$$

$$akb + ad - bka - bc$$

$$= ad - bc$$

⇨ $L.H.S = R.H.S$

8. $let\ A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

=> $|A| = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$

$$= (a_{11})(a_{22})(a_{33}) = a_{11}a_{22}a_{33}$$

Ad joint of a square matrix of order $n \geq 3$

$if\ A = [a_{ij}]\ be\ a\ square\ matrix\ of\ order\ n,$ Then $[a_{ij}]$ is matrix of cofactors, $adjoint\ of\ \ A\ is\ denoted\ by\ adjA\ and\ defined\ as$

For example

$$if\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\ then$$

$$adjA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^t$$

$$adjA = \begin{bmatrix} A_{11} & a_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Inverse of a square matrix of order $n \geq 3$

$if\ A\ is\ non-singular\ matrix\ of\ order\ n$ $then\ its\ inverse$ is denoted by A^{-1} $and\ defined$ AS

$$A^{-1} = \frac{1}{|A|}adjA$$

Exercise 3.3

Q.1 evaluate the following determinants.

$$i)\ \begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix} \quad expanding\ by\ R_1$$

$$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$5(-2+3) + 2(6-6) - 4(3-2)$$

$$5(1) + 2(6-6) - 4(3-2)$$

$$5(1) + 2(0) - 4(1) = 5 + 0 - 4 = 1$$

ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix} \quad expanding\ by\ R_1$$

$$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$5(2-1) - 2(-6+2) - 3(3-2)$$

$$5(1) - 2(-6+2) - 3(3-2)$$

$$5(1) - 2(-4) - 3(1) = 5 + 8 - 3 = 10$$

iii)

$$\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix} \quad expanding\ by\ R_1$$

$$= 1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 1(18-20) - 2(-6+8) - 3(-5+6)$$

$$= 1(-2) - 2(2) - 3(1) = -2 - 4 - 3 = -9$$

iv)

$$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$$

Solution:

$$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix} \quad expanding\ by\ R_1$$

$$= a + l\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix} - (a-l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix} + a\begin{vmatrix} a & a-l \\ a-l & a \end{vmatrix}$$

$$(a+l)[(a+l)^2 - a(a-l)] - (a-l)[a(a+l) - (a-l)^2] + a[a^2 - (a-l)(a+l)$$

$$(a+l)[a^2 + l^2 + 2al - a^2 + al] - (a-l)[a^2 + al - a^2 - l^2 + 2al] + a[a^2 - A^2 + l^2]$$

$$= (a+l)(l^2 + 3al) - (a-l)(3al - l^2)al^2$$

$$al^2+3a^2l+l^3+3al^2 - (3al-al^2-3al^2+l^3+al^2)$$
$$al^2+3a^2l+l^3+3al^2-3a^2l+al^2+3al^2-l^3+al^2 = 9al^2$$

v)

$$\begin{vmatrix}1 & 2 & -2\\ -1 & 1 & -3\\ 2 & 4 & -1\end{vmatrix}$$

Solution:

$$\begin{vmatrix}1 & 2 & -2\\ -1 & 1 & -3\\ 2 & 4 & -1\end{vmatrix} \quad expanding\ by\ R_1$$
$$=1\begin{vmatrix}1 & -3\\ 4 & -1\end{vmatrix}-2\begin{vmatrix}-1 & -3\\ 2 & -1\end{vmatrix}-2\begin{vmatrix}-1 & 1\\ 2 & 4\end{vmatrix}$$
$$=1(-1+12)-2(1+6)-2(-4-2)$$
$$1(11)-2(7)-2(-6)$$
$$=11-14+12$$
$$=9$$

(vi)

$$\begin{vmatrix}2a & a & a\\ b & 2b & b\\ c & c & 2c\end{vmatrix}$$

Solution:

$$\begin{vmatrix}2a & a & a\\ b & 2b & b\\ c & c & 2c\end{vmatrix} \quad expanding\ by\ R_1$$
$$=2a\begin{vmatrix}2b & b\\ c & 2c\end{vmatrix}-a\begin{vmatrix}b & b\\ c & 2c\end{vmatrix}+a\begin{vmatrix}b & 2b\\ c & c\end{vmatrix}$$
$$=2a(4bc-bc)-a(2bc-bc)+a(bc-2bc)$$
$$2a(3bc)-a(bc)+a(-2bc)$$
$$=6abc-abc-2abc=4abc$$

Q2.
Without expansion show that
(i)

$$\begin{vmatrix}6 & 7 & 8\\ 3 & 4 & 5\\ 2 & 3 & 4\end{vmatrix}=0$$

Solution:

$$L.H.S=\begin{vmatrix}6 & 7 & 8\\ 3 & 4 & 5\\ 2 & 3 & 4\end{vmatrix}$$
$$\begin{vmatrix}6 & 7-6 & 8-7\\ 3 & 4-3 & 5-4\\ 2 & 3-2 & 4-3\end{vmatrix} \quad C_2-C_1, C_3-C_2$$
$$\begin{vmatrix}6 & 1 & 1\\ 3 & 1 & 1\\ 2 & 1 & 1\end{vmatrix} \quad C_2\ C_3 are\ identical$$
$$=\mathbf{0}=\boldsymbol{R.H.S}$$

ii)

$$\begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{vmatrix}=0$$

Solution:

$$L.H.S=\begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{vmatrix}$$
$$\begin{vmatrix}2 & 3 & -1+3\\ 1 & 1 & 0+1\\ 2 & -3 & 5-3\end{vmatrix} \quad C_3+C_2$$
$$\begin{vmatrix}2 & 3 & 2\\ 1 & 1 & 1\\ 2 & -3 & 2\end{vmatrix} \quad C_1\ and\ C_3 are\ identical$$
$$=\mathbf{0}=\boldsymbol{R.H.S}$$ **Hence proved.**

iii)

$$\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=0$$

Solution:

$$L.H.S=\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}$$
$$\begin{vmatrix}1 & 1 & 1\\ 4 & 1 & 1\\ 7 & 1 & 1\end{vmatrix} \quad C_2-C_1, C_3-C_2$$
$$\begin{vmatrix}2 & 3 & 2\\ 1 & 1 & 1\\ 2 & -3 & 2\end{vmatrix} \quad C_2\ and\ C_3 are\ identical$$
$$=\mathbf{0}=\boldsymbol{R.H.S}$$ **Hence proved.**

Q.3
Show That
i)

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}+\alpha_{13}\\ a_{21} & a_{22} & a_{23}+\alpha_{23}\\ a_{31} & a_{32} & a_{33}+\alpha_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix}a_{11} & a_{12} & \alpha_{13}\\ a_{21} & a_{22} & \alpha_{23}\\ a_{31} & a_{32} & \alpha_{33}\end{vmatrix}$$

Solution:

$$L.H.S=\begin{vmatrix}a_{11} & a_{12} & a_{13}+\alpha_{13}\\ a_{21} & a_{22} & a_{23}+\alpha_{23}\\ a_{31} & a_{32} & a_{33}+\alpha_{33}\end{vmatrix}$$
$$Opening\ from\ C_3$$
$$=(a_{13}+\alpha_{13})\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}-(a_{23}+\alpha_{23})\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}+(a_{23}+\alpha_{33})\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$
$$a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}-a_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}+a_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$
$$+\alpha_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}-\alpha_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}+\alpha_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$
$$=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix}a_{11} & a_{12} & \alpha_{13}\\ a_{21} & a_{22} & \alpha_{23}\\ a_{31} & a_{32} & \alpha_{33}\end{vmatrix}$$
$$=R.H.S\ hence\ proved$$

ii)

$$\begin{vmatrix}2 & 3 & 0\\ 3 & 9 & 6\\ 2 & 15 & 1\end{vmatrix}=9\begin{vmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ 2 & 5 & 1\end{vmatrix}$$

Solution:

$$L.H.S=\begin{vmatrix}2 & 3 & 0\\ 3 & 9 & 6\\ 2 & 15 & 1\end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1 \end{vmatrix} \; take\ 3\ common\ from\ R_2$$

$$= 3.3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix} \; take\ 3\ common\ from\ C_2$$

$$= 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix} = R.H.S$$

Hence proved.

(iii)

$$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$$

Solution:

$$L.H.S = \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$= \begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \; R_1 + (R_2 + R_3)$$

$$= 3a+l\begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \; take\ common\ (3a + l) from R_1$$

$$= 3a+l\begin{vmatrix} 1 & 1 & 1 \\ a & l & 0 \\ a & 0 & l \end{vmatrix} \; C_2 - C_1 \;\; and\ C_3 - C_1$$

$$(3a+l)\left[1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0 + 0\right] \; Expanding\ By\ R_1$$

$$= l^2(3a+l) = R.H.S \quad hence\ proved.$$

(iv)

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Solution:

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yzx & zxy & xyz \end{vmatrix} \; xC_1, yC_2, zC_3$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \; take\ common\ xyz\ from\ R_3$$

$$= -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix} \; intercahging\ R_2\ and\ R_3$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \; intercahging\ R_2\ and\ R_1$$

$$R.H.S \quad hence\ proved.$$

(v)

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Solution:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expand by R_1

$$(b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$(b+c)[(c+a)(a+b) - bc] - a[b(a+b) - bc] + a[bc - cc + a)]$$

$$(b+c)[ac + bc + a^2 + ab - bc] - a[b(a+b) - bc] + a[bc - c^2 - ac]$$

$$abc + a^2b + ab^2 + ac^2 + a^2c + abc - a^2b - ab^2 + abc + abc - ac^2 - a^2c$$

$$= 4abc = R.H.S$$

$$\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$$

Solution :

$$R.H.S = \begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix}$$

Expand by R_1

$$b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} + 1\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$$

$$= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b)$$

$$= b^3 + ab + a^3 - ab$$

$$a^3 + b^3 = R.H.S$$

Hence proved.

vii)

$$\begin{vmatrix} rcos\emptyset & 1 & -sin\emptyset \\ 0 & 1 & 0 \\ rsin\emptyset & 0 & cos\emptyset \end{vmatrix} = r$$

Solution:

$$L.H.S = \begin{vmatrix} rcos\emptyset & 1 & -sin\emptyset \\ 0 & 1 & 0 \\ rsin\emptyset & 0 & cos\emptyset \end{vmatrix}$$

$$Expand\ by\ R_2$$

$$= -0 + 1\begin{vmatrix} cos\emptyset & -rsin\emptyset \\ rsin\emptyset & cos\emptyset \end{vmatrix}$$

$$= rcos^2\emptyset + rsin^2\emptyset = r(1)$$

$$= r = R.H.S$$

Hence proved.

viii)

$$\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^2 - 3abc$$

Solution:

$$= L.H.S\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$$

$$\begin{vmatrix} a+b+c+\lambda & b & c \\ a+b+c+\lambda & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \; C_1 + (C_2 + C_3)$$

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix} \; taking\ (a+b+c+\lambda)$$

Common from C_1

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \; R_2 - R_1, R_3 - R_1$$

Expanding by C_1

$$(a+b+c+\lambda)\left\{1\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0\right\}$$

$$(a+b+c+\lambda)(\lambda^2 - 0)$$

$\lambda^2(a+b+c+\lambda)$ *hence proved.*

(x)

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution:

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} C_2 - C_1 \;, C_3 - C_1$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

Taking common $(b-a)$ from C_2 $\; and \; (c-a) from \; C_3$

$$(b-a)(c-a)\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ a^2 & (b+a) & (c+a) \end{vmatrix} \; Expanding \; by \; R_1$$

$$= (b-a)(c-a)\left\{1\begin{vmatrix} 1 & o \\ b+a & c+a \end{vmatrix} - 0 + 0\right\}$$

$$(b-a)(c-a)\{(c+a-(b+a)\}$$

$$(b-a)(c-a)(c+a-b-a)$$

$$(b-a)(c-a)(c-b)$$

$$[-(a-b)](c-a)[-(b-c)]$$

$$(a-b)(b-c)(c-a) = R.H.S$$

Hence Proved.

(xi)

$$\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$

Solution:

$$= \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a & a^2 \\ c+b+a & b & b^2 \\ a+c+b & c & c^2 \end{vmatrix} C_1 + C_2$$

Taking $(a+b+c)$ *common from* C_1

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} R_2 - R_1 and \; R_3 - R_1,$$

Expanding By C_1

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a+b+c)\left\{1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0\right\}$$

$$= (a+b+c)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$$

take common $(b-a)$ *from* R_1 *and* $(c-a)$ *from* R_2

$$(a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$(a+b+c)(b-a)(c-a)\{c+a-b-a\}$$

$$= (a+b+c)(b-a)(c-a)(c-b)$$

$$= (a+b+c)[-(a-b)](c-a)[-(b-c)]$$

$$(a+b+c)(a-b)(b-c)(c-a)$$

$$= R.H.S \; hence \; proved.$$

Q.4

$$if \; A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix} and \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$$

Then find (i) A_{12}, A_{22}, A_{32} *and* $|A|$

ii) B_{21}, B_{22}, B_{23} *and* $|B|$

Solution:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1(-2+0) - 2(0+0) - 3(0-4)$$

$$|A| = -2 - 0 + 12 = 10$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} = -(0+0) = 0$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = (1-6) = -5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)(0-0) = 0$$

ii)

$$B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{vmatrix}$$

$$|B| = 5\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ -2 & -2 \end{vmatrix} + (5)\begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= 5(2-4) + 2(-6+8) + 5(3-2)$$

$$= -10 + 4 + 5 = -1$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix} = -(4-5) = 1$$

$$B_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix} = (-10+10) = 0$$

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = -(5-4) = -1$$

Q.5

Without expansion verify

$$i)\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$$

Solution:

$$= L.H.S\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$$

$$\begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \gamma+\alpha+\beta & \gamma+\alpha & 1 \\ \alpha+\beta+\gamma & \alpha+\beta & 1 \end{vmatrix} \quad C_1 + C_2$$

Taking $(\alpha+\beta+\gamma) common \; from \; C_1$

$$= \alpha+\beta+\gamma\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix}$$

$$\alpha+\beta+\gamma(0) = 0 \; (\; C_1 \; and \; C_2 \; are \; identicle$$

ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

solution:

$$L.H.S = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$$

$$Take\ 3x\ common\ from C_3$$

$$= 3x \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$

$$= 3x(0) = 0 = R.H.S$$

$$hence\ proved.$$

iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

Solution:

$$\text{L.H.S=} \begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 1 & a^2 & \frac{a(abc)}{bc} \\ 1 & b^2 & \frac{b(abc)}{ca} \\ 1 & c^2 & \frac{c(abc)}{ab} \end{vmatrix}$$

$$\times R_2\ by\ abc\ and\ \div outside$$

$$\frac{1}{abc} \begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix} = \frac{1}{abc}(0) = 0 = R.H.S$$

Hence proved.

(iv)

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Solution:

$$L.H.S = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$C_1 + (C_2 + C_3)$$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-a & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 = R.H.S$$

Hence proved C_1 $is\ Zero.$

(v)

$$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

Solution:

$$\text{L.H.S=} \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

$$\frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ a & b & c \end{vmatrix} \times R_2 by\ abc\ and \div outside$$

$$\frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ ab & ca & ab \\ a & b & c \end{vmatrix} R_1 and\ R_2\ are\ identicle.$$

$$\frac{1}{abc}(0) = 0 + R.H.S$$

(vi)

$$\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

Solution:

$$\text{L.H.S=} \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$$

$$= \frac{1}{lmn} \begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix} lR_1, mR_2, nR_3$$

$$= \frac{lmn}{lmn} \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} taking\ lmn\ common\ from C_1$$

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} = R.H.S$$

Hence proved.

(vii)

$$\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$$

Solution:

$$L.H.S = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$2 \begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} take\ 2\ common\ from\ R_1$$

$$= 2 \begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix} R_2 - R_1, R_3 - R_1$$

$$= 2bc \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} take\ common\ b\ from R_2\ and\ c\ from R_3$$

$$= 2bc(0) = R.H.S\ (R_2\ and\ R_3\ are\ identicle)$$

Hence proved.

(viii)

$$\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

Solution:

$$R.H.S \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix} Add\ C_3 of\ both$$

$$= \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = L.H.S \quad Hence\ proved.$$

ix)

$$\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$$

Solution:

$$L.H.S = \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$$

$$\frac{1}{abc}\begin{vmatrix} -ab & 0 & cb \\ 0 & ac & -bc \\ ba & -ca & 0 \end{vmatrix} bR_1, cR_2, aR_3$$

$$\frac{1}{abc}\begin{vmatrix} -ab+ab & ac-ac & cb-bc \\ 0 & ac & -bc \\ ba & -ca & 0 \end{vmatrix} R_1 + (R_2 + R_3)$$

$$\frac{1}{abc}\begin{vmatrix} 0 & 0 & 0 \\ 0 & ac & -bc \\ ba & -ca & 0 \end{vmatrix} = \frac{1}{abc}(0) = 0\ (R_1 is\ zero)$$

$$= R.H.S. \quad hence\ proved.$$

Q.6

Find the valves of $x\ if$

i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

Solution:

$$\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$$

⇨ $3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$

⇨ $3(0\text{-}4) - 1(0 - 4x) + x(-1 - 3x) = -30$

⇨ $-12 + 4x - x - 3x^2 = -30$

⇨ $-3x^2 + 3x + 18 = 0$

⇨ $x^2 - x - 6 = 0$

⇨ $x^2 - 3x + 2x - 6 = 0$

⇨ $x(x-3) + 2(x-3) = 0$

⇨ $(x+2)(x-3) = 0$

⇨ $X = 3, x = -2$

ii)

$$\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$$

⇨ $1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$

$$(x^2 + x + 4) - x(x-1)(-x-4) + 3(2 - 2x - 2) = 0$$

$$x^2 + x + 4 - (-x^2 - 4x + x + 4) + 6 - 6x - 6 = 0$$

$$x^2 + x + 4 + x^2 + 4x - x - 4 - 6x = 0$$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0$$

$$2x = 0. x - 1 = 0$$

$$x = 0, x = 1$$

iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$$

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$$(x^2 - 12) - 2(2x - 6) + (12 - 3x) = 0$$

$$x^2 - 12 - 4x + 12 + 12 - 3x = 0$$

$$x^2 - 7x + 12 = 0$$

⇨ $x^2 - 4x - 3x + 12 = 0$

⇨ $x(x-4) - 3(x-4) = 0$

⇨ $(x-4)(x-3) = 0$

⇨ $x - 4 = 0 \qquad x - 3 = 0$

⇨ $x = 4\ \ and\ x = 3$

Q7. Evaluate the following determinants

$$\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix} R_1 - R_2$$

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 7 & -4 & -1 \\ 0 & 3 & -5 & 1 \\ 0 & 5 & -10 & -10 \end{vmatrix} R_2 - 2R_1, R_3 - R_1, R_4 - 4R_1$$

$$= 1\begin{vmatrix} 7 & -4 & -5 \\ 3 & -5 & 1 \\ 5 & -10 & -10 \end{vmatrix} - 0 + 0 - 0\ expand\ byC_1$$

$$7\begin{vmatrix} -5 & 1 \\ -10 & -10 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 1 \\ 5 & -10 \end{vmatrix} + (-5)\begin{vmatrix} 3 & -5 \\ 5 & -10 \end{vmatrix}$$

$$= 7(50 + 10) + 4(-30 - 5) - 5(-30 + 25)$$

$$= 420 - 140 + 25 = 305$$

ii)

$$\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix} R_2 + R_1, R_3 + 6R_1, R_4 - 2R_1$$

$$(-1)\begin{vmatrix} 6 & 3 & 3 \\ 17 & 20 & 5 \\ -1 & -13 & 0 \end{vmatrix} + 0 - 0 + 0\ Expand\ from\ C_4$$

$$= \begin{vmatrix} 6 & 3 & 3 \\ 17 & 20 & 5 \\ -1 & -13 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} 17 & 20 \\ -1 & -13 \end{vmatrix} - 5\begin{vmatrix} 6 & 3 \\ -1 & -13 \end{vmatrix} + 0 \;\; Expand\ by\ C_3$$

$$= 3(-221+20) - 5(-78+3)$$

$$= 3(-201) - 5(-75) = -603 + 375 = -228$$

iii)

$$\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 2 \\ -2 & 0 & 1 & -1 \end{vmatrix}$$

Solution:

Solution:

$$\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 2 \\ -2 & 0 & 1 & -1 \end{vmatrix} R_2 + R_1, R_3 + R_1, R_4 - R_1$$

$$= \begin{vmatrix} -3 & 12 & 3 \\ 6 & 16 & 2 \\ 1 & 9 & -2 \end{vmatrix} - 0 + 0 - 0 \ Expand\ from\ C_3$$

$$= 3\begin{vmatrix} 16 & 2 \\ -9 & -2 \end{vmatrix} - 12\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 6 & 6 \\ 1 & -7 \end{vmatrix}$$

$$= -3(-32+18) - 12(-12-2) + 3(-54-16)$$

$$= -3(-14) - 12(-14) + 3(-70)$$

$$= 42 + 168 - 210 = 0$$

Q8.

Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-3)^3$

Solution:

$$L.H.S = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+1+1+1 & 1 & 1 & 1 \\ 1+x+1+1 & x & 1 & 1 \\ 1+1+x+1 & 1 & x & 1 \\ 1+1+1+x & 1 & 1 & x \end{vmatrix} C_1 + (C_2 + C_3 + C_4)$$

$$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & x & 1 \\ x+3 & 1 & 1 & x \end{vmatrix}$$

$Take\ common\ (x+3) from\ C_1$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1, R_4 - R_1$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 1 \\ 0 & 1 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix}$$

$Expanding\ by\ C_1$

$$= (x+3)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} - 0 + 0 - 0$$

$$= (x+3)(x-1)^3\ by\ determinant\ property$$

$$= R.H.S$$

Q.9 $find\ |AA^t| and\ |A^tA|\ if$

i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}, A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

$$|AA^t| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix} = 196 - 25 = 171$$

And $A^tA = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$A^tA = \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

$$|A^tA| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 673 - 616 - 21 = 0$$

(ii) $AA^t = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$

$$AA^t = \begin{bmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 5+12 & 4+3 & 2+3 & 4+6 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & -7 & 5 & 13 \end{bmatrix}$$

$$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & -7 & 5 & -13 \end{vmatrix}$$

$$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 25 & 10 & 7 & 18 \end{vmatrix} \quad R_4 + R_3$$

$$= 0(R_1 and\ R_4\ are\ identicel)$$

And $A^tA = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

$$A^tA = \begin{bmatrix} 9+4+1+4 & 12+2+1+6 \\ 16+1+1+9 & 12+2+1+6 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 18 & 21 \\ 21 & 27 \end{bmatrix}$$

$$|A^tA| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix} = 486 - 441 = 45$$

Q.10 if A is a square matrix of order 3 then show that

$|KA| = k^3|A|$

Solution:

$$let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$KA = K \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$KA = \begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{bmatrix}$$

$$|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$$

$$taking\ common\ K\ from\ R_1, R_2, R_3$$

$$|KA| = K.K.K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= K^3|A| = R.H.S \quad hence\ proved.$$

Q.11

Find the valves of λ *if A and B are singular*.

$$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

Solution:

Given matrix is singular so,

$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$|A| = 4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix}$$

$$4(3-18) - \lambda(7-12) + 3(21-6) = 0$$

$$-60 + 5\lambda + 45 = 0$$

$$5\lambda - 15 = 0$$

⇨ $5\lambda = 15$

⇨ $\lambda = 3$

ii)

given matrix is singular so

$$|B| = \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

$R_3 - R_2, R_4 - 3R_2$

$$⇨ \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ -5 & 0 & -5 & 0 \\ -22 & \lambda-6 & -16 & 0 \end{vmatrix} = 0$$

Expand byC_4

$$0 + 1\begin{vmatrix} 5 & 1 & 2 \\ -5 & 0 & -5 \\ -22 & \lambda-6 & -16 \end{vmatrix} - 0 + 0 = 0$$

Taking -5 common from R_2

$$-5\begin{vmatrix} 5 & 1 & 2 \\ 1 & 0 & 1 \\ -22 & \lambda-6 & -16 \end{vmatrix} = 0$$

Expand by R_2

$$-5\left\{-1\begin{vmatrix} 1 & -3 \\ \lambda-6 & 6 \end{vmatrix} + 0 - 0\right\} = 0$$

$$5(6 + 3(\lambda - 6) = 0$$

$$5(6 + 3\lambda - 18) = 0$$

$$5(6 + 3\lambda - 18) = 0$$

$$5(3\lambda - 12) = 0$$

$$3\lambda - 12 = 0$$

$$3\lambda = 12$$

$$\lambda = \frac{12}{3} = 4$$

Q12. Which of the following matrices are singular and which of them are non-singular?

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

Solution:

Let $|A| = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0 + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 1(4+2)3(6-0)$$

$$|A| = 6 + 18 = 24 \neq 0$$

⇨ *A is non − singular*

ii) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

Solution:

Let B= $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

$$|B| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$2\begin{vmatrix}1&0\\-3&5\end{vmatrix}-3\begin{vmatrix}1&0\\2&5\end{vmatrix}+(-1)\begin{vmatrix}1&1\\-2&3\end{vmatrix}$$
$$=2(5-0)-3(5-0)-1(-3-2)$$
$$|B|=10-15+5=0$$

⇨ B is singular.

iii) $\begin{bmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{bmatrix}$

Solution:

$$let\ C=\begin{bmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{bmatrix}$$

⇨ $|C|=\begin{vmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{vmatrix}$

$$R_2-R_1,R_3-2R_1,R_4-3R_1$$

⇨ $=\begin{vmatrix}1&1&2&-1\\0&1&-3&-3\\0&1&-3&4\\0&-41&-3&7\end{vmatrix}$

$$=\begin{vmatrix}1&-3&-2\\1&-3&4\\-4&-3&7\end{vmatrix}-0+0-0\ expand\ by C_1$$

$$\begin{vmatrix}1&-3&-2\\0&0&4\\0&-15&7\end{vmatrix}R_2-R_1,R_3+4R_1$$

$$=1\begin{vmatrix}0&6\\-15&-1\end{vmatrix}-0+0\quad Expand\ by\ C_1$$
$$|C|=0+90=90\neq 0$$
$$C\ is\ not\ singular$$

Q.13. $find\ \ the\ inverse\ of\ A=\begin{bmatrix}2&1&0\\1&1&0\\2&-3&5\end{bmatrix}$ $and\ show$

That $A^{-1}A=I_3$

Solution:

$$|A|=\begin{vmatrix}2&1&0\\1&1&0\\2&-3&5\end{vmatrix}$$
$$=2\begin{vmatrix}1&0\\-3&5\end{vmatrix}-1\begin{vmatrix}1&0\\2&5\end{vmatrix}+0\begin{vmatrix}1&1\\2&-3\end{vmatrix}$$
$$2(5-0)-1(5-0)=10-5=5$$
$$A_{11}=(-1)^{1+1}\begin{vmatrix}1&0\\-3&5\end{vmatrix}=(5-0)=5$$
$$A_{12}=(-1)^{1+2}\begin{vmatrix}1&0\\2&5\end{vmatrix}=-(5-0)=-5$$
$$A_{13}=(-1)^{1+3}\begin{vmatrix}1&1\\2&-3\end{vmatrix}=(-3-2)=-5$$
$$A_{21}=(-1)^{2+1}\begin{vmatrix}1&0\\-3&5\end{vmatrix}=-(5-0)=-5$$
$$A_{22}=(-1)^{2+2}\begin{vmatrix}2&0\\2&5\end{vmatrix}=(10-0)=10$$
$$A_{23}=(-1)^{2+3}\begin{vmatrix}2&1\\2&-3\end{vmatrix}=-(-6-2)=8$$
$$A_{31}=(-1)^{3+1}\begin{vmatrix}1&0\\1&0\end{vmatrix}=(0-0)=0$$
$$A_{32}=(-1)^{3+2}\begin{vmatrix}2&0\\1&0\end{vmatrix}=(0-0)=0$$
$$A_{33}=(-1)^{3+3}\begin{vmatrix}2&1\\1&1\end{vmatrix}=(2-1)=1$$
$$cofector\ of\ A=\begin{bmatrix}-5&-5&0\\-5&10&0\\0&0&1\end{bmatrix}$$
$$AdjA=\begin{bmatrix}5&-5&0\\-5&10&0\\-5&8&1\end{bmatrix}$$
$$A^{-1}=\frac{adjA}{A}=\frac{1}{5}\begin{bmatrix}5&-5&0\\-5&10&0\\-5&8&1\end{bmatrix}$$

Now $A^{-1}A=\frac{1}{5}\begin{bmatrix}5&-5&0\\-5&10&0\\-5&8&1\end{bmatrix}\begin{bmatrix}2&1&0\\1&1&0\\2&-3&5\end{bmatrix}$

$$=\frac{1}{5}\begin{bmatrix}10-5+0&5-5+0&0-0+0\\-10+10+0&-5+10-0&0+0+0\\-10+8+2&-5+8-3&0+0+5\end{bmatrix}$$
$$=\frac{1}{5}\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$
$$\frac{5}{5}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3$$
$$hence\ A^{-1}A=I_3$$

Q.14

Verify that

$$(AB)^{-1}=B^{-1}A^{-1}\ if$$
$$A=\begin{bmatrix}1&2\\-1&0\end{bmatrix},B=\begin{bmatrix}-3&1\\4&1\end{bmatrix}$$

Solution:

$$AB=\begin{bmatrix}-3+8&1-2\\3+0&-1-0\end{bmatrix}=\begin{bmatrix}5&-1\\3&-1\end{bmatrix}$$
$$|AB|=\begin{vmatrix}5&-1\\3&-1\end{vmatrix}=-5+3=-2\neq 0$$
$$Adj(AB)=\begin{bmatrix}-1&1\\-3&5\end{bmatrix}$$

$L.H.S=(AB)^{-1}=\frac{adj(AB)}{|AB|}$

Now for B^{-1}

$$|B|=\begin{vmatrix}-3&1\\-4&-1\end{vmatrix}=3-4=-1\neq 0$$
$$adjB=\begin{bmatrix}-1&-1\\-4&-3\end{bmatrix}$$

⇨ $B^{-1}=\frac{adjB}{|B|}=\frac{\begin{bmatrix}-1&-1\\-4&-3\end{bmatrix}}{-1}$

⇨ $B^{-1}=\begin{bmatrix}1&1\\4&3\end{bmatrix}$

For A^{-1}, $|A|=\begin{vmatrix}1&2\\-1&0\end{vmatrix}=0+2=2\neq 0$

$$adjA=\begin{bmatrix}0&-2\\1&1\end{bmatrix}$$
$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{2}\begin{bmatrix}0&-2\\1&1\end{bmatrix}$$

R.H.S=$B^{-1}A^{-1}=\begin{bmatrix}1&1\\4&3\end{bmatrix}\frac{1}{2}\begin{bmatrix}0&-2\\1&1\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}1&1\\4&3\end{bmatrix}\begin{bmatrix}0&-2\\1&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0+1&-2+1\\0+3&-8+3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&-1\\3&-5\end{bmatrix}$$
$$=\begin{bmatrix}1\backslash 2&-1\backslash 2\\3\backslash 2&-5\backslash 2\end{bmatrix}=R.H.S$$
$$L.H.S=R.H.S\quad hence\ proved.$$

ii) $A=\begin{bmatrix}5&1\\2&2\end{bmatrix},B=\begin{bmatrix}4&3\\2&1\end{bmatrix}$

Solution:

$$AB=\begin{bmatrix}20+2&15+1\\8+4&6+2\end{bmatrix}=\begin{bmatrix}5&-1\\3&-1\end{bmatrix}$$
$$|AB|=\begin{vmatrix}22&16\\12&8\end{vmatrix}=176-192=-16\neq 0$$
$$Adj(AB)=\begin{bmatrix}8&16\\-12&22\end{bmatrix}$$

$$L.H.S=(AB)^{-1}=\frac{adj(AB)}{|AB|}$$
$$=\frac{1}{-16}\begin{bmatrix}8 & 16\\ -12 & 22\end{bmatrix}$$
$$=\begin{bmatrix}\frac{8}{-16} & -\frac{16}{-16}\\ -\frac{12}{-16} & \frac{22}{-16}\end{bmatrix}=\begin{bmatrix}-1\backslash 2 & 1\\ 3/4 & -11/8\end{bmatrix}$$

Now for B^{-1}

$$|B|=\begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix}=4-6=-2\neq 0$$
$$adjB=\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$$
$$⇨ \quad B^{-1}=\frac{adjB}{|B|}=\frac{\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}}{-2}$$
$$⇨ \quad B^{-1}=\frac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$$

For $A^{-1}, |A|=\begin{vmatrix}5 & 1\\ 2 & 2\end{vmatrix}=10-2=8\neq 0$

$$adjA=\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$$
$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$$

R.H.S=$B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}\frac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix}\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix}=\frac{1}{-16}\begin{bmatrix}2+6 & -1-15\\ -4-8 & 2+20\end{bmatrix}$$
$$=-\frac{1}{16}\begin{bmatrix}8 & -16\\ -12 & 22\end{bmatrix}=\begin{bmatrix}-\frac{8}{16} & \frac{-16}{-16}\\ -\frac{12}{-16} & \frac{22}{-16}\end{bmatrix}$$
$$=\begin{bmatrix}-1/2 & 1\\ 3/4 & -11/8\end{bmatrix}=R.H.S$$
$$L.H.S=R.H.S \quad hence\ proved.$$

Q15.

Verify that $(AB)^t=B^tA^t$

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$$

Solution:

$$L.H.S=(AB)^t$$
$$=\left(\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}\right)^t$$
$$\begin{bmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{bmatrix}^t=\begin{bmatrix}-2 & 9\\ 9 & 5\end{bmatrix}^t$$
$$=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}=L.H.S$$
$$R.H.S=B^tA^t$$
$$=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}^t\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}^t$$
$$=\begin{bmatrix}1 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$$
$$R.H.S=\begin{bmatrix}1-3+0 & 0+9+0\\ 1-2-2 & 0+6-1\end{bmatrix}=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$$
$$Hence\ L.H.S=R.H.S$$

Q.16

$$if\ A=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix} verify\ that\ (A^{-1})^t=(A^t)^{-1}$$

Solution:

$$A=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix}$$
$$|A|=\begin{vmatrix}2 & -1\\ 3 & 1\end{vmatrix}=2+3=5\neq 0$$
$$adjA=\begin{bmatrix}1 & 1\\ -3 & 2\end{bmatrix}$$
$$thus\ A^{-1}=\frac{adjA}{|A|}=\frac{1}{5}\begin{bmatrix}1 & 1\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1/5 & 1/5\\ -3/5 & 2/5\end{bmatrix}$$
$$(A^{-1})^t=\begin{bmatrix}1/5 & -3/5\\ 1/5 & 2/5\end{bmatrix}\to(1)$$

Now $A^t=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$

$$|A^t|=\begin{vmatrix}2 & 3\\ -1 & 1\end{vmatrix}=2+3=5$$
$$adj(A^t)=\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}$$
$$(A^t)^{-1}=\frac{1}{|A^t|}adjA^t$$
$$\frac{1}{5}\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & -\frac{3}{5}\\ \frac{1}{5} & \frac{2}{5}\end{bmatrix}\to(2)$$
$$by\ (1) and\ (2)\ L.H.S=R.H.S$$

Q17. If A and B are non-singular matrix then show that

$$(AB)^{-1}=B^{-1}A^{-1}$$

Solution:

$$(AB)^{-1}=B^{-1}A^{-1}$$

We know that

$$(AB)(AB)^{-1}=I$$

Pre multiplying by A^{-1}

$$A^{-1}(AB)(AB)^{-1}=A^{-1}I$$
$$(A^{-1}A)B(AB)^{-1}=A^{-1}(Associative\ law)$$
$$IB(AB)^{-1}=A^{-1}$$
$$B(AB)^{-1}=A^{-1}$$

Pre-multiplying by B^{-1}

$$B^{-1}.B(AB)^{-1}=B^{-1}A^{-1}$$
$$(B^{-1}B)(AB)^{-1}=B^{-1}A^{-1}$$
$$I(AB)^{-1}=B^{-1}A^{-1}$$

$(AB)^{-1}=B^{-1}A^{-1}$ Hence proved.

ii)

$$(A^{-1})^{-1}=A$$

We know that

$$I=AA^{-1}$$
$$post\ multiplying\ by\ (A^{-1})^{-1}$$
$$I\ (A^{-1})^{-1}=(AA^{-1})(A^{-1})^{-1}$$
$$(A^{-1})^{-1}=A[A^{-1}((A^{-1})^{-1})]$$
$$(A^{-1})^{-1}=AI$$
$$(A^{-1})^{-1}=A \quad hence\ proved$$

https://newsongoogle.com/

Elementary Row and Column Operations on a Matrix:

Row Operation:

The following three operations on a matrix are called elementary row operation.

i) Interchange of any two rows.
ii) Multiplication of a row by any non-zero number.
iii) Addition of any multiply of one row to another.

Column Operation:

The following three operations on a matrix are called are elementary column operation.

i) Interchange of any two columns.
ii) Multiplication of a column by a non-zero number.
iii) Addition of any multiply of one column to another column.

Upper Triangular Matrix:

A square matrix $A = [a_{ij}]$ is called upper triangular matrix if all elements below the main diagonal are zero.

$i.e\ a_{ij} = 0 \forall\ i > j$

$$i.e\ ; \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower Triangular Matrix:

A square matrix $A = [a_{ij}]$ is called lower triangular matrix if all elements above the main diagonal are zero.

$i.e\ a_{ij} = 0\ \forall\ i > j$

$$i.e\ ; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 6 \end{bmatrix}$$

Triangular Matrix:

A square matrix A is said to be triangular matrix if it is upper triangular or lower triangular while both upper and lower are called triangular matrix.

Note: diagonal matrices are both upper triangular and lower triangular.

Symmetric Matrix:

A square matrices $A = [a_{ij}]_{n\times m}$ is called symmetric matrix if

$A^t = A$ $e.g;$

$$if\ A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\Rightarrow\ A^t = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = A$$

Skew Symmetric Matrix:

A Square matrix $A = [a_{ij}]_{m\times n}$ is called skew – symmetric matrix.

If $A^t = A$

$e.g$

$$A = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$\Rightarrow\ A^t = -A$

Hermitian Matrix:

A square matrix A is said to be Hermitian matrix if $(\bar{A})^t = A$

$e.g;$

$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}, \bar{A} = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} = A$$

Skew –Hermitian Matrix:

A square matrix **A is said to be skew –Hermitian matrix**

If $(\bar{A})^t = -A$

$e.g;$

$$A = \begin{bmatrix} 0 & -2+3i \\ -2+3i & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} 0 & -2+3i \\ 2+3i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2-3i \\ -2-3i & 0 \end{bmatrix}$$

$$(\bar{A})^t = -A$$

Zero Row:

If all entries of a row are zero then this row is called zero row, otherwise row non-zero row.

$e.g;$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\ R_2 is\ zero\ row\ R_1 and\ R_3 are\ non\ zero\ rows$$

Leading entry:

In any non-zero row, the first non-zero element is the leading entry of that row;

$$\begin{bmatrix} 1 & 0 & 5 & -4 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 8 & 7 \end{bmatrix}$$

In

$R_1 leading\ entry\ is\ 1. in\ R_2\ leading\ entry\ entry\ is\ 2. in\ R_3 Leadind\ entry\ is\ 8.$

Leading Zeros:

The zeros before the leading entry of a row are called leading zeros

$$e.g;\ \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 8 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$in\ R_1\ there\ is\ no\ leading\ zero. In\ R_2 only\ one\ zero\ leading\ zero. in R_3 two\ zeros\ are\ leading\ zeros. in\ R_4 only\ one\ zero$ Is leading zero.

Echelon Form of a Matrix:

A matrix is said to be Echelon form if

1) 1 is the leading entry of each non-zero row.
2) In each row, the number of leading zeros is greater than the preceding row.

e.g;

$$A=\begin{bmatrix}0&1&-2&4\\0&0&1&2\\0&0&0&0\end{bmatrix},B=\begin{bmatrix}1&2&-3&4\\0&0&1&2\\0&0&0&1\end{bmatrix}$$

A and B are in Echelon form

$$C=\begin{bmatrix}0&0&1&2\\0&1&3&-1\\0&0&0&0\end{bmatrix},D=\begin{bmatrix}0&1&-2\\0&0&-1\\0&0&4\end{bmatrix}$$

C is not in Echelon form

$\because$ *leading zero in* $R_1 >$ *leading zero in* R_2.

D is not Echelon form $\because$ *leading entry in* R_2 *and* R_3 *is not* 1.

Reduced Echelon form of Matrix:

A matrix is said to be in Reduce Echelon form if

i) It is in Echelon form.

ii) In the column of leading entry all elements above and below leading entry (1) must be zero.

$$\begin{bmatrix}0&1&0&4\\0&0&1&2\\0&0&0&0\end{bmatrix},B=\begin{bmatrix}1&2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

A and B are in Reduce Echelon form

$$C=\begin{bmatrix}1&-4&-7\\0&1&5\\0&0&0\\0&0&0\end{bmatrix},D=\begin{bmatrix}1&-1&2&-3\\0&0&4&7\\0&0&4&0\end{bmatrix}$$

C and D are in Echelon form

but not in Reduce echelon form.

Rank of Matrix:

The number of non-zero rows in echelon form or reduce Echelon form of a matrix is called rank of a matrix.

A^{-1} *by Row operation*

- For a non-singular matrix A

If $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ *we find* A^{-1} *using row*

Operation as Make inC_1

First $a_{11}=1$ *then* $a_{21}=0$ $a_{31}=0$ *make in* C_2

first $a_{22}=1$ *then* a_{12} *and* $a_{22}=0$

make in C_3

first $a_{33}=1$ *then* $a_{13}=0$ *and* $a_{23}=0$

A^{-1} *by column operation*

- **For a** non-singular matrix A if

$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ then we find A^{-1}

using column operation as

make in R_1

first $a_{11}=1$ *then* $a_{12}=0$ *and* $a_{13}=0$

make in R_2

first $a_{22}=1$ *then* $a_{21}=0$ *and* $a_{23}=0$

make in R_3

first $a_{33}=1$ *then* $a_{31}=0$ *and* $a_{32}=0$

Note:

If we reduce it into reduce echelon form then no change occurs in the rank of matrix as

$$\sim R\begin{bmatrix}1&-1+1&2+3/2&-3-1/2\\0&1&3/2&-1/2\\0&0&0&0\end{bmatrix}R_1+R_2$$

$$\sim R\begin{bmatrix}1&0&\frac{7}{2}&-\frac{7}{2}\\0&1&3/2&-1/2\\0&0&0&0\end{bmatrix}$$

Which is in reduce Echelon form No. of non-Zero rows=2

So Rank=2

Exercise 3.4

Q1.

if $A=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}$ *and* $B=\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$

Then A+B is symmetric.

Solution:

$$A+B=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}+\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$$

$$A+B=\begin{bmatrix}1-3&-2+1&5-2\\-2+1&3+1&-1-1\\5-2&-1-1&0+2\end{bmatrix}$$

$$A+B=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}$$

$$(A+B)^t=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}^t=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}$$

$=A+B$ *hence* $(A+B)$ *is symmetric*

Q2. *if* $A=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}$ *show that*

$(i)A+A^t$ *is symmetric.*

$(ii)A-A^t$ *is skew symmetric.*

Solution:

$(i)A+A^t$ *is symmetric.*

$$A+A^t=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}+\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}^t$$

$$A+A^t=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}+\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

$$A+A^t=\begin{bmatrix}1+1&2+3&0-1\\3+2&2+2&-1+3\\-1+0&3-1&2+2\end{bmatrix}$$

$$A+A^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}$$

$$\Rightarrow\ (A+A^t)^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}^t$$

⇨ $(A+A^t)^t = \begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix} = A+A^t$

hence $A+A^t$ is symmtric

(ii) $A-A^t$ is skew symmetric

$$A-A^t = \begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix} - \begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}^t$$

$$A+A^t = \begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix} - \begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

$$A+A^t = \begin{bmatrix}1-1&2-3&0+1\\3-2&2-2&-1-3\\-1-0&3+1&2-2\end{bmatrix}$$

$$A-A^t = \begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}$$

⇨ $(A-A^t)^t = \begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}^t$

⇨ $(A+A^t)^t = \begin{bmatrix}0&1&-1\\-1&0&4\\1&-4&0\end{bmatrix}$

⇨ $(A+A^t)^t = -\begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}$

⇨ $= -(A-A^t)$

hence $A-A^t$ is skew symmtric

Q3. *if $0 > A$ is any square matrix of order* 3, *,show that (i) $A+A^t$ is symmetric (ii) $A-A^t$ is skew symmetric*

Solution:

(i) $A+A^t$ is symmetric

Let $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

$$A+A^t = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix} + \begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

$$A+A^t = \begin{bmatrix}a_{11}+a_{11}&a_{12}+a_{21}&a_{13}+a_{31}\\a_{21}+a_{12}&a_{22}+a_{22}&a_{23}+a_{32}\\a_{31}+a_{13}&a_{32}+a_{23}&a_{33}+a_{33}\end{bmatrix}$$

$$(A+A^t)^t = \begin{bmatrix}a_{11}+a_{11}&a_{12}+a_{21}&a_{13}+a_{31}\\a_{21}+a_{12}&a_{22}+a_{22}&a_{23}+a_{32}\\a_{31}+a_{13}&a_{32}+a_{23}&a_{33}+a_{33}\end{bmatrix}^t$$

$$(A+A^t)^t = \begin{bmatrix}a_{11}+a_{11}&a_{21}+a_{12}&a_{31}+a_{13}\\a_{12}+a_{21}&a_{22}+a_{22}&a_{32}+a_{23}\\a_{13}+a_{31}&a_{23}+a_{32}&a_{33}+a_{33}\end{bmatrix}$$

$$= A+A^t$$

(i) $A-A^t$ is skew symmetric

Let $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

$$A-A^t = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix} - \begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

$$A-A^t = \begin{bmatrix}a_{11}-a_{11}&a_{12}-a_{21}&a_{13}-a_{31}\\a_{21}-a_{12}&a_{22}-a_{22}&a_{23}-a_{32}\\a_{31}-a_{13}&a_{32}-a_{23}&a_{33}-a_{33}\end{bmatrix}$$

$$(A-A^t)^t = \begin{bmatrix}0&a_{12}-a_{21}&a_{13}-a_{31}\\a_{21}-a_{12}&0&a_{23}-a_{32}\\a_{31}-a_{13}&a_{32}-a_{23}&0\end{bmatrix}^t$$

$$(A-A^t)^t = \begin{bmatrix}0&a_{21}-a_{12}&a_{31}-a_{13}\\a_{12}-a_{21}&0&a_{32}-a_{23}\\a_{13}-a_{31}&a_{23}-a_{32}&0\end{bmatrix}$$

$$= -\begin{bmatrix}0&-a_{21}+a_{12}&-a_{31}+a_{13}\\-a_{12}+a_{21}&0&-a_{32}+a_{23}\\-a_{13}+a_{31}&-a_{23}+a_{32}&0\end{bmatrix}$$

$$= -(A-A^t)$$

hence $A-A^t$ is skew sysmmetric

Q4. *if the matrix A and B are symmetric and $AB = BA$, show that AB is symmetric.*

Solution:

given $A^t = A, B^t = B, AB = BA$

Now $(AB)^t = B^tA^t$

$= BA \quad \because B^t = B, A^t = A$

⇨ $(AB)^t$ =AB $\quad \because BA = AB$

⇨ Hence AB is symmetric.

Q5. Show that

AA^t and A^tA are symmetric for any matrix of order 2×3

Solution:

let $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}_{2\times 3}$

$$A^t = \begin{bmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{bmatrix}$$

$$AA^t = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}\begin{bmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{bmatrix}$$

$$AA^t = \begin{bmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}\\a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2\end{bmatrix}$$

$$(AA^t)^t = \begin{bmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}\\a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2\end{bmatrix}$$

$$= AA^t$$

hence AA^t is symmetric.

Also

$$A^tA = \begin{bmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}^2+a_{12}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{33}\\a_{12}a_{11}+a_{22}a_{21} & a_{12}^2+a_{22}^2 & a_{12}a_{23}+a_{32}a_{23}\\a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{23} & a_{13}^2+a_{23}^2\end{bmatrix}$$

⇨ $(A^tA)^t$

$$\begin{bmatrix}a_{11}^2+a_{12}^2 & a_{12}a_{11}+a_{22}a_{21} & a_{13}a_{11}+a_{23}a_{21}\\a_{11}a_{12}+a_{21}a_{22} & a_{12}^2+a_{22}^2 & a_{13}a_{12}+a_{23}a_{23}\\a_{11}a_{13}+a_{21}a_{33} & a_{12}a_{23}+a_{32}a_{23} & a_{13}^2+a_{23}^2\end{bmatrix}$$

$$= A^tA$$

⇨ *A^tA is symmetric.*

Q6. *if* $A = \begin{bmatrix}i&1+i\\1&-i\end{bmatrix}$ *show that*

i) $A+(\bar{A})^t$ is hermition.

ii) $A-(\bar{A})^t$ is skew − hermition

Solution:

i) $A+(\bar{A})^t$ is hermition.

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}, \bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2-i \\ 2 & 0 \end{bmatrix}$$

$$\overline{A + (\bar{A})^t} = \begin{bmatrix} 0 & 2 \\ 2-i & 0 \end{bmatrix} = A + (\bar{A})^t$$

So $A + (\bar{A})^t$ is Hermitian.

$ii)\ A - (\bar{A})^t$ *is skew hermition.*

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}, \bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A - (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A - (\bar{A})^t = \begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\overline{A - (\bar{A})^t} = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$-(A - (\bar{A})^t)$$

So $A - (\bar{A})^t$is skew Hermitian.

Q7. ***if A is symmetric or skew symmetric show that A^2 is symmetric***

Solution:

Give that $A^t = A$ Or $A^t = -A$

Now $(A^2)^t = (A.A)^t$

also $(A^2)^t = (A.A)^t$

$$A^t.A^t \quad or = A^t.A^t$$

$$= A.A \qquad \because (A^t = A)$$

⇨ A^2 *is symmetric*

or $(-A)(-A) \qquad \because (A^t = -A)$

⇨ A^2 *is skew symmetric*

Q8. $\boldsymbol{if A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}}$ $\boldsymbol{find\ A(\bar{A})^t}$

$$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}, \bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

$$(\bar{A})^t = [1 \quad 1-i \quad -i]$$

$$A(\bar{A})^t = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix} [1 \quad 1-i \quad -i]$$

$$A(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -(-1) \end{bmatrix}$$

$$A(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-(-1) & -i-(-1) \\ i & i-(-1) & -(-1) \end{bmatrix}$$

$$A(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & 1+i & 1 \end{bmatrix}$$

Q9. Find the inverse of the following matrices. Also find their inverse by row and column operations:

i) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{-2} & \mathbf{2} \end{bmatrix}$

Solution:

$$let\ A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix}$$

expand by R_2

$$= 0 = (-2)\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} - 0$$

$$|A| = -2(2-6) = -2(-4) = 8$$

$$|A| \neq 0\ so\ A^{-1}\ exist.$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = (-4+0) = -4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = -(0+0) = 0$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} = 0-4 = -4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = -(4-6) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = 2-6 = -4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = -(-2+4) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = 0-6 = -6$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = -(0+0) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2-0 = -2$$

$$adjA = \begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$$

$$\because A^{-1} = \frac{1}{|A|}\ adjA = \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{4}{8} & \frac{2}{8} & -\frac{6}{8} \\ 0 & -\frac{4}{8} & 0 \\ -\frac{4}{8} & -\frac{2}{8} & -\frac{2}{8} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

BY Row operation;

$$\because A = AI$$

$$⇨\ A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2+2 & -2+4 & 2-6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0+2 & 0 & 1 \end{bmatrix} R_3 + 2R_1$$

$$\sim R \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\sim R\begin{bmatrix}1&2&-3\\0&1&0\\0&2&-4\end{bmatrix}\begin{bmatrix}1&0&0\\0&-\frac{1}{2}&0\\2&0&1\end{bmatrix}-\frac{1}{2}R_2$$

$$\sim R\begin{bmatrix}1&2-2&-3\\0&1&0\\0&2-2&-4\end{bmatrix}\begin{bmatrix}1&0+1&0\\0&-\frac{1}{2}&0\\2&0+1&1\end{bmatrix}R_1-2R_2, R_3-2R_2$$

$$\sim R\begin{bmatrix}1&0&-3\\0&1&0\\0&0&-4\end{bmatrix}\begin{bmatrix}1&1&0\\0&-\frac{1}{2}&0\\2&1&1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&0&-3\\0&1&0\\0&0&-4\end{bmatrix}\begin{bmatrix}1&1&0\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}-\frac{1}{4}R_3$$

$$\sim R\begin{bmatrix}1&0&-3+3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1-\frac{3}{2}&1-\frac{3}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}R_1+3R_3$$

Type equation here.

$$\sim R\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

$$\Rightarrow\ A^{-1}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

By column operation:

$\because A = AI$

$$A=\begin{bmatrix}1&2&-3\\0&-2&0\\-2&-2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\sim C\begin{bmatrix}1&2-2&-3+3\\0&-2&0\\-2&-2+4&2-6\end{bmatrix}\begin{bmatrix}1&0-2&0+3\\0&1&0\\0&0&1\end{bmatrix}C_2-2C_1, C_3+3C_1$$

$$\sim C\begin{bmatrix}1&0&0\\0&-2&0\\-2&2&-4\end{bmatrix}\begin{bmatrix}1&-2&3\\0&1&0\\0&0&1\end{bmatrix}$$

$$\sim C\begin{bmatrix}1&0&0\\0&1&0\\-2&-1&-4\end{bmatrix}\begin{bmatrix}1&1&3\\0&-\frac{1}{2}&0\\0&0&1\end{bmatrix}-\frac{1}{2}C_2$$

$$\sim C\begin{bmatrix}1&0&0\\0&1&0\\-2&-1&1\end{bmatrix}\begin{bmatrix}1&1&-3/4\\0&-\frac{1}{2}&0\\0&0&-1/4\end{bmatrix}-\frac{1}{4}C_3$$

$$\sim C\begin{bmatrix}1&0&0\\0&1&0\\-2+2&-1+1&1\end{bmatrix}\begin{bmatrix}1-\frac{3}{2}&1-\frac{3}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\0-\frac{1}{2}&0-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}C_1+2C_3, C_2+C_3$$

$$\sim C\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}&-\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

$$\Rightarrow\ A^{-1}=\begin{bmatrix}-\frac{1}{2}&-\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

ii) $let\ A=\begin{bmatrix}1&2&-1\\0&-1&3\\1&0&2\end{bmatrix}$

$$|A|=\begin{vmatrix}1&2&-1\\0&-1&3\\1&0&2\end{vmatrix}$$

$expand\ by\ R_3-R_1$

$$\begin{vmatrix}1&2&-1\\0&-1&3\\0&2&3\end{vmatrix}$$

$$=(1)\begin{vmatrix}-1&3\\-2&3\end{vmatrix}-0+0\ \ Expand\ by\ C_1$$

$$|A|=(-3+6)=3\neq 0$$

$$|A|\neq 0\ so\ A^{-1}\ exist.$$

By Adjoins Method

$$A_{11}=(-1)^{1+1}\begin{vmatrix}-1&3\\0&2\end{vmatrix}=(-2-0)=-2$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}0&3\\1&2\end{vmatrix}=-(0-3)=3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}0&-1\\1&0\end{vmatrix}=(0+1)=1$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2&-1\\0&2\end{vmatrix}=-(4+0)=-4$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1&-1\\1&2\end{vmatrix}=(2+1)=3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&2\\1&0\end{vmatrix}=-(0-2)=2$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2&-1\\-1&3\end{vmatrix}=(6-1)=5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1&-1\\0&3\end{vmatrix}=-(3+0)=-3$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1&2\\0&-1\end{vmatrix}=-1-0=-1$$

$$adjA=\begin{bmatrix}-2&-4&5\\3&3&-3\\1&2&-1\end{bmatrix}$$

$$adjA=\begin{bmatrix}-2&-4&5\\3&3&-3\\1&2&-1\end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}adjA=\frac{1}{3}\begin{bmatrix}-2&-4&5\\3&3&-3\\1&2&-1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\\1&1&-1\\1/3&2/3&-1/3\end{bmatrix}$$

By Row operation:

$\because A = AI$

$$\Rightarrow A = \begin{bmatrix}1 & 2 & -1\\0 & -1 & 3\\1 & `0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1\\0 & -1 & 3\\1-1 & `0-2 & 2+1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0-1 & 0 & 1\end{bmatrix} R_3 - R_1$$

$$\sim R\begin{bmatrix}1 & 2 & -1\\0 & -1 & 3\\0 & `-2 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1\\0 & 1 & -3\\0 & `-2 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\-1 & 0 & 1\end{bmatrix}(-1)R_2$$

$$\sim R\begin{bmatrix}1 & 2-2 & -1+6\\0 & 1 & -3\\0 & `-2+2 & 3-6\end{bmatrix}\begin{bmatrix}1 & 0+2 & 0\\0 & -1 & 0\\-1 & 0-2 & 1\end{bmatrix} R_1 - 2R_2, R_3 + 2R_2$$

$$\sim R\begin{bmatrix}1 & 0 & 5\\0 & 1 & -3\\0 & `0 & -3\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\0 & -1 & 0\\-1 & -2 & 1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 0 & 5\\0 & 1 & -3\\0 & `0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\0 & -1 & 0\\\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix} -\frac{1}{3}R_3$$

$$\sim R\begin{bmatrix}1 & 0 & 5-5\\0 & 1 & -3+3\\0 & `0 & 1\end{bmatrix}\begin{bmatrix}1-\frac{5}{3} & 2-\frac{10}{3} & 0+\frac{5}{3}\\0+1 & -1+2 & 0\\\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix} R_1 - 5R_3, R_2 + 3R_3$$

$$\sim R\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & `0 & 1\end{bmatrix}\begin{bmatrix}-2/3 & -4/3 & 5/3\\1 & 1 & -1\\\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}-2/3 & -4/3 & 5/3\\1 & 1 & -1\\\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

By column operation:

$$\because A = AI$$

$$\Rightarrow A = \begin{bmatrix}1 & 2 & -1\\0 & -1 & 3\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim C\begin{bmatrix}1 & 2-2 & -1+1\\0 & -1 & 3\\1 & 0-2 & 2+1\end{bmatrix}\begin{bmatrix}1 & 0-2 & 0+1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} C_2 - 2C_1, C_3 + C_1$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & -1 & 3\\1 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & 1 & 3-3\\1 & 2 & 3-6\end{bmatrix}\begin{bmatrix}1 & -2 & 1-6\\0 & 1 & 0+3\\0 & 0 & 1\end{bmatrix} C_3 - 3C_2$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 2 & -3\end{bmatrix}\begin{bmatrix}1 & -2 & -5\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 5/3\\0 & 1 & -1\\0 & 0 & -1/3\end{bmatrix} -\frac{1}{3}C_3$$

$$C_1 - C_3, C_2 - 2C_3$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1-1 & 2-2 & 1\end{bmatrix}\begin{bmatrix}1-\frac{5}{3} & -2-\frac{10}{3} & \frac{5}{3}\\0+1 & -1+2 & -1\\0+\frac{1}{3} & 0+\frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\sim C\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2/3 & -4/3 & 5/3\\1 & 1 & -1\\1/3 & 2/3 & -1/3\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}-2/3 & -4/3 & 5/3\\1 & 1 & -1\\1/3 & 2/3 & -1/3\end{bmatrix}$$

iii) $let\ A = \begin{bmatrix}1 & -3 & 2\\2 & 1 & 0\\0 & -1 & 1\end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix}1 & -3 & 2\\2 & 1 & 0\\0 & -1 & 1\end{vmatrix}$$

$$expand\ by\ R_3$$

$$= 0 - 0 + 1\begin{vmatrix}1 & -1\\2 & 1\end{vmatrix}$$

$$|A| = (1+2) = 3 \neq 0$$

$$|A| \neq 0\ so\ A^{-1}\ exist.$$

By Adjoins Method

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & 0\\-1 & 1\end{vmatrix} = (1+0) == 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} = -(2-0) = -2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} = (-3+2) = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}2 & 1\\0 & -1\end{vmatrix} = -(-3+2) = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix} = (1-0) = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & -3\\0 & -1\end{vmatrix} = -(-1+0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}-3 & 2\\1 & o\end{vmatrix} = -(0-2) = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 2\\2 & 0\end{vmatrix} = -(0-4) = 4$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & -3\\2 & 1\end{vmatrix} = (1+6) = 7$$

$$adjA = \begin{bmatrix}1 & 1 & -2\\-2 & 1 & 4\\-2 & 1 & 7\end{bmatrix}$$

$$\because A^{-1} = \frac{1}{|A|}adjA = \frac{1}{3}\begin{bmatrix}1 & 1 & -2\\-2 & 1 & 4\\-2 & 1 & 7\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}1/3 & 1/3 & -2/3\\-2/3 & 1/3 & 4/3\\-2/3 & 1/3 & 7/3\end{bmatrix}$$

By row operation $\because A = AI$

$$\Rightarrow A = \begin{bmatrix}1 & -3 & 2\\2 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -3 & 2\\2-2 & 1+6 & 0-4\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0-2 & 1 & 0\\0 & 0 & 1\end{bmatrix} R_2 - 2R_1$$

$$\sim R\begin{bmatrix}1 & -3 & 2\\0 & 7 & -4\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -3 & 2\\0 & 7 & -4\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix} R_2 interchange\ R_3$$

$$\sim R\begin{bmatrix}1 & -3 & 2\\0 & -1 & 1\\0 & 7 & -4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\-2 & 1 & 0\end{bmatrix}$$

$$R_1 + 3R_2, R_3 - 7R_2$$

$$\sim R \begin{bmatrix} 1+0 & -3+3 & 2-3 \\ 0 & 1 & -1 \\ 0 & 7-7 & -4+7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0-3 \\ 0 & 0 & 1 \\ -2 & 1 & 0+7 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ -2 & 1 & 7 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \frac{1}{3}R_3$$

$$R_1 + R_3, R_2 + R_3$$

$$\sim R \begin{bmatrix} 1 & 0 & -1+1 \\ 0 & 1 & -1+1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{2}{3} & 0+\frac{1}{3} & -3+\frac{7}{3} \\ 0-\frac{2}{3} & 0+\frac{1}{3} & -1+\frac{7}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 0 & -1+1 \\ 0 & 1 & -1+1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & +\frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & +\frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

By Column Operation:

$$\because A = AI$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 + 3C_1, C_3 - 2C_1$$

$$\sim C \begin{bmatrix} 1 & -3+3 & 2-2 \\ 2 & 1+6 & 0-4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0+3 & 0-2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7-8 & -4 \\ 0 & -1+2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3-4 & -2 \\ 0 & 1 & 0 \\ 0 & 0+2 & 1 \end{bmatrix} C_2 + 2C_3$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} (-1)C_2$$

$$C_1 - 2C_2, C_3 + 4C_2$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 2-2 & 1 & -4+4 \\ 0+2 & -1 & 1-4 \end{bmatrix} \begin{bmatrix} 1-2 & 1 & -2+4 \\ 0+2 & -1 & 0-4 \\ 0+4 & -2 & 1-8 \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -4 \\ 4 & -2 & -7 \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & -\frac{3}{-3} \end{bmatrix} \begin{bmatrix} -1 & 1 & \frac{2}{3} \\ 2 & -1 & -\frac{4}{-3} \\ 4 & -2 & -\frac{7}{-3} \end{bmatrix} -\frac{1}{3}C_3$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -\frac{2}{3} \\ 2 & -1 & \frac{4}{3} \\ 4 & -2 & \frac{7}{3} \end{bmatrix}$$

$$C_1 - 2C_3, C_2 + C_3$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2-2 & -1+1 & 1 \end{bmatrix} \begin{bmatrix} -1+\frac{4}{3} & 1-\frac{2}{3} & -\frac{2}{3} \\ 2-\frac{8}{3} & -1+\frac{4}{3} & \frac{4}{3} \\ 4-\frac{14}{3} & -2+\frac{7}{3} & \frac{7}{3} \end{bmatrix}$$

$$\sim C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Q10. Find the rank of the following matrices.

$$i) \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2-2 & -6+2 & 5-4 & 1-2 \\ 3-3 & 5+3 & 4-6 & -3-3 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & 6 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8 & -2 & 6 \end{bmatrix} -\frac{1}{4}R_2$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8-8 & -2+2 & -6-2 \end{bmatrix} R_3 - 8R_2$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -1 & 2 & 1\\0 & 1 & -\frac{1}{4} & \frac{1}{4}\\0 & 0 & 0 & 1\end{bmatrix} -\frac{1}{8}R_3$$

Which is in Echelon form No. of non − Zero rows = 3

So Rank = 3

ii) $\begin{bmatrix}1 & -4 & -7\\2 & -5 & 1\\1 & -2 & 3\\3 & -7 & 4\end{bmatrix}$

Solution: $\begin{bmatrix}1 & -4 & -7\\2 & -5 & 1\\1 & -2 & 3\\3 & -7 & 4\end{bmatrix}$

$R_2 - 2R_1, R_3 - R_1, R_4 - 3R_1$

$$\sim R\begin{bmatrix}1 & -4 & -7\\2-2 & -5+8 & 15\\1-1 & -2+4 & 10\\3-3 & -7+12 & 25\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -4 & -7\\0 & 3 & 15\\0 & 2 & 10\\0 & 5 & 25\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -4 & -7\\0 & 1 & 5\\0 & 2 & 10\\0 & 5 & 25\end{bmatrix}\frac{1}{3}R_2$$

$R_3 - 2R_2, R_4 - 5R_2$

$$\sim R\begin{bmatrix}1 & -4 & -7\\0 & 1 & 5\\0 & 2-2 & 10-10\\0 & 5-5 & 25-25\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & -4 & -7\\0 & 1 & 5\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Which is Echelon form No. of non-Zero rows=2

iii)

$$\begin{bmatrix}3 & -1 & 3 & 0 & -1\\1 & 2 & -1 & -3 & -2\\2 & 3 & 4 & 2 & 5\\2 & 5 & -2 & -3 & 3\end{bmatrix}$$

Solution:

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\3 & -1 & 3 & 0 & -1\\2 & 3 & 4 & 2 & 5\\2 & 5 & -2 & -3 & 3\end{bmatrix}R_1 \leftrightarrow R_2$$

$R_2 - 3R_1, R_3 - 2R_1, R_4 - 2R_1$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\3-3 & -1-6 & 3+3 & 0+9 & -1+6\\2-2 & 3-4 & 4+2 & 2+6 & 5+4\\2-2 & 5-4 & -2+2 & -3+9 & 3+4\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & -7 & 6 & 9 & 5\\0 & -1 & 6 & 8 & 9\\0 & 1 & 0 & 3 & 7\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & -1 & 6 & 8 & 9\\0 & -7 & 6 & 9 & 5\end{bmatrix}R_2 \leftrightarrow R_4$$

$R_3 + R_2, R_4 + 7R_2$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & -1+1 & 6+0 & 8+3 & 9+7\\0 & -7+7 & 6 & 9+21 & 5+49\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 6 & 11 & 16\\0 & 0 & 6 & 30 & 54\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 6 & 11 & 16\\0 & 0 & 1 & 5 & 9\end{bmatrix}\frac{1}{6}R_4$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 1 & 5 & 9\\0 & 0 & 6 & 11 & 16\end{bmatrix}R_3 \leftrightarrow R_4$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 1 & 5 & 9\\0 & 0 & 6-6 & 11-30 & 16-54\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 1 & 5 & 9\\0 & 0 & 0 & -19 & -38\end{bmatrix}$$

$$R\begin{bmatrix}1 & 2 & -1 & -3 & -2\\0 & 1 & 0 & 3 & 7\\0 & 0 & 1 & 5 & 9\\0 & 0 & 0 & 1 & 2\end{bmatrix} -\frac{1}{19}R_4$$

Which is in Echelon form.

No of non zero rows = 4 so rank = 4

System of linear Equations:

i) The equation $ax + by = k$ *where* $a \neq 0$ $b \neq 0, k \neq 0$ is called a non - homogenous linear equation in two variables x and y.

If $ax + by = 0$ *then it is called Homgenous linear* Equation.

ii) The equations

$$\left.\begin{matrix}a_1x + b_1y = k_1\\a_2x + b_2y = k_2\end{matrix}\right\} \text{are called system of}$$

Linear equations in two variables x and y. if k_1, k_2 Are not both zero or at least one of k_1, k_2 *is non − zero.*

If $\left.\begin{matrix}a_1x + b_1y = 0\\a_2x + b_2y = 0_2\end{matrix}\right\}$ *then it is called system of* Homogenous linear equations.

iii) The equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=k_1 \\ a_2x+b_2y+c_2z=k_2 \\ a_3x+b_3y+c_3z=k_3 \end{array}\right\}$$

are called system of non – homgenous linear Equations in three variables x, y *and* z *if* k_1, k_2 *and* k_3 *are not at all zero*

If

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \\ a_3x+b_3y+c_3z=0 \end{array}\right\} \text{ then it is called system of}$$

Homogenous linear equations.

System of Homogenous Linear Equations:

Consider,

$a_{11}x_1+a_{12}x_2+x_{13}x_3=b_1$
$a_{21}x_2+a_{22}x_2+a_{23}x_3=b_2$
$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then $AX=B$

Where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}, X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

here A is called matrix of cofficieents and

$$A_b=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \vdots & b_2 \\ a_{31} & a_{23} & a_{33} & \vdots & b_3 \end{bmatrix}$$

A_b *is called augmented matrix.*

Consistency of a System:

A system of linear equations is said to be consistent if

i) It has unique solution or
ii) It has an unlimited number of solution.

Inconsistency of a system:

A system of linear equations is said to be inconsistent if it has no solution.

Remember some important note:

i) If a system of linear equations is consistent and has unique solution then

$Rank(A)=Rank(A_b)$

ii) If a system of linear equations is consistent and has unlimited solution then

$Rank(A)=Rank(A_b)$

Also

$Rank\ (A) <$ *No. of variables used in the system.*

iii) If a system of linear equations is inconsistent.

i.e it has no solution then $Rank\ (A) \neq Rank(A_b)$

Trivial Solution:

if we solve a system and get valves of all variables zero, then the solution is called trivial solution.

For trivial solution $|A| \neq 0$

Non-trivial solution:

Solutions in which at least one of the variables has a valve different from zero is called non-trivial solution.

For non-trivial solution

$|A|=0$ *also*

$Rank\ (A) <$ *no. of varibles used in the system*

How to solve Homogenous Linear Equations:

Non-homogenous linear equations can be solved following three methods.

i) using matrices
ii) Using Echelon and Reduce echelon form.
iii) Using crammer's Rule.

Cramer's Rule:

$a_{11}x_1+a_{12}x_2+x_{13}x_3=b_1$
$a_{21}x_2+a_{22}x_2+a_{23}x_3=b_2$
$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then $AX=B \Rightarrow X=BA^{-1}$

Where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}, X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Matrix of cofactor $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{23} & A_{33} \end{bmatrix}$

$$adjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$X=\frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\frac{1}{|A|}\begin{bmatrix} b_1A_{11}+b_2A_{21}+b_3A_{31} \\ b_1A_{12}+b_2A_{22}+b_3A_{32} \\ b_1A_{13}+b_2A_{23}+b_3A_{33} \end{bmatrix}$$

$$\Rightarrow \boldsymbol{x_1}=\frac{b_1A_{11}+b_2A_{21}+b_3A_{31}}{\boldsymbol{A}}$$

Or $x_1=\dfrac{\begin{vmatrix} b_1 & a_{12} & a_{12} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$

Or $x_1=\dfrac{|A_1|}{|A|}$ $\quad A_1=\begin{vmatrix} b_1 & a_{12} & a_{12} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

Also

$$x_2=\frac{b_1A_{12}+b_2A_{22}+b_3A_{32}}{|A|}$$

$$x_2=\frac{\begin{vmatrix}a_{11} & b_1 & a_{13}\\ a_{21} & b_2 & a_{23}\\ a_{31} & b_3 & a_{33}\end{vmatrix}}{|A|}$$

$$x_2=\frac{|A_2|}{|A|}\quad ;|A_2|=\begin{vmatrix}a_{11} & b_1 & a_{13}\\ a_{21} & b_2 & a_{23}\\ a_{31} & b_3 & a_{33}\end{vmatrix}$$

$$x_3=\frac{b_1A_{13}+b_2A_{23}+b_3A_{33}}{|A|}$$

$$x_3=\frac{\begin{vmatrix}a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\\ a_{31} & a_{32} & b_3\end{vmatrix}}{|A|}$$

$$x_3=\frac{|A_3|}{|A|}\quad ;|A_3|=\begin{vmatrix}a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\\ a_{31} & a_{32} & b_3\end{vmatrix}$$

$$thus\ x_1=\frac{|A_1|}{|A|},\quad x_2=\frac{|A_2|}{|A|},x_3=\frac{|A_3|}{|A|}$$

https://newsongoogle.com/

Exercise 3.5

Q1. Solve the following systems of linear equations by Cramer's rule.

i) $2x+2y+z=3$
$3x-2y-2z=1$
$5x+y-3z=2$

Solution: in matrix form.

$$\begin{bmatrix}2 & 2 & 1\\ 3 & -2 & -2\\ 5 & 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

$$AX=B$$

Where

$$A=\begin{bmatrix}2 & 2 & 1\\ 3 & -2 & -2\\ 5 & 1 & -3\end{bmatrix},X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},B=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & 2 & 1\\ 3 & -2 & -2\\ 5 & 1 & -3\end{vmatrix}$$

$$2\begin{vmatrix}-2 & -2\\ 1 & -3\end{vmatrix}-2\begin{vmatrix}3 & -2\\ 5 & -3\end{vmatrix}+3\begin{vmatrix}3 & -2\\ 5 & 1\end{vmatrix}$$

$$=2(6+2)-2(-9+10)+1(3+10)$$

$$=2(8)-2(1)+(13)$$

$$=16-2(1)+13$$

$$|A|=16-2+13=27\neq 0$$

So solution exists. Now

$$|A_1|=\begin{vmatrix}3 & 2 & 1\\ 1 & -2 & -2\\ 2 & 1 & -3\end{vmatrix}$$

$$3\begin{vmatrix}-2 & -2\\ 1 & -3\end{vmatrix}-2\begin{vmatrix}1 & -2\\ 2 & -3\end{vmatrix}+3\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}$$

$$=3(6+2)-2(-3+4)+1(1+4)$$

$$=3(8)-2(1)+1(5)$$

$$|A_1|=24-2+5=27$$

$$\because x=\frac{|A_1|}{|A|}=\frac{27}{27}=1$$

$$|A_2|=\begin{vmatrix}2 & 3 & 1\\ 3 & 1 & -2\\ 5 & 2 & -3\end{vmatrix}$$

$$2\begin{vmatrix}1 & -2\\ 2 & -3\end{vmatrix}-3\begin{vmatrix}3 & -2\\ 5 & -3\end{vmatrix}+1\begin{vmatrix}3 & 1\\ 5 & 2\end{vmatrix}$$

$$=2(-3+4)-3(-9+10)+1(6-5)$$

$$=2(1)-3(1)+1(1)$$

$$|A_2|=2-3+1=0$$

$$\because y=\frac{|A_2|}{|A|}=\frac{0}{27}=0$$

$$|A_3|=\begin{vmatrix}2 & 2 & 3\\ 3 & -2 & 1\\ 5 & 1 & 2\end{vmatrix}$$

$$2\begin{vmatrix}-2 & 1\\ 1 & 2\end{vmatrix}-2\begin{vmatrix}3 & 1\\ 5 & 2\end{vmatrix}+3\begin{vmatrix}3 & -2\\ 5 & 1\end{vmatrix}$$

$$=2(-4-1)-2(6-5)+3(3+10)$$

$$=2(-5)-2(1)+3(13)$$

$$|A_3|=-10-2+39=27$$

$$\because z=\frac{|A_3|}{|A|}=\frac{27}{27}=1$$

Hence $x=1,y=0,z=1$

ii) $2x_1-x_2+x_3=5$

$$4x_1 + 2x_2 + 3x_3 = 8$$
$$3x_1 - 4x_2 - x_3 = 3$$

Solution:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$
$$AX = B$$
$$where\ A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$
$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - 4\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 2(-2 + 12) - 4(1 + 4) + 3(-3 - 2)$$
$$= 2(10) - 4(5) + 3(-5)$$
$$|A| = 20 - 20 - 12 = -15 \neq 0$$
$$So\ solution\ exist. Now$$
$$|A_1| = \begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$
$$5\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - 8\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 5(-2 + 12) - 8(1 + 4) + 3(-3 - 2)$$
$$= 5(10) - 8(5) + 3(-5)$$
$$|A_1| = 50 - 40 - 15 = -5$$
$$\because x = \frac{|A_1|}{|A|} = \frac{-5}{-15} = \frac{1}{3}$$
$$|A_2| = \begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}$$
$$2\begin{vmatrix} 8 & 3 \\ 3 & -1 \end{vmatrix} - 4\begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 8 & 3 \end{vmatrix}$$
$$= 2(-8 - 9) - 4(-5 - 3) + 3(15 - 8)$$
$$= 2(-17) - 4(-8) + 3(7)$$
$$|A_2| = -34 - 32 + 21 = 19$$
$$\because y = \frac{|A_2|}{|A|} = \frac{19}{-15}$$
$$|A_3| = \begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 0 \\ 3 & -4 & 3 \end{vmatrix}$$
$$2\begin{vmatrix} 2 & 8 \\ -4 & 3 \end{vmatrix} - 4\begin{vmatrix} -1 & 5 \\ -4 & 3 \end{vmatrix} + 3\begin{vmatrix} -1 & 5 \\ 2 & 8 \end{vmatrix}$$
$$= 2(6 + 32) - 4(3 + 20) + 3(-8 - 10)$$
$$= 2(38) - 4(17) + 3(18)$$
$$|A_3| = 76 - 68 - 54 = 76 - 122 = -46$$
$$\because z = \frac{|A_3|}{|A|} = \frac{-46}{-15} = \frac{46}{15}$$
$$x_1 = \frac{1}{3}, x_2 = -\frac{19}{15}, x_3 = \frac{46}{15}$$

iii) $2x_1 - x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 6$

$x_1 - 2x_2 - x_3 = 1$

Solution: in Matrix form

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B\begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$
$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix}$$
$$= 2(-2 + 4) - 1(1 + 2) + 1(-2 - 2)$$
$$= 2(2) - 1(3) + 1(-4)$$
$$|A| = 4 - 3 - 4 = -3 \neq 0$$
$$so\ solution\ exist$$

Now

$$|A_1| = \begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$
$$= 8\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 6\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$
$$= 8(-2 + 4) - 6(1 + 2) + 1(-2 - 2)$$
$$= 8(2) - 6(3) + 1(-4)$$
$$|A_1| = 16 - 8 - 4 = -6$$
$$\because x_1 = \frac{|A_1|}{|A|} = -\frac{6}{-3} = 2$$
$$|A_2| = \begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 6 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 8 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}$$
$$= 2(-6 - 2) - 1(-8 - 1) + 1(16 - 6)$$
$$= 2(-8) - 1(-8 - 1) + 1(16 - 6)$$
$$= 2(-8) - 1(-9) + 1(10)$$
$$|A_2| = 2(-8) - 1(-9) + 1(10)$$
$$|A_2| = -16 + 9 + 10 = 3$$
$$\because x_2 = \frac{|A_2|}{|A|} = \frac{3}{-3} = -1$$
$$|A_3| = \begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 8 \\ -2 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 8 \\ 2 & 6 \end{vmatrix}$$
$$= 2(2 + 12 - 1(-1 + 16) + 1(-6 - 16)$$
$$= 2(14) - 1(15) + (-22)$$
$$|A_3| = 28 - 15 - 22 = -9$$
$$\because x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$
$$hence\ x_1 = 2, x_2 = -1, x_3 = 3$$

Q3. Use matrix to solve the following systems:

i) $X - 2y + z = -1$

$3x + y - 2z = 4$

$y - z = 1$

Solution:

In Matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

Where

$$A=\begin{bmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{bmatrix},X=\begin{bmatrix}x\\y\\z\end{bmatrix},B=\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

$$|A|=\begin{vmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{vmatrix}$$

$$=1\begin{vmatrix}1&-2\\1&-1\end{vmatrix}-3\begin{vmatrix}-2&1\\1&-1\end{vmatrix}+0$$

$$=1(1+2)-3(2-1)=1(1)-3(1)$$

$$|A|=1-3=-2\neq 0\ so\ solution\ exist.$$

$$\because AX=B$$

$$X=A^{-1}B\rightarrow(1)$$

For A^{-1}

$$A_{11}=(-1)^{1+1}\begin{vmatrix}1&-2\\1&-1\end{vmatrix}=-1+2=1$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}3&-2\\0&1\end{vmatrix}=-(3+0)=-3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3&1\\0&1\end{vmatrix}=3-0=3$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-2&1\\1&-1\end{vmatrix}=-(2-1)=-1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1&1\\0&-1\end{vmatrix}=(-1-0)=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&-2\\0&-1\end{vmatrix}=-(-1-0)=1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2&1\\1&-2\end{vmatrix}=4-1=3$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1&1\\3&-2\end{vmatrix}=-(-2-3)=5$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1&-2\\3&1\end{vmatrix}=(1+6)=7$$

Matrix of cofactor $=\begin{bmatrix}1&3&3\\-1&-1&-1\\3&5&7\end{bmatrix}$

$$adj(A)=(matrix\ of\ cofector)^t$$

$$adjA=\begin{bmatrix}1&3&3\\3&-1&-1\\3&5&7\end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}adjA=\frac{1}{-2}\begin{bmatrix}1&3&3\\3&-1&-1\\3&5&7\end{bmatrix}$$

So (i)

$$X=\frac{1}{-2}\begin{bmatrix}1&3&3\\3&-1&-1\\3&5&7\end{bmatrix}\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{-2}\begin{bmatrix}-1-4+3\\-3-4+5\\-3-4+7\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2\\-2\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\Rightarrow\ x=1,y=1,z=0$$

ii)

$$2x_1+x_2+3x_2=3$$
$$x_1+x_2-2x_3=0$$
$$-3x_1-x_1+2x_3=-4$$

Solution:

$$\begin{bmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3\\0\\-4\end{bmatrix}$$

Where

$$A=\begin{bmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{bmatrix},X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},B=\begin{bmatrix}3\\0\\-4\end{bmatrix}$$

$$|A|=\begin{vmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{vmatrix}$$

$$=2\begin{vmatrix}1&-2\\-1&2\end{vmatrix}-1\begin{vmatrix}1&-2\\-3&2\end{vmatrix}+3\begin{vmatrix}1&1\\-3&-1\end{vmatrix}$$

$$=2(2-2)-1(2-6)+3(-1+3)$$

$$=2(0)-1(-4)+3(2)$$

$$|A|=4+6=10\neq 0\ solution\ is\ exist.$$

$$\because AX=B$$

$$\Rightarrow\ X=A^{-1}B\rightarrow(1)$$

For A^{-1}

$$A_{11}=(-1)^{1+1}\begin{vmatrix}1&-2\\-1&2\end{vmatrix}=(2-2)=0$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}1&-2\\-3&2\end{vmatrix}=-(2-6)4$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}1&1\\-3&-1\end{vmatrix}=-1+3=2$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1&3\\-1&2\end{vmatrix}=-(2+3)=-5$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}2&3\\-3&2\end{vmatrix}=4+9=13$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}2&1\\-3&-1\end{vmatrix}=-(-2+3)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1&3\\1&-2\end{vmatrix}=-2-3=-5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}2&3\\1&-2\end{vmatrix}=-(-4-3)7$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}2&1\\1&1\end{vmatrix}=2-1=1$$

$$Matrix\ of\ cofactor=\begin{bmatrix}0&4&2\\-5&13&-1\\-5&7&1\end{bmatrix}$$

$$adjA=(matrix\ of\ cofector)^t$$

$$adjA=\begin{bmatrix}0&-5&-5\\4&-13&7\\2&-1&1\end{bmatrix}$$

$$\because A^{-1}=\frac{adjA}{|A|}=\frac{1}{10}\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&-1\end{bmatrix}$$

So (i)

$$X=\frac{1}{10}\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&-1\end{bmatrix}\begin{bmatrix}3\\0\\-4\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{10}\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&-1\end{bmatrix}\begin{bmatrix}-3\\0\\-4\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{10}\begin{bmatrix}0-0+20\\12+0-28\\-6-0-4\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\-40\\-10\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-4\\-1\end{bmatrix}\quad so\ x_1=2,x_2=-4,x_3=-1$$

iii)

$$x+y=2$$
$$2x-z=1$$
$$2y-3z=-1$$

Solution:

In matrix form

$$\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Where

$$A=\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix},X=\begin{bmatrix}x\\y\\z\end{bmatrix},B=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$|A|=\begin{vmatrix}1&1&0\\2&0&-1\\0&2&-3\end{vmatrix}$$

$$=1\begin{vmatrix}0&-1\\2&-3\end{vmatrix}-2\begin{vmatrix}1&0\\2&-3\end{vmatrix}+0$$

$$|A|=(0+2)-2(-3-0)=2+6=8\neq 0$$

So solution exist.

$$\because AX=B$$

$$\Rightarrow\ X=A^{-1}B\rightarrow(1)$$

$for\ A^{-1}$

$$A_{11}=(-1)^{1+1}\begin{vmatrix}0&-1\\2&-3\end{vmatrix}=(0+2)=2$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}2&-1\\0&-3\end{vmatrix}=-(-6+0)=6$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}2&0\\0&2\end{vmatrix}=4$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1&6\\2&-3\end{vmatrix}=-(-3-0)=3$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1&0\\0&-3\end{vmatrix}=-(-3-0)=3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&1\\0&2\end{vmatrix}=-(2-0)=-2$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1&0\\0&-1\end{vmatrix}=-1$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1&0\\2&-1\end{vmatrix}=-(-1-0)=1$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1&1\\2&0\end{vmatrix}=(0-2)=-2$$

Matrix of cofactor $=\begin{bmatrix}2&6&4\\3&-3&-2\\-1&1&-2\end{bmatrix}$

$$adjA=(matrix\ of\ cofector\ )^t$$

$$adjA=\begin{bmatrix}2&3&-1\\6&-3&1\\4&-2&-2\end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}adjA=\frac{1}{8}\begin{bmatrix}2&3&-1\\6&-3&1\\4&-2&-2\end{bmatrix}$$

So (i)

$$X=\frac{1}{8}\begin{bmatrix}2&3&-1\\6&-3&1\\4&-2&-2\end{bmatrix}\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}4+3+1\\12-3-1\\8-2+2\end{bmatrix}=\frac{1}{8}\begin{bmatrix}8\\8\\8\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\Rightarrow\ x=1,y=1,z=1$$

Q3.

Solve the following systems by reducing their augmented matrix to echelon form and the reduced echelon form.

i)

$$x_1-2x_2-2x_3=-1$$
$$2x_1+3x_2+x_3=1$$
$$5x_1-4x_2-3x_3=1$$

Solution: Solution of Echelon form

The augment matrix is

$$A_b=\begin{bmatrix}1&-2&-2&:&-1\\2&3&1&:&1\\5&-4&-3&:&1\end{bmatrix}$$

$$R_2-2R_1,R_3-5R_1$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\2-2&3+4&1+4&:&1+2\\5-5&-4+10&-3+10&:&1+5\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\0&7&5&:&3\\0&6&7&:&6\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\0&7-6&5-7&:&3-6\\0&6&7&:&6\end{bmatrix}R_2-R_3$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\0&1&-2&:&-3\\0&6-6&7+12&:&6+18\end{bmatrix}R_3-6R_2$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\0&1&-2&:&-3\\0&0&19&:&24\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&-2&-2&:&-1\\0&1&-2&:&-3\\0&0&19&:&24\end{bmatrix}\frac{1}{19}R_3$$

$$\Rightarrow\ x_1-2x_2-2x_3=-1\rightarrow(1)$$
$$x_2-2x_3=-3\rightarrow(2)$$
$$x_3=\frac{24}{19}\rightarrow(3)$$

$$put\ x_3=\frac{24}{19}\ in\ (2)$$

$$x_2-2\left(\frac{24}{19}\right)=-3\Rightarrow x_2=-3+\frac{48}{19}$$

$$x_2=\frac{-57+48}{19}=-\frac{9}{19}$$

$$put\ x_3=\frac{24}{19}\ and\ x_2=-\frac{9}{19}\ in(1)$$

$$x_1-2\left(-\frac{9}{19}\right)-2\left(\frac{24}{19}\right)=-1$$

$$x_1+\frac{18}{19}-\frac{48}{19}+1=0$$

$$x_1+\frac{(18-48+19)}{19}=0$$

$$x_1-\frac{11}{19}=0\Rightarrow x_1=\frac{11}{19}$$

$$hence\ x_1=\frac{11}{19},x_2=-\frac{9}{19},x_3=\frac{24}{19}$$

Solution by Reduce Echelon form

we reduce $\begin{bmatrix}1&-2&-2&:&-1\\0&1&-2&:&-3\\0&0&1&:&\frac{24}{9}\end{bmatrix}$ into reduce echelon form.

$$\sim R\begin{bmatrix}1&-2+2&-2-4&:&-1-6\\0&1&-2&:&-3\\0&0&1&:&\frac{24}{9}\end{bmatrix} R_1+2R_2$$

$$\sim R\begin{bmatrix}1&0&-6&:&-7\\0&1&-2&:&-3\\0&0&1&:&\frac{24}{9}\end{bmatrix}$$

R_1+6R_3, R_2+2R_3

$$\sim R\begin{bmatrix}1&0&-6+6&:&-7+6\left(\frac{24}{19}\right)\\0&1&-2+2&:&-3+2\left(\frac{24}{14}\right)\\0&0&1&:&\frac{24}{9}\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&0&0&:&\frac{-133+144}{19}\\0&1&0&:&\frac{-57+48}{19}\\0&0&1&:&\frac{24}{9}\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&0&0&:&\frac{11}{19}\\0&1&0&:&\frac{-9}{19}\\0&0&1&:&\frac{24}{9}\end{bmatrix}$$

$$x_1=\frac{11}{19}, x_2=-\frac{9}{19}, x_3=\frac{24}{19}$$

ii)

$x+2y+z=2$

$2x+y+2z=-1$

$2x+3y-z=9$

Solution:

Solution by Echelon form

The augmented matrix is

$$A_b=\begin{bmatrix}1&2&1&:&2\\2&1&2&:&-1\\2&3&-1&:&9\end{bmatrix}$$

R_2-2R_1, R_3-2R_1

$$\sim R\begin{bmatrix}1&2&1&:&2\\2-2&1-3&2-2&:&-1-4\\2-2&3-4&-1-2&:&9-4\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&-2&0&:&-5\\0&-1&-3&:&5\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&-2&0&:&-5\\0&-1&-3&:&5\end{bmatrix} R_2\leftrightarrow R_3$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&-3&0&:&-5\end{bmatrix}(-1)R_2$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&-3+3&0+9&:&-5-15\end{bmatrix} R_3+3R_2$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&0&9&:&-20\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&0&9&:&-20\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&0&1&:&-\frac{20}{9}\end{bmatrix}\frac{1}{9}R_3$$

⇨ $x+2y+z=2\rightarrow(1)$

$y+3z=-5\rightarrow(2)$

$Z=-\frac{20}{9}\rightarrow(3)$

$$put\ z=-\frac{20}{9}in\ (2)$$

$$y+3\left(\frac{-20}{9}\right)=-5\Rightarrow y=-5+\frac{60}{9}$$

$$=y=\frac{-45+60}{9}=\frac{15}{9}=\frac{5}{3}$$

Put $z=-\frac{20}{9}$ $and\ y=\frac{5}{3}\ in\ (1)$

$$x+2\left(\frac{5}{3}\right)-\frac{20}{9}=2$$

$$x+\frac{10}{3}-\frac{20}{9}-2=0$$

$$x+\frac{30-20-18}{9}=0\Rightarrow x-\frac{8}{9}=0\Rightarrow x=\frac{8}{9}$$

$$hence\ x=\frac{8}{9}, y=\frac{5}{3}, z=-\frac{20}{9}$$

Solution by reduced Echelon form:

We reduce

$\begin{bmatrix}1&2&1&:&2\\0&1&3&:&-5\\0&0&1&:&-\frac{20}{9}\end{bmatrix}$ $in\ to\ reduce\ echelon\ form$

$$\sim R\begin{bmatrix}1&2-2&1-6&:&2+10\\0&1&3&:&-5\\0&0&1&:&-\frac{20}{9}\end{bmatrix} R_1-2R_2$$

$$\sim R\begin{bmatrix}1&0&-5&:&12\\0&1&3&:&-5\\0&0&1&:&-\frac{20}{9}\end{bmatrix}$$

$$\sim R\begin{bmatrix}1&0&-5+5&:&12+5\left(-\frac{20}{9}\right)\\0&1&3-3&:&-5-3\left(-\frac{20}{9}\right)\\0&0&1&:&-\frac{20}{9}\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 0 & 0 & : & \frac{108-100}{9}\\ 0 & 1 & 0 & : & \frac{-45+60}{9}\\ 0 & 0 & 1 & : & -\frac{20}{9}\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 0 & 0 & : & \frac{8}{9}\\ 0 & 1 & 0 & : & \frac{15}{9}\\ 0 & 0 & 1 & : & -\frac{20}{9}\end{bmatrix}$$

⇨ $x_1 = \frac{8}{9}, x_2 = \frac{15}{9}, x_3 = -\frac{20}{9}$

iii)

$$x_1 + 4x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 9$$
$$3x_1 + 2x_2 - 2x_3 = 12$$

Solution:

Solution by Echelon form;

The augmented matrix is

$$A_b\begin{bmatrix}1 & 4 & 2 & : & 2\\ 2 & 1 & -2 & : & 9\\ 3 & 2 & -2 & : & 12\end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 2-2 & 1-8 & -2-4 & : & 9-4\\ 3-3 & 2-12 & -2-6 & : & 12-6\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & -7 & -6 & : & 5\\ 0 & -10 & -8 & : & 6\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 21 & 18 & : & -15\\ 0 & 30 & -16 & : & 12\end{bmatrix} (-3)R_2, 2R_3$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -2\\ 0 & -20 & -16 & : & 12\end{bmatrix} R_2 - R_3$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -2\\ 0 & -20 & -16 & : & 12\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -2\\ 0 & -20+20 & -16+40 & : & 12-60\end{bmatrix} R_3 + 20R_2$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -3\\ 0 & 0 & 24 & : & -48\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -3\\ 0 & 0 & 1 & : & -2\end{bmatrix} \frac{1}{24}R_3$$

$$x_1 + 4x_2 + 2x_3 = 2 \rightarrow (1)$$
$$x_2 + 2x_3 = -3 \rightarrow (2)$$
$$x_3 = -2 \rightarrow (3)$$
$$put\ x_3 = -2\ in\ (2)$$
$$x_2 + 2(-2) = -3 \Rightarrow x_2 = -3 + 4 = 1$$
$$x_2 = 1$$
$$put\ x_2 = 1\ and\ x_3 = -2\ in\ (1)$$
$$x_1 + 4(1) + 2(-2) = 2$$
$$x_1 + 4 - 4 = 2 \Rightarrow x_1 = 2$$

Hence $x_1 = 2\ x_2 = 1\ x_3 = -2$

Solution by reduced Echelon form

We reduce

$$\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 2 & : & -3\\ 0 & 0 & 1 & : & -2\end{bmatrix} into\ reduce\ echelon\ form.$$

$$\sim R\begin{bmatrix}1 & 4-4 & 2-8 & : & 2+12\\ 0 & 1 & 2 & : & -3\\ 0 & 0 & 1 & : & -2\end{bmatrix} R_1 - 4R_2$$

$$\sim R\begin{bmatrix}1 & 0 & -6 & : & 14\\ 0 & 1 & 2 & : & -3\\ 0 & 0 & 1 & : & -2\end{bmatrix}$$

$$R_1 + 6R_2, R_2 - 2R_3$$

$$\sim R\begin{bmatrix}1 & 0 & -6+6 & : & 14-12\\ 0 & 1 & 2-2 & : & -3+4\\ 0 & 0 & 1 & : & -2\end{bmatrix}$$

$$\sim R\begin{bmatrix}1 & 0 & 0 & : & 2\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & -2\end{bmatrix}$$

⇨ $x_1 = 2, x_2 = 1, x_3 = -2$

Q4. Solve the following systems of homogenous linear equations.

$$x + 2y - 2z = 0 \rightarrow (1)$$
$$2x + y + 5z = 0 \rightarrow (2)$$
$$5x + 4y + 8z = 0 \rightarrow (3)$$

Solution:

In matrix form

$$\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 5\\ 5 & 4 & 8\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Where A= $\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 5\\ 5 & 4 & 8\end{bmatrix}$

$$= 1\begin{vmatrix}1 & 5\\ 4 & 8\end{vmatrix} - 2\begin{vmatrix}2 & -2\\ 4 & 8\end{vmatrix} + 5\begin{vmatrix}2 & -2\\ 1 & 5\end{vmatrix}$$
$$= 1(8-20) - 2(16+8) + 5(10+2)$$
$$= 1(-12) - 2(24) + 5(12)$$
$$|A| = -12 - 48 + 60 = 0$$
$$|A| = 0$$

Hence system has non-trivial solution now we solve (1) and (2) to find $x_1 and\ x_2$

$$by (1) - 2(2)$$

⇨ $x + y - 2z = 0$

$\pm 4x \pm 2y \pm 10z = 0$

$$-3x - 12z = 0$$

⇨ $3x = -12 \Rightarrow x = -4z$

$by\ (2) - 2(1)$

⇨ $2x + y + 5z = 0$

$\pm 2x \pm 4y \mp 4z = 0$

$$\mathbf{-3y + 9z = 0}$$
$$\mathbf{-3y = -9z = 0 \Rightarrow y = 3z}$$
$$put\ x = -4z\ and\ y = 32\ in (3)$$
$$5(-4z) + 4(32) + 8z = 0$$
$$-20z + 12Z + 8z = 0$$

⇨ $0 = 0$

$$Eq\ (3)\ \ is\ stisfied$$
$$Let\ z = t\ , t \in \mathbb{R}\ then$$
Then $x = -4t\ and\ y = 3t$
$so\ x = -4t\ , y = 3t\ , and\ z = t$
Hence the system has unlimited solutions.

ii)
$x_1 + 4x_2 + 2x_3 = 0 \rightarrow (1)$
$2x_1 + x_2 - 3x_2 = 0 \rightarrow (2)$
$3x_1 + 2x_2 - 4x_3 = 0 \rightarrow (3)$

Solution:
In matrix form:
$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$where\ A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$
$$= 1\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ 2 & -4 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix}$$
$$= 1(-4+6) - 2(-16-4) + 3(-12-2)$$
$$2 - 2(-20) + 3(-14) = 2 + 40 - 42 = 0$$
$$|A| = 0$$
Hence system has non-trivial solution now we solve (1) and (2) to find $x_1 and\ x_2$

$$by (1) - 4(2)$$
⇨ $x_1 + 4x_2 + 2x_3 = 0$
$\pm 8x_1 \pm 4x_2 \mp 12x_3 = 0$

$-7x_1 + 14x_3 = 0$
⇨ $x_1 = 2x_3$
$by\ (2) - 2(1)$
⇨ $2x_1 + x_2 - 3x_3 = 0$
$\pm 2x_1 \pm 8x_2 \pm 4x_3 = 0$

$\boldsymbol{-7x_2 - 7x_3 = 0}$
$\Rightarrow \boldsymbol{x_2 = -x_3}$
$put\ x_1 = 2x_3\ and\ x_2 = -x_3\ in(3)$
⇨ $3(2x_3) + 2(-x_3) - 4x_3 = 0$
⇨ $6x_3 - 2x_3 - 4x_3 = 0$
⇨ $0 = 0$
$$Eq\ (3)\ \ is\ stisfied$$
$Let\ x_3 = t\ , t \in \mathbb{R}\ then\ x_1 = 2t\ \ and\ x_2 = -t$
$hence\ x_1 = 2t, x_2 = -t, x_3 = t$
$$\because\ system\ has\ unlimited\ solution$$

iii)

$$x_1 - 2x_2 - x_3 = 0 \rightarrow (1)$$
$$x_1 + x_2 + 5x_3 = 0 \rightarrow (2)$$
$$2x_1 - x_2 + 4x_3 = 0 \rightarrow (3)$$

Solution:
In matrix form
$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$
$$= 1\begin{vmatrix} 1 & 5 \\ -1 & 4 \end{vmatrix} - 1\begin{vmatrix} -2 & -1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix}$$
$$= (4+5) - 1(-8-1) + 2(-10+1)$$
$$= 9 + 9 - 18 = 0$$
$|A| = 0$
Hence system has non-trivial solution now we solve (1) and (2) to find $x_1 and\ x_2$
$$by(1) + \ 2(2)$$
⇨ $x_1 - 2x_2 - x_3 = 0$
$+2x_1 + x_2 + 10x_3 = 0$

$3x_1 + 9x_3 = 0$
⇨ $x_1 = -3x_3$
$by\ (2) - 2(1)$
⇨ $x_1 + x_2 + 5x_3 = 0$
$\pm x_1 \mp 8x_2 \mp x_3 = 0$

$\boldsymbol{3x_2 + 6x_3 = 0}$
$\Rightarrow \boldsymbol{x_2 = -2x_3}$
$put\ x_1 = -2x_3\ and\ x_2 = -2x_3\ in(3)$
⇨ $2(-3x_3) - 2(-x_3) + 4x_3 = 0$
⇨ $-6x_3 + 2x_3 + 4x_3 = 0$
⇨ $0 = 0$
$$Eq\ (3)\ \ is\ stisfied$$
$Let\ x_3 = t\ , t \in \mathbb{R}\ then\ x_1 = -3t\ \ and\ x_2 = -2t$
$hence\ x_1 = -3t, x_2 = -2t, x_3 = -t$
$$\because\ system\ has\ unlimited\ solution$$

Q5. Find the value of λ for which the following systems have non-trivial solution. Also solve the system for the valves of λ

i)
$$x + y + z = 0$$
$$2x + 2y - 2z = 0$$
$$x + 2y - 2z = 0$$

Solution:
In matrix form
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Where
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\because\ system\ has\ non-trivial\ solution\ so, |A| = 0$
⇨ $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$
⇨ $1\begin{vmatrix} 1 & -\lambda \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} = 0$
⇨ $-2 + 2\lambda - 2(-2-2) + 1(-\lambda - 1) = 0$

$$-2+2\lambda+8+\lambda-1=0$$
$$\lambda+5=0 \Longrightarrow \lambda=-5$$

For $\lambda=-5$ *given system becomes*

$$x+y+z=0 \rightarrow (1)$$
$$2x+y+5z=0 \rightarrow (2)$$
$$x+2y-2z=0 \rightarrow (3)$$

we solve (1) and (2) to find $x and\ y$

$$by (2)-(1)$$

⇨ $2x+y+5z=0$

$\pm x \pm y \pm z=0$

$$x+4z=0$$

⇨ $x=-4z$

$by\ (2)-2(1)$

⇨ $2x+y+5z=0$

$\pm 2x \pm 2y \pm 2z=0$

$$\boldsymbol{-y+3z=0}$$
$$\Rightarrow \boldsymbol{y=3z}$$

$$put\ x=-4z\ and\ y=3z\ in (3)$$
$$(-4z)+2(3z)+2z=0$$
$$-4z+6Z-2z=0$$

⇨ $0=0$

$$Eq\ (3)\ is\ stisfied$$
$$Let\ z=t\ , t\in\mathbb{R}\ then\ x=-4t\ , y=3t$$
$$hence\ x=-4t, y=3t, z=t$$
$$\because system\ has\ unlimited\ solution:$$

ii)

$$x_1+4x_2+\lambda x_3=0$$
$$2x_1+x_2-3x_3=0$$
$$3x_1+\lambda x_2-4x_3=0$$

Solution:

In matrix form

$$\begin{bmatrix}1&4&\lambda\\2&1&-3\\3&\lambda&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Where $A=\begin{bmatrix}1&4&\lambda\\2&1&-3\\3&\lambda&-4\end{bmatrix}, X=\begin{bmatrix}x_1\\x_2\\x3\end{bmatrix}, O=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Since the system has non-trivial solution.

So $|A|=0$

$$\begin{vmatrix}1&4&\lambda\\2&1&-3=0\\3&\lambda&-4\end{vmatrix}$$

⇨ $1\begin{vmatrix}1&-3\\\lambda&-4\end{vmatrix}-2\begin{vmatrix}4&\lambda\\\lambda&-4\end{vmatrix}+3\begin{vmatrix}4&\lambda\\1&-3\end{vmatrix}=0$

⇨ $-4-3\lambda-2(-16-\lambda^2)+3(12-\lambda)=0$

$$-4+3\lambda+32+2\lambda^2-36-3\lambda=0$$
$$2\lambda^2-8=0 \Longrightarrow 2\lambda^2=8 \Rightarrow \lambda^2=4$$
$$\lambda=\pm 2$$

For $\lambda=2$ *given system becomes as*

$$x_1+4x_2+2x_3=0 \rightarrow (1)$$
$$2x_1+x_2-3x_3=0 \rightarrow (2)$$
$$3x_1+2x_2-4x_3=0 \rightarrow (3)$$

we solve (1) -4 (2) to find $x_1 and\ x_2$

$$by(1)+\ 2(2)$$

⇨ $x_1+4x_2+2x_3=0$

$+8x_1+4x_2-12x_3=0$

$-7x_1+14x_3=0$

⇨ $x_1=2x_3$

$by\ (2)-2(1)$

⇨ $2x_1+x_2-3x_3=0$

$\pm 2x_1 \pm 8x_2 \pm 4x_3=0$

$$\boldsymbol{-7x_2-7x_3=0}$$

$\Rightarrow \boldsymbol{x_2=-x_3}$

$$put\ x_1=2x_3\ and\ x_2=-x_3\ in(3)$$

⇨ $3(2x_3)+2(-x_3)-4x_3=0$

⇨ $6x_3-2x_3-4x_3=0$

⇨ $0=0$

$$Eq\ (3)\ is\ stisfied$$

$Let\ x_3=t\ , t\in\mathbb{R}\ then\ x_1=2t\ \ and\ x_2=-t$

$hence\ x_1=2t, x_2=-t, x_3=t$

$$\because system\ has\ unlimited\ solution$$

For $\boldsymbol{\lambda=-2}$

$$x_1+4x_2-2x_3=0 \rightarrow (1)$$
$$2x_1+x_2-3x_3=0 \rightarrow (2)$$
$$3x_1-2x_2-4x_3=0 \rightarrow (3)$$

we solve (1) and (2) to find $x_1 and\ x_2$

$$by(1)-4(2)$$

⇨ $x_1+4x_2-2x_3=0$

$\pm 8x_1 \pm 4x_2 \mp 12x_3=0$

$$-7x_1+10x_3=0$$

⇨ $x_2=\frac{10}{7}x_3$

$by\ (2)-2(1)$

⇨ $2x_1+x_2-3x_3=0$

$\pm 2x_1 \pm 8x_2 \mp 4x_3=0$

$$\boldsymbol{-7x_2+x_3=0}$$

$\Rightarrow \boldsymbol{x_2=\frac{1}{7}x_3}$

$$put\ x_1=\frac{10}{7}x_3\ and\ x_2=\frac{1}{7}x_3\ in(3)$$

⇨ $3\left(\frac{10}{7}x_3\right)-2\left(\frac{1}{7}x_3\right)-4x_3=0$

⇨ $\frac{30}{7}x_3-\frac{2}{7}x_3-4x_3=0$

⇨ $\frac{30x_2-2x_2-28x_2}{7}=0$

⇨ $0=0$

$$Eq\ (3)\ is\ stisfied$$

$Let\ x_3=t\ , t\in\mathbb{R}\ then\ x_1=\frac{10}{7}t\ \ and\ x_2=-\frac{1}{7}t$

$hence\ x_1=\frac{10}{7}t, x_2=-\frac{1}{7}t, x_3=t$

$$\because system\ has\ unlimited\ solution$$

Important note:

If a system does not possess unique solution it means that it has unlimited solutions. We know already a system has unlimited solution if

$$Rank(A)=Rank(A_b) and$$
$$Rank\ (A)<no. of\ variables\ used\ in\ the\ system.$$

Q6. Find the value of λ for which the following system does not possess unique solution. Also solve the system for the value of solve the system for the value of λ

$$x_1 + 4x_2 + \lambda x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 11$$
$$3x_1 + 2x_2 - 2x_3 = 16$$

Solution:

Augmented matrix is

$$A_b = \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1$$

$$\sim R \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 2-2 & 1-8 & -2-2\lambda & : & 11-4 \\ 3-3 & 2-12 & -2-2\lambda & : & 16-6 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & -7 & -2(1+\lambda) & : & 7 \\ 0 & -10 & -(2+3\lambda & : & 10 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & -7 & -\frac{2}{7}(1+\lambda) & : & -1 \\ 0 & -10 & \frac{6-\lambda}{7} & : & 0 \end{bmatrix} R_3 + 10R_2$$

$$\rightarrow (A)$$

∵System does not possess unique solution for

$$\frac{6-\lambda}{7} = 0$$

⇨ $6 - \lambda = 0$

⇨ $\lambda = 6$

For $\lambda = 6$ $(A) becomes$

$$\sim R \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & \frac{2+2(6)}{7} & : & -1 \\ 0 & 0 & \frac{6-6}{7} & : & 0 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x_1 + 4x_2 + 6x_3 = 2 \rightarrow (1)$$
$$x_2 + 2x_3 = -1$$

⇨ $x_2 = -1 - 2x_3 \; put \; in(1)$

$$x_1 + 4(-1 - 2x_3) + 6x_3 = 2$$
$$x_1 - 4 - 8x_3 + 6x_3 = 2$$
$$x_1 - 2x_3 + 6$$

⇨ $x_1 = 2x_3 + 6$

$$let \; x_3 = t \; then \; x_1 = 2t + 6$$
$$t\varepsilon R \; and \; x_2 = 2t - 1$$

Hence $x_1 = 2t + 6, x_2 = -2t - 1 \; and \; x_3 = t$

$\because$ *system has unlimited solutions.*